Sufficient condition for local invertibility of spatio-temporal 4D B-spline deformations

Se Young Chun[a], Colas Schretter[b], and Jeffrey A. Fessler[c]

[a]Massachusetts General Hospital and Harvard Medical School, USA,
[b]RWTH Aachen University, Aachen, Germany,
[c]University of Michigan, Ann Arbor, USA

Motivation

There has been much research on achieving diffeomorphism (or local invertibility) of 2D and 3D B-spline deformations. Sufficient conditions for local invertibility of 2D and 3D cubic B-spline were proposed by Choi *et al.*. These determine specific parameter values to guarantee local invertibility. We also proposed larger sufficient conditions for local invertibility of 2D and 3D *n*th-order B-spline.
Recently, many researchers use 4D B-spline deformations and diffeomorphism (local invertibility) is still attractive for them. However, there is no specific criteria to guarantee local invertibility of 4D B-spline deformations so far. We investigated this specific guideline for 4D B-spline deformations.

Mathematical model for image registration

A 4D nonrigid transformation $\underline{T} : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3 \times \mathbf{R}$ can be written

$$\underline{T}(\underline{r}, t) = \begin{bmatrix} \underline{r} + \underline{d}(\underline{r}, t) \\ t \end{bmatrix}, \qquad (1)$$

where $\underline{r} = (x, y, z)^T$ and $\underline{d}(\underline{r}, t)$ is a 3D deformation $\underline{d} = (d^x, d^y, d^z)^T$ with the time axis:

$$d^q(\underline{r}, t) = \sum_{ijkl} c^q_{ijkl} \beta(x/m_x - i)\, \beta(y/m_y - j)\, \beta(z/m_z - k)\, \phi(t/m_t - l),$$

where $q \in \{x, y, z\}$, m_q is knot spacing in q direction, β is a nth-order B-spline basis, and ϕ is a basis that satisfies the following conditions:

$$\sum_l \phi(t/m_t - l) = 1, \quad \forall t,$$
$$\phi(t/m_t - l) \geq 0, \quad \forall t.$$

The goal in 4D image registration is to estimate the deformation coefficients $\underline{c} = \{c^q_{i,j,k,l}\}$ by maximizing a similarity metric Ψ:

$$\hat{\underline{c}} = \arg\max_{\underline{c}} \Psi[g(\cdot), f(\underline{T}(\cdot; \underline{c}))] \qquad (2)$$

where $g(\underline{r}, t)$ / $f(\underline{r})$ denote an image sequence / a 3D or 4D image.

References

1. Y. Choi and S. Lee, "Injectivity conditions of 2D and 3D uniform cubic B-spline functions," Graphical Models 62(6), pp. 411-27, 2000.
2. S.Y. Chun and J. A. Fessler, "A simple regularizer for B-spline nonrigid image registration that encourages local invertibility," IEEE J. Sel. Top. Sig. Proc. 3(1), pp. 159-69, Feb. 2009.
3. Michaël Sdika, "A fast nonrigid image registration with constraints on the Jacobian using large scale constrained optimization," IEEE Trans. Med. Imaging 27(2), pp. 271-81, 2008.

Counterexample

An image sequence (at discrete time points) is usually given and we may try to enforce local invertibility of spatial deformations (2D or 3D) at each time points. Does it guarantee local invertibility of *spatial and temporal* deformations? This simple mathematical example shows that it is not always true.

For example, assume that we have two invertible transformations at $t = 1$ and 2 such that

$$\underline{T}(x, y, 1) = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \underline{T}(x, y, 2) = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We can easily show that $|\nabla \underline{T}(x, y, 1)| = |\nabla \underline{T}(x, y, 2)| = 1 > 0$. However, if we interpolate a $\underline{T}(x, y, t)$ value at $t = 1/2$ by using a usual linear interpolation, we can show that $\underline{T}(x, y, 1/2) = \underline{T}(x, y, 1)/2 + \underline{T}(x, y, 2)/2$ and $|\nabla \underline{T}(x, y, 1/2)| = 0$ for all (x, y). Therefore, $|\nabla \underline{T}(\underline{r}, t_n)| > 0$ for all $\underline{r}$ and $t_1, \cdots, t_N$ does not imply that $|\nabla \underline{T}(\underline{r}, t)| > 0$ for all $\underline{r}$ in general when one has invertible transformations at discrete time points and one uses an overlapped time bases for the time axis interpolation.

Proposed 4D sufficient condition

Suppose $0 \leq k_q < \frac{1}{2}$ for $q \in \{x, y, z\}$. Define:

$$\begin{aligned} C_5 \triangleq \{\underline{c} :& -m_x k_x \leq c^x_{i+1,j,k,l} - c^x_{i,j,k,l} \leq m_x K_x, \\ & -m_y k_y \leq c^y_{i,j+1,k,l} - c^y_{i,j,k,l} \leq m_y K_y, \\ & -m_z k_z \leq c^z_{i,j,k+1,l} - c^z_{i,j,k,l} \leq m_z K_z, \\ & |c^q_{i+1,j,k,l} - c^q_{i,j,k,l}| \leq m_q k_q \text{ for } q = y, z, \\ & |c^q_{i,j+1,k,l} - c^q_{i,j,k,l}| \leq m_q k_q \text{ for } q = x, z, \\ & |c^q_{i,j,k+1,l} - c^q_{i,j,k,l}| \leq m_q k_q \text{ for } q = x, y, \forall i, j, k, l\}. \end{aligned}$$

In (1), if $\underline{c} \in C_5$ then $1 - (k_x + k_y + k_z) \leq |\nabla \underline{T}(\underline{r}, t)| \leq (1 + K_x)(1 + K_y)(1 + K_z) + (1 + K_x)k_y k_z + k_x(1 + K_y)k_z + k_x k_y(1 + K_z)$ $\forall \underline{r} \in \mathbf{R}^3$ and $\forall t \in \mathbf{R}$. Moreover, if $k_x + k_y + k_z < 1$, then the transformation (1) is locally invertible everywhere.

Simulation results

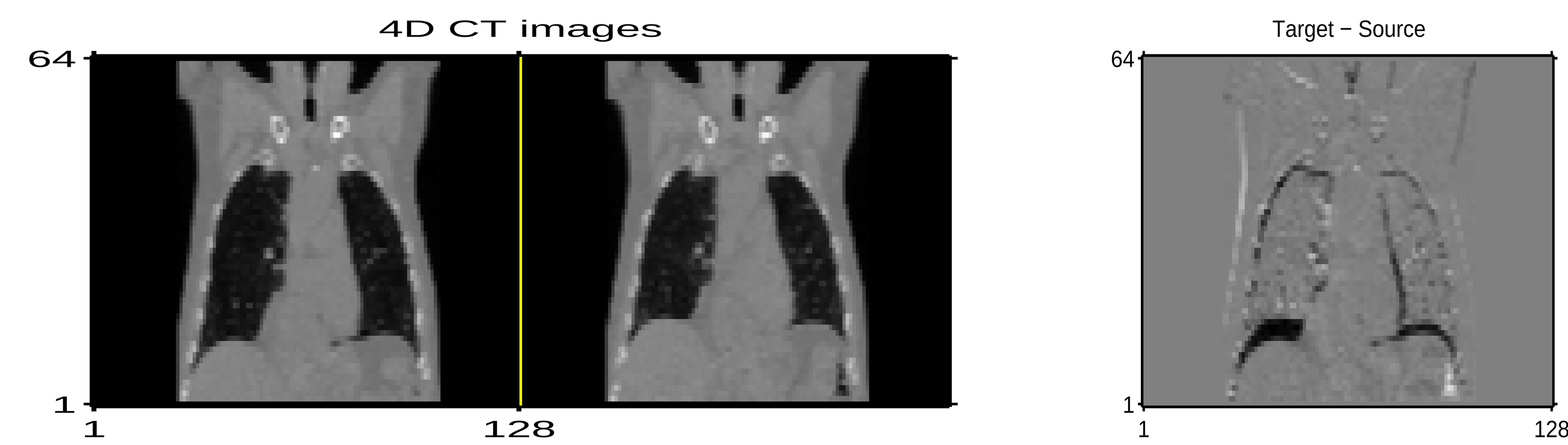

Figure: 4D CT image sequence (128 × 128 × 64 × 10, left) and difference image between a target (frame 7) and a source (right).

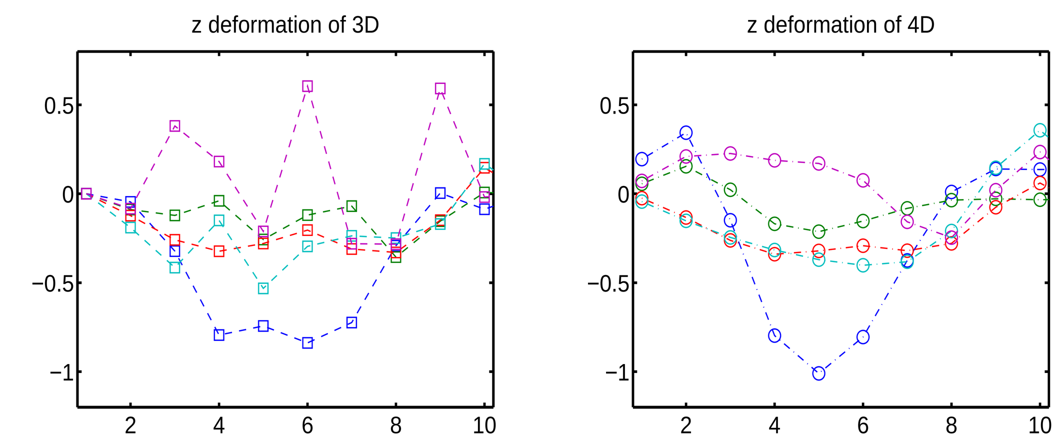

Figure: Z direction of 3D (left) and 4D (right) deformations over one cycle of breathing at 5 different voxels.

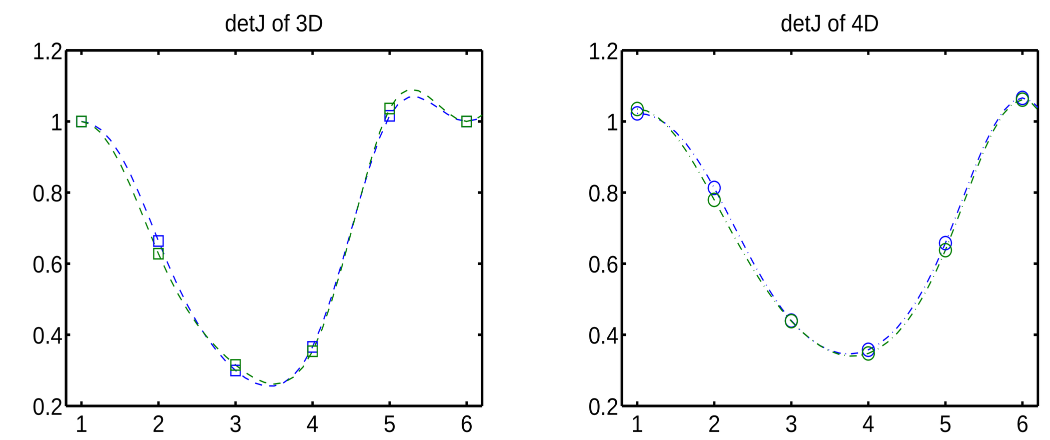

Figure: Jacobian determinants at control points (square, circle) of 3D (left) and 4D (right) over time and inbetween values at 2 different voxels.

Conclusion

Enforcing the local invertibility of 4D deformations at discrete times does not guarantee the local invertibility of 4D deformations over continuous time. One may achieve local invertibility using additional temporal regularization or smoothing, but it is challenging to determine *how much* one has to regularize. Instead of using sufficiently small 4D deformations for local invertibility, our proposed sufficient condition provides a guideline to achieve local invertibility of spatio-temporal 4D B-spline deformations over continuous space and time.

sychun@pet.mgh.harvard.edu