Determining the Automorphism Group of the Linear Ordering Polytope

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Abstract

In this paper we explore the combinatorial automorphism group of the linear ordering polytope $\mathbf{P}^n_{\mathrm{LO}}$ for each $n > 1$. We establish that this group is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$ if $n > 2$ (and to $\mathbb{Z}_2$ if $n = 2$). Doing so, we provide a simple and unified interpretation of all the automorphisms.

Key words: Linear ordering polytope, automorphism group, facets

1 Introduction

The *linear ordering polytope* is a familiar object from polyhedral combinatorics. It is defined as the convex hull of the 0/1–vectors encoding linear orders (or total orders) on a given base set. Exploiting results on the facial structure of this family of polytopes and using advanced techniques in linear programming, efficient algorithms could be designed to solve real-world instances of some hard combinatorial optimization problems. For example, the *triangulation problem for input-output tables* can be formulated as a linear program on the linear ordering polytope. This problem asks, given a matrix of $n \times n$ positive real numbers, to permute simultaneously the rows and columns of this matrix in order to maximize the sum of the entries above the main diagonal. Using results on the linear ordering polytope, Reinelt [13] was able to solve triangulation problems up to $n = 60$.

In our case, the *base set* is the set $\mathbf{n} = \{1, 2, \ldots, n\}$. Its members are called *elements*. We denote by $\mathcal{A}_n$ (or simply $\mathcal{A}$ when no confusion can occur) the set of *arcs* on the base set: $\mathcal{A}_n = \{(i,j) \mid i, j \in \mathbf{n}, \text{ with } i \neq j\}$. Here, a *linear order* (on the base set) is a complete, antisymmetric and transitive subset of $\mathcal{A}$. Each linear order is encoded by means of its 'characteristic vector' in the real vector space $\mathbb{R}^{\mathcal{A}}$. This vector space has dimension $|\mathcal{A}| = n(n-1)$ and its canonical basis is $\{e_{(i,j)} \mid (i,j) \in \mathcal{A}\}$. Moreover, if $x \in \mathbb{R}^{\mathcal{A}}$ then $x_{(i,j)}$ is the coordinate of x associated to the arc (i,j). Given a subset R of $\mathcal{A}$ (that is, an antireflexive relation R on the base set), the *characteristic vector* $\chi(R)$ of R is defined by $\chi(R) = \sum_{(i,j)\in R} e_{(i,j)}$. In order to keep concise notations, we use upper case letters for relations and lower case letters for their characteristic vectors: if $R \subseteq \mathcal{A}$, then r is the same thing as $\chi(R)$. As stated above, the linear ordering polytope $\mathbf{P}^n_{\mathrm{LO}}$ is the convex hull in $\mathbb{R}^{\mathcal{A}}$ of characteristic vectors of all linear orders on the base set:

$$\mathbf{P}^n_{\mathrm{LO}} = \mathrm{Conv}\{l \in \mathbb{R}^{\mathcal{A}} \mid L \text{ is a linear order on } \mathbf{n}\}.$$

Many authors have studied the facial structure of the linear ordering polytope, e.g. Grötschel *et al.* [9], Reinelt [13], Koppen [10], Leung and Lee [11], Goemans and Hall [8], Girlich, Kovalev and Nalivaiko [7]. A complete linear description of $\mathbf{P}^n_{\mathrm{LO}}$ is known up to $n = 7$ (see [14]), and 488 602 996 facets were found for $n = 8$ (see [2] – this description is conjectured to be complete). Wide families of facet-defining inequalities are known as for example: the k-fence inequalities, the k-wheel inequalities, the Z_k inequalities, the Möbius ladder inequalities [9,13], the α-critical fence inequalities [10], the (m,k)-fence inequalities [1], the t-reinforced k-fence inequalities, the augmented t-reinforced k-fence inequalities [11], the Paley inequalities [8] and the ∇-extended Möbius ladders inequalities [7]. The computational complexity of some of the corresponding separation problems is studied in [13,12].

Recently, Bolotashvili, Girlich and Kovalev [1] used symmetries of the linear ordering polytope to obtain new facets. Symmetries of a polytope are usually called *automorphisms*. They can be of different kinds [15]: e.g. vectorial, affine, euclidian. In this paper, we focus on the most general kind of automorphisms: *combinatorial* automorphisms, which are simply defined as permutations of the vertex set mapping each face to a face. More precisely, identifying any face with its set of vertices, we define a combinatorial automorphism of a polytope P as a permutation α of the set $\mathrm{Vert}(P)$ (the vertex set of P) such that $\alpha(F) = \{\alpha(v) \mid v \in F\}$ is a face of P for each face F of P. Affine (resp. euclidian) automorphisms of a polytope P in $\mathbb{R}^d$ are permutations of its vertices induced by affine permutations (resp. isometries) of $\mathbb{R}^d$ mapping P to itself. Clearly, every euclidian automorphism is an affine automorphism and every affine automorphism is a combinatorial automorphism.

Subsequently, we will only consider combinatorial automorphisms (they will be called simply 'automorphisms'). The automorphisms of a polytope P form the group $\mathrm{Aut}(P)$ (for an excellent introduction to group theory, we recommend Fraleigh's book [6]). Knowing the full automorphism group of a polytope helps studying its facial structure: automorphisms can create links between families of facets that seem unrelated at first sight, and sometimes 'old' facets can be recycled to produce new ones. For instance, Doignon and Regenwetter [3,4] were able to produce further facets of the approval voting polytope using some 'strange' automorphisms.

In this paper, we show that *all* automorphisms of $\mathbf{P}^n_{\mathrm{LO}}$ can be obtained by composing the obvious ones (which were known to Reinelt [13]) with the automorphisms found by Bolotashvili, Girlich and Kovalev [1] (definitions are recalled in the next paragraph). Moreover, we provide a simple and unified interpretation for the automorphisms by extending the base set (see Section 3). It turns out that every automorphism of the linear ordering polytope is an euclidian automorphism.

We now define the 'relabeling', 'duality' (called *arc reversal* in [13]) and 'exchange' automorphisms. Each permutation π of the base set yields a linear permutation R_π of $\mathbb{R}^{\mathcal{A}}$ defined by $R_\pi(e_{(i,j)}) = e_{(\pi(i),\pi(j))}$, which induces the *relabeling* automorphism R_π (no ambiguity should arise from the use of the same notation). The *duality* automorphism D simply maps the vertex corresponding to a linear order L to the vertex corresponding to its dual linear order $L^{-1} = \{(y,x) \mid (x,y) \in L\}$. It is also induced by a linear permutation D of $\mathbb{R}^{\mathcal{A}}$, defined by $D(e_{(i,j)}) = e_{(j,i)}$. For each fixed element r, the affine permutation E_r defined by

$$E_r(x)_{(i,r)} = x_{(r,i)},$$

$$E_r(x)_{(r,j)} = x_{(j,r)},$$

$$E_r(x)_{(i,j)} = x_{(i,j)} + x_{(j,r)} + x_{(r,i)} - 1$$

(for all distinct elements i, j with $i, j \neq r$), induces the *exchange* automorphism E_r. These were recently introduced by Bolotashvili, Girlich and Kovalev [1]. As will be shown later, the relabeling, duality and exchange automorphisms span the whole automorphism group of the linear ordering polytope and this group is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$ when $n > 2$. If X is any set, $\mathrm{Sym}(X)$ denotes the symmetric group on X – the group of all permutations of the set X – and if k is an integer greater than zero, then $\mathrm{Sym}(k) = \mathrm{Sym}(\{1, 2, \ldots, k\})$. Note that every relabeling automorphism is a composition of exchange automorphisms, so duality and exchange automorphisms also span $\mathrm{Aut}(\mathbf{P}^n_{\mathrm{LO}})$. As $\mathbf{P}^2_{\mathrm{LO}}$ is a closed segment, its automorphism group is isomorphic to $\mathbb{Z}_2$. From now on, we assume $n > 2$.

2 Trivial and 3-cycle facets

It is well known that the inequalities $x_{(i,j)} \leq 1$ and $x_{(i,j)} + x_{(j,k)} + x_{(k,i)} \leq 2$ are facet-defining for $\mathbf{P}^n_{\mathrm{LO}}$ (i, j and k are three distinct elements). They are respectively called *trivial* and *3-cycle* inequalities. In this section, we study some properties of trivial and 3-cycle facets (i.e. facets induced by trivial and 3-cycle inequalities). Together with the minimal equation system for $\mathbf{P}^n_{\mathrm{LO}}$

$$x_{(i,j)} + x_{(j,i)} = 1 \quad (i < j), \tag{1}$$

these facets provide a complete linear description of $\mathbf{P}^n_{\mathrm{LO}}$ up to $n = 5$ (it was even conjectured that they were sufficient to completely describe $\mathbf{P}^n_{\mathrm{LO}}$ for each n – this is false when n is greater than 5, see [13]).

An *antisymmetric inequality* (in $\mathbb{R}^{\mathcal{A}}$) is an inequality $\sum_{(i,j)\in\mathcal{A}} a_{(i,j)}\, x_{(i,j)} \leq b$ with $a_{(j,i)} = -a_{(i,j)}$ for all (i,j) in $\mathcal{A}$.

Lemma 1 *Every facet of $\mathbf{P}^n_{\mathrm{LO}}$ can be defined by an antisymmetric inequality. Moreover,* all *antisymmetric inequalities defining a certain facet can be obtained from* any *antisymmetric inequality defining this facet by scaling (multiplication by positive real numbers).*

PROOF. Given a facet-defining inequality $\sum_{(i,j)\in\mathcal{A}} a_{(i,j)}\, x_{(i,j)} \leq b$, we can easily symmetrize it using the minimal equation system for $\mathbf{P}^n_{\mathrm{LO}}$ by substracting $(a_{(i,j)} + a_{(j,i)})/2$ times the equation $x_{(i,j)} + x_{(j,i)} = 1$ from the inequality for each couple $\{i,j\}$. The new inequality

$$\sum_{(i,j)\in\mathcal{A}} \frac{a_{(i,j)} - a_{(j,i)}}{2}\, x_{(i,j)} \leq b - \sum_{i<j} \frac{a_{(i,j)} + a_{(j,i)}}{2}$$

is obviously antisymmetric and defines the same facet of $\mathbf{P}^n_{\mathrm{LO}}$ as the original one. Moreover, adding to this inequality any nontrivial combination of equalities from equation system (1) yields an inequality which is not antisymmetric. In conclusion, any facet of $\mathbf{P}^n_{\mathrm{LO}}$ can be defined by an antisymmetric inequality unique up to multiplication by a positive real number. $\square$

If R is an antireflexive relation on the base set, its *Hasse relation* $H(R)$ is defined by $H(R) = \{(i,k) \in R \mid \text{there is no } j \text{ s.t. } (i,j),(j,k) \in R\}$. If L is a linear order and if $(i,j) \in H(L)$ then the relation $L^{j\prec i} = L \setminus \{(i,j)\} \cup \{(j,i)\}$ obtained from L by switching i and j is again a linear order.

A trivial or 3-cycle facet contains exactly $n!/2$ vertices (a vertex belongs to such a facet iff its dual vertex does not). This property characterizes the trivial and 3-cycle facets.

Lemma 2 *No facet of $\mathbf{P}^n_{\mathrm{LO}}$ contains more than $n!/2$ vertices. Moreover the facets of $\mathbf{P}^n_{\mathrm{LO}}$ containing exactly $n!/2$ vertices are precisely the trivial and 3-cycle facets.*

PROOF. Let F be a facet of $\mathbf{P}^n_{\mathrm{LO}}$ defined by an antisymmetric inequality $\sum_{(i,j)\in\mathcal{A}} a_{(i,j)}\, x_{(i,j)} \leq b$. We claim that F cannot contain a vertex and its dual. Suppose that F contains a vertex l and its dual $D(l)$. Then, for any other vertex l' in F we have $l + D(l) = l' + D(l')$, thus $D(l')$ also belongs to F. As a consequence, F is stable by duality thus the inequality $\sum_{(i,j)\in\mathcal{A}} a_{(j,i)}\, x_{(i,j)} \leq b$ also defines F and is antisymmetric, so by Lemma 1 the latter is a positive multiple of the first inequality. This implies $a_{(i,j)} = 0$ for all (i,j) in $\mathcal{A}$, a contradiction.

In conclusion, if a vertex belongs to F then its dual is not in F. Therefore, F cannot contain more than $n!/2$ vertices. Now suppose that F contains exactly $n!/2$ vertices. Then the image of F by duality, $D(F)$, is a facet disjoint from F. Moreover the vertices of $\mathbf{P}^n_{\mathrm{LO}}$ belong either to F or to $D(F)$. This implies that the linear form defined by $f(x) = \sum_{(i,j)\in\mathcal{A}} a_{(i,j)}\, x_{(i,j)}$ takes exactly two values on the set of vertices of $\mathbf{P}^n_{\mathrm{LO}}$: b and $-b$. Note that b must be positive because $f(x) \leq b$ is valid of $\mathbf{P}^n_{\mathrm{LO}}$.

Now fix $(i,j) \in \mathcal{A}$ with $a_{(i,j)} > 0$ and take a linear order L on the base set such that $(i,j) \in H(L)$, then the linear order $L^{j\prec i}$ obtained from L by switching i and j gives

$$f(l^{j\prec i}) = f(l - e_{(i,j)} + e_{(j,i)}) = f(l) - a_{(i,j)} + a_{(j,i)} = f(l) - 2\, a_{(i,j)},$$

thus $f(l)$ and $a_{(i,j)}$ must be equal to b. By antisymmetry, it follows that if $a_{(i,j)} < 0$ then $a_{(i,j)} = -b$ and $f(l) = -b$ for each vertex l such that (i,j) is in the Hasse relation of the associated linear order. In particular, $a_{(i,j)} \in \{-b, 0, b\}$ for each arc (i,j).

Let i, j and k be three distinct elements with $a_{(i,j)} > 0$. Consider a linear order L such that $(i,k), (k,j) \in H(L)$. Then $f(l^{j\prec k}) = f(l^{k\prec i}) = b$ because $a_{(i,j)} > 0$ and $(i,j) \in H(L^{j\prec k}) \cap H(L^{k\prec i})$. So $a_{(j,k)}$ and $a_{(k,i)}$ must be nonnegative for otherwise $f(l^{j\prec k})$ or $f(l^{k\prec i})$ would be equal to $-b$, a contradiction. From

$$\underbrace{f(l^{j\prec k} - e_{(j,k)} + e_{(k,j)})}_{b-2\,a_{(j,k)}} = f(l) = \underbrace{f(l^{k\prec i} - e_{(k,i)} + e_{(i,k)})}_{b-2\,a_{(k,i)}}$$

we conclude that $a_{(j,k)} = a_{(k,i)}$. Therefore, either $a_{(i,j)} = b$ and $a_{(j,k)} = a_{(k,i)} = 0$, or $a_{(i,j)} = a_{(j,k)} = a_{(k,i)} = b$.

The relation $R = \{(i,j) \in \mathcal{A} \mid a_{(i,j)} > 0\}$ entirely characterizes f and satisfies the following properties (i, j and k are any three distinct elements): (i) R is not empty; (ii) R is antisymmetric; (iii) $(i,j) \in R$ implies $(k,j) \notin R$, $(i,k) \notin R$ and $(j,k) \in R$ iff $(k,i) \in R$. The proof is completed by checking that R consists of isolated arcs and 3-cycles, and that $f(x) \leq b$ is valid and facet-defining for $\mathbf{P}^n_{\mathrm{LO}}$ iff R is reduced to one arc or to one 3-cycle. $\square$

3 Extending the base set

In this section we give a simple interpretation of the relabeling, exchange and duality automorphisms by extending the base set. This extension allows us to introduce unified notations for the trivial and 3-cycle facets: we note C_{ijk} the facet defined by the 3-cycle inequality $x_{(i,j)} + x_{(j,k)} + x_{(k,i)} \leq 2$, and $C_{ij\infty}$ the facet defined by the trivial inequality $x_{(i,j)} \leq 1$. For any u, v, w distinct in $\mathbf{n} \cup \{\infty\}$, C_{uvw}, C_{vwu} and C_{wuv} denote the same facet (indices can be permuted cyclically). The set $\mathbf{n}_\infty = \mathbf{n} \cup \{\infty\}$ is the *extended base set.*

Let $\mathfrak{T}$ be the set of all trivial and 3-cycle facets. By Lemma 2, every automorphism of $\mathbf{P}^n_{\mathrm{LO}}$ permutes among themselves the trivial and 3-cycle facets. Thus any automorphism α of $\mathbf{P}^n_{\mathrm{LO}}$ determines the permutation $\psi(\alpha)$ of the set $\mathfrak{T}$ defined by $\psi(\alpha)(C_{uvw}) = \alpha(C_{uvw})$. The map $\psi : \mathrm{Aut}(\mathbf{P}^n_{\mathrm{LO}}) \to \mathrm{Sym}(\mathfrak{T})$ is a one-to-one group homomorphism. What are the images of the relabeling, exchange and duality automorphisms under this homomorphism? In other words, how do relabeling, exchange and duality automorphisms permute trivial and 3-cycle facets?

If π is a permutation of the set $\mathbf{n}$ and i, j, k are three distinct elements, we have

$$R_\pi(C_{ij\infty}) = C_{\pi(i)\pi(j)\infty},\ R_\pi(C_{ijk}) = C_{\pi(i)\pi(j)\pi(k)},$$

$$D(C_{ij\infty}) = C_{\infty ji} \text{ and } D(C_{ijk}) = C_{kji}.$$

Now let r, i, j and k be four distinct elements. A quick computation tells us that

$$E_r(C_{ij\infty}) = C_{ijr},\ E_r(C_{ijk}) = C_{ijk},$$

$$E_r(C_{ir\infty}) = C_{i\infty r} \text{ and } E_r(C_{rj\infty}) = C_{\infty jr}.$$

The action of the relabeling, exchange and duality automorphism on the set $\mathfrak{T}$ is recognized by means of a second one-to-one homomorphism:

$$\phi : \mathbb{Z}_2 \times \mathrm{Sym}(\mathbf{n}_\infty) \to \mathrm{Sym}(\mathfrak{T}), \text{ with}$$

$$\text{and} \quad \begin{aligned} \phi(0,\sigma)(C_{uvw}) &= C_{\sigma(u)\sigma(v)\sigma(w)} \\ \phi(1,\sigma)(C_{uvw}) &= C_{\sigma(w)\sigma(v)\sigma(u)}. \end{aligned}$$

Clearly, if σ is a permutation of the extended base set fixing ∞ and r is any element, then $\phi(0,\sigma) = \psi(R_{\sigma|_{\mathbf{n}}})$, $\phi(0,(r,\infty)) = \psi(E_r)$ and $\phi(1, id) = \psi(D)$.

Consider an automorphism α of $\mathbf{P}^n_{\mathrm{LO}}$ which is a product of relabeling, duality and exchange automorphisms, say $\alpha = \alpha_1 \circ \ldots \circ \alpha_n$. Because ψ is an homomorphism, $\psi(\alpha)$ is equal to the product $\psi(\alpha_1) \circ \ldots \psi(\alpha_n)$ in which each factor

$\psi(\alpha_i)$ can be rewritten $\phi(\varepsilon_i, \sigma_i)$ for some ε_i in $\mathbb{Z}_2$ and some σ_i in $\mathrm{Sym}(\mathbf{n}_\infty)$. Using the fact that ϕ is an homomorphism, we see that the action of α on the set $\mathfrak{T}$ of trivial and 3-cycle facets can be identified in a natural way with the action of some element of $\mathbb{Z}_2 \times \mathrm{Sym}(\mathbf{n}_\infty)$ on $\mathfrak{T}$. Thus the two one-to-one homorphisms ψ and ϕ shed some light on the subgroup of $\mathrm{Aut}(\mathbf{P}^n_{\mathrm{LO}})$ spanned by relabeling, exchange and duality automorphisms: this subgroup is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(\mathbf{n}_\infty)$ (which is clearly isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$).

4 The combinatorial automorphism group

Lemma 3 *The following table gives the number of vertices in the intersection of any two trivial or 3-cycle facets (u, v, w, u', v', w' are distinct elements of $\mathbf{n}_\infty$).*

intersection	*# vertices*
$C_{uvw} \cap C_{wvu}$	*0*
$C_{uvw} \cap C_{uvw'}$	$n!/3$
$C_{uvw} \cap C_{vuw'}$	$n!/6$
$C_{uvw} \cap C_{uv'w'}$	$n!/4$
$C_{uvw} \cap C_{u'v'w'}$	$n!/4$

PROOF. Straightforward (by standard counting arguments). □

Theorem 4 *The combinatorial automorphism group of $\mathbf{P}^n_{\mathrm{LO}}$ is spanned by the exchange and duality automorphisms. Moreover, it is isomorphic to the direct product $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$.*

PROOF. Let $\mathcal{A}_\infty$ be the set of arcs on the extended base set, α be an automorphism of $\mathbf{P}^n_{\mathrm{LO}}$ and (u, v) be an arc in $\mathcal{A}_\infty$. Choose two elements s, t distinct from u and v in the extended base set. By Lemma 3, the face $C_{uvs} \cap C_{uvt}$ has $n!/3$ vertices, so its image $\alpha(C_{uvs}) \cap \alpha(C_{uvt})$ also has $n!/3$ vertices. Lemmas 2 and 3 together imply that there exist distinct elements s', t', u' and v' of the extended base set such that $\alpha(C_{uvs}) = C_{u'v's'}$ and $\alpha(C_{uvt}) = C_{u'v't'}$. Set $\overline{\alpha}(u, v) = (u', v')$.

We claim that $\overline{\alpha}$ is a well defined permutation of $\mathcal{A}_\infty$. If r is an element distinct from u, v, s and t, then $\alpha(C_{uvr})$ is a facet having exactly $n!/3$ vertices in common with $C_{u'v's'}$ and $C_{u'v't'}$, so it must be equal to $C_{u'v'r'}$ for a certain

r'. This implies that $\overline{\alpha}$ is a well defined map from $\mathcal{A}_\infty$ to $\mathcal{A}_\infty$. Now if β is another automorphism of $\mathbf{P}^n_{\mathrm{LO}}$, then $\overline{\beta}$ and $\overline{\beta \circ \alpha}$ can be defined exactly the same way, and it can be easily checked that $\overline{\beta \circ \alpha} = \overline{\beta} \circ \overline{\alpha}$. When $\beta = \alpha^{-1}$, we have $\overline{\beta} \circ \overline{\alpha} = id$, so $\overline{\alpha}$ is a permutation. In conclusion, $\overline{\alpha}$ is a well defined permutation of $\mathcal{A}_\infty$. The two following statements are consequences of Lemma 3: (i) $\overline{\alpha}$ commutes with 'arc duality', (ii) $\overline{\alpha}$ maps 3-cycles to 3-cycles. They are proved in the next paragraph.

As before, take four distinct elements u, v, s and t in the extended base set. Let $\alpha(C_{uvs}) = C_{u'v's'}$ and $\alpha(C_{uvt}) = C_{u'v't'}$. A trivial or 3-cycle facet has a non-empty intersection with every other trivial or 3-cycle facet except its image by the duality automorphism D. Thus $\alpha(C_{svu}) = C_{s'v'u'}$ and $\alpha(C_{tvu}) = C_{t'v'u'}$, so $\overline{\alpha}(v,u) = (v',u')$. Therefore, $\overline{\alpha}$ commutes with the *arc duality mapping* δ defined by $\delta(u,v) = (v,u)$ for each arc (u,v) (clearly, $\delta = \bar{D}$). As automorphisms preserve the number of vertices in a face, the faces $\alpha(C_{vst}) \cap C_{u'v's'}$ and $\alpha(C_{vst}) \cap C_{u'v't'}$ have respectively $n!/3$ and $n!/6$ vertices. So $\alpha(C_{vst})$ is either equal to $C_{v's't'}$ or to $C_{s'u't'}$ and $\overline{\alpha}(v,s)$ is either equal to (v',s') or to (s',u'). Using the same argument once more with C_{vst} replaced by C_{sut}, we see that $\overline{\alpha}(s,u)$ is either equal to (s',u') or to (v',s'). In both cases we have $\overline{\alpha}(\{(u,v),(v,s),(s,u)\}) = \{(u',v'),(v',s'),(s',u')\}$, thus $\overline{\alpha}$ maps 3-cycles to 3-cycles.

A *k-cycle* in $\mathcal{A}_\infty$ is any set $C = \{(u_1,u_2),(u_2,u_3),\ldots,(u_{k-1},u_k),(u_k,u_1)\}$ of k arcs belonging to $\mathcal{A}_\infty$. Moreover, if $u_1, \ldots, u_k$ are all distinct then C is a *simple k-cycle*. It follows from (i) and (ii), by induction on k, that for any automorphism α of $\mathbf{P}^n_{\mathrm{LO}}$, $\overline{\alpha}$ maps simple k-cycles to simple k-cycles (for all $k \geq 2$).

We constructed a one-to-one homomorphism $f : \alpha \mapsto \overline{\alpha}$ from $\mathrm{Aut}(\mathbf{P}^n_{\mathrm{LO}})$ into the group of permutations of $\mathcal{A}_\infty$ mapping simple k-cycles to simple k-cycles (for all $k \geq 2$). The latter is easily seen to be isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$ (see e.g. Fiorini [5]). But we saw in Section 3 that the group $\mathrm{Aut}(\mathbf{P}^n_{\mathrm{LO}})$ contains an isomorphic copy of $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$, so f is an isomorphism, $\mathrm{Aut}(\mathbf{P}^n_{\mathrm{LO}})$ is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$ and is spanned by the relabeling, exchange and duality automorphisms. Because every relabeling automorphism is a composition of exchange automorphisms, $\mathrm{Aut}(\mathbf{P}^n_{\mathrm{LO}})$ is also spanned by duality and exchange automorphisms. $\square$

Acknowledgements

We wish to thank J.P. Doignon for so many fruitful discussions, and three anonymous referees for their useful remarks.

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