THE APPROVAL-VOTING POLYTOPE: COMBINATORIAL INTERPRETATION OF THE FACETS

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SUMMARY – Doignon and Fiorini (2003) determine all facets of the approval-voting polytope, thus offering a characterization of the size-independent model for approval voting of Falmagne and Regenwetter (1996). The present paper is a follow-up. It first provides an alternate proof of the basic result, which is more direct and at the same time constructive. Then, the combinatorial interpretation of the facets of the approval-voting polytope is further investigated. Finally, we derive a linear description of the polytope in case the number of alternatives equals 6.

KEYWORDS – Approval-voting polytope, Facet, Network flow, Stable set in a graph

RÉSUMÉ – Le polytope du vote par approbation : interprétation combinatoire des facettes.
Doignon et Fiorini (2003) déterminent toutes les facettes du polytope du vote approbatoire. Ils livrent ainsi une caractérisation d'un modèle probabiliste dû à Falmagne et Regenwetter (1996) : le modèle sous indépendance de taille pour le vote approbatoire. Le présent texte est un complément. Il donne d'abord une preuve alternative du résultat central, plus directe mais aussi constructive. L'interprétation combinatoire des facettes du polytope du vote approbatoire est ensuite étudiée. Enfin, une description linéaire du polytope est obtenue dans le cas où le nombre d'alternatives vaut 6.

MOTS-CLÉS – Polytope du vote approbatoire, Facette, Flot dans un réseau, Sous-ensemble stable d'un graphe

1. INTRODUCTION

In order to give a concrete approach to our subject, we start with an example in which there are only three alternatives a, b and c; the general theory will handle a general, finite set S of n *alternatives.* Consider the following 0/1-matrix M having

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one row per subset of $S = \{a, b, c\}$, and one column per ranking of S:

	$\langle a,b,c\rangle$	$\langle a,c,b\rangle$	$\langle b,a,c\rangle$	$\langle b,c,a\rangle$	$\langle c,a,b\rangle$	$\langle c,b,a\rangle$
$\varnothing$	1	1	1	1	1	1
$\{a\}$	1	1	0	0	0	0
$\{b\}$	0	0	1	1	0	0
$\{c\}$	0	0	0	0	1	1
$\{a,b\}$	1	0	1	0	0	0
$\{a,c\}$	0	1	0	0	1	0
$\{b,c\}$	0	0	0	1	0	1
S	1	1	1	1	1	1

The entry $M_{X,\pi}$ equals 1 if the ranking π of S places the alternatives of X before the other alternatives from S, and equals 0 otherwise. The convex hull of the column vectors of M forms a 0/1-polytope, the so-called *approval-voting polytope* P^3_{AV}. We will investigate the facets of the polytopes P^n_{AV}.

The question of finding all (or many) facets of some definite 0/1-polytope repeatedly appears in combinatorial optimization. It is usually motivated by the application of linear programming to an optimization problem. The problem of determining all facets of the approval-voting polytopes P^n_{AV} stems instead from mathematical psychology: as shown in Doignon and Regenwetter [8], the complete collection of facets entails a characterization of the Falmagne and Regenwetter [9] model of 'approval voting'. The latter term refers to a voting procedure according to which each voter announces the set of alternatives she/he approves of. This procedure is advocated by e.g. Brams and Fishburn [1] as being less sensitive to vote manipulations. Several data were collected for such elections performed in various scientific societies, and are subject to statistical studies which test possible probabilistic models (see e.g. Regenwetter and Grofman [16]). The size-independent model is central among these probabilistic models (see also Doignon, Pekeč and Regenwetter [7]).

In a sense, the problem of finding all facets of P^n_{AV} was solved by Doignon and Fiorini [6]: they exhibit a one-to-one correspondence between the facets and certain antichains of the power set of S. The present work is to be seen as a complement to this paper. First, an alternate proof of the basic result is built on the Max Flow Min Cut Theorem; it is thus a proof which can be easily turned into an algorithm for expressing any point in P^n_{AV} as a convex combination of the vertices. Second, the combinatorial interpretation of the facets is further investigated in terms of (hyper)graph theory. As a byproduct, we derive in the final section a list of all facets of P^n_{AV} when $n \leq 6$. In particular, we confirm by theoretical arguments the results produced for $n \leq 5$ by the `Porta` software [3] and reported in Doignon and Regenwetter [8].

2. A LINEAR DESCRIPTION OF THE APPROVAL-VOTING POLYTOPE

Given a set S of n alternatives, with $n \geq 2$, we denote by $\mathcal{P}(S)$ the power set of S, and by $\mathcal{P}(S, k)$ the collection of all k-sets contained in S, for $0 \leq k \leq n$. Moreover, we let Π denote the collection of all rankings (or linear orderings) of S, and Π_X the collection of all rankings starting with the subset X of S (these are the rankings in which the elements of X precede those from $S \setminus X$).

We create a 0/1-matrix M having one row per subset X of S, and one column per ranking π of S, by setting

$$M_{X,\pi} \quad = \quad \begin{cases} 1 & \text{if } \pi \in \Pi_X, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The *approval voting polytope* P^n_{AV} is the convex hull of the set of column vectors of M. This polytope lies in a real space of dimension 2^n having one coordinate per subset of S, but its affine dimension equals only $2^n - n - 1$ (see Doignon and Regenwetter [8]).

To describe the approval voting polytope P^n_{AV} by a system of linear equations and inequalities, we need some further notions. In $\mathcal{P}(S)$ partially ordered by inclusion, an *antichain* $\mathcal{A}$ is a collection $\mathcal{A}$ of subsets of S such that for every pair of distinct sets in $\mathcal{A}$, none is included in the other. For each subset $\mathcal{A}$ of $\mathcal{P}(S)$, we set $x(\mathcal{A}) = \sum_{X \in \mathcal{A}} x_X$. It is easily seen that all vertices of P^n_{AV} satisfy the following (in)equalities:

$$\begin{aligned} x(\mathcal{P}(S,k)) &= 1, && \text{for } k = 0, 1, \ldots, n; && (2)\\ x_Y &\geq 0, && \text{for } Y \in \mathcal{P}(S), && (3)\\ x(\mathcal{A}) &\leq 1, && \text{for any antichain } \mathcal{A} \text{ of } \mathcal{P}(S). && (4) \end{aligned}$$

These (in)equalities are thus satisfied by the whole polytope P^n_{AV}. According to Doignon and Fiorini [6], they completely describe the polytope.

PROPOSITION 1. *The approval-voting polytope P^n_{AV} consists exactly of the solutions to the (redundant) system of (in)equalities formed by Equations (2), (3), and (4).*

The proof of Proposition 1 given in Doignon and Fiorini [6] is based on two classical results: first, a polyhedral characterization of perfect graphs, due to Fulkerson [10, 11, 12], Lovász [14] and Chvátal [4] and second, Dilworth's Theorem [5] on chain coverings of posets. We now present an alternate proof, which can be summarized as follows. Let x be a vector satisfying Inequalities (2), (3), and (4). Replacing each element Y of the power set of S with two successive elements Y^- and Y^+, we get a network with source $\varnothing^-$ and sink X^+. Our assumption implies that the minimum of all cut capacities equals 1. The Max Flow Min Cut Theorem delivers a flow with value 1. We then conclude either by applying the Flow Decomposition Theorem, or by giving an explicit expression of x as a convex combination of the polytope vertices.

We write $X \sqsubset Y$ when X is *covered* by Y, that is $X \subset Y$ and $|X| + 1 = |Y|$.

Proof. As mentioned above, the approval-voting polytope is easily seen to satisfy (In)equalities (2)-(4). We now prove conversely that any vector x with real components indexed by $\mathcal{P}(S)$ and satisfying these (in)equalities is a convex combination of the columns of the matrix M. We will derive the coefficients of a desired convex combination from quantities $f(X, Y)$ defined for all pairs (X, Y) of subsets of S

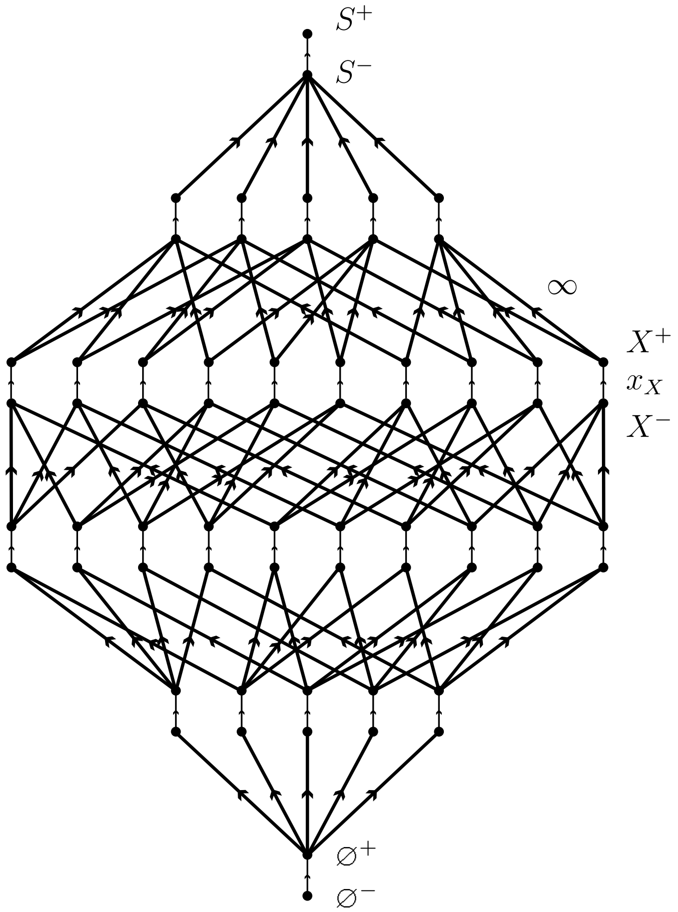

Figure 1. The digraph G from the proof of Proposition 1, when $n = 5$.

with $X \sqsubset Y$. The quantities $f(X, Y)$ will be obtained as flow values on a certain digraph G (see Korte and Vygen [13] for terminology).

We now build the digraph G by doubling the elements of the power set and defining appropriate arcs e with capacities $v(e)$. An illustration of such a digraph, in case $n = 5$, is given in Figure 1. The digraph G has two vertices Y^- and Y^+ for each subset Y of S, and two types of arcs. First, it has an arc (Y^-, Y^+) for each $Y \in \mathcal{P}(S)$, with capacity $v(Y^-, Y^+) = x_Y$. Second, for each pair (X, Y) of subsets of S with $X \sqsubset Y$, the digraph G has an arc (X^+, Y^-) with capacity $v(X^+, Y^-) = \infty$. We turn G into a network by selecting $\varnothing^-$ as the source, and S^+ as the sink. To apply the Max Flow Min Cut Theorem, we proceed to show that the minimum value of a $(\varnothing^-, S^+)$-cut of the network $(G, v, \varnothing^-, S^+)$ equals 1. From now on, we will always abbreviate '$(\varnothing^-, S^+)$-cut of G' into 'cut'.

Notice that the cut value $v(\delta^+(\Gamma))$ equals 1 in case $\Gamma = \{Y^- \in G \,|\, |Y| = k + 1\} \cup \{X^-, X^+ \in G \,|\, |X| \leq k\}$ for some $k = -1, 0, \ldots, n-1$ because then $\delta^+(\Gamma) = \{(Y^-, Y^+) \,|\, Y \in \mathcal{P}(S, k+1)\}$ and Equation (2) applies.

Take now any cut $\delta^+(\Gamma)$. If $\delta^+(\Gamma)$ contains some arc of the type (X^+, Y^-), the cut value $\delta^+(\Gamma)$ is infinite. We may thus assume that $\delta^+(\Gamma)$ contains only arcs of the type (Y^-, Y^+). Let $\Gamma_\ell = \{Y \in \mathcal{P}(S, \ell) \,|\, Y^- \in \Gamma \text{ and } Y^+ \notin \Gamma\}$, and $k = -1 + \max\{\ell \,|\, \Gamma_\ell \neq \varnothing\}$. Notice that $Y^+ \in \Gamma$ implies $|Y| \leq k$. Moreover, in case $\mathcal{P}(S, k+1) = \Gamma_{k+1}$, we know $v(\delta^+(\Gamma)) \geq 1$. In the other case, we will build another cut Γ' with $v(\delta^+(\Gamma')) \leq v(\delta^+(\Gamma))$ and with a lesser value of k. By decreasing recurrence on k, we can then conclude that any cut has value at least 1.

Define the following subsets of $\mathcal{P}(S)$ (see Figure 2 for a sketch):

$$\begin{aligned} \mathcal{A}_{k+1} &= \{Y \in \mathcal{P}(S, k+1) \,|\, Y^- \notin \Gamma\}, \\ \mathcal{B}_{k+1} &= \{Y \in \mathcal{P}(S, k+1) \,|\, Y^- \in \Gamma\}, \\ \mathcal{A}_k &= \{X \in \mathcal{P}(S, k) \,|\, Y^-, Y^+ \in \Gamma\}. \end{aligned}$$

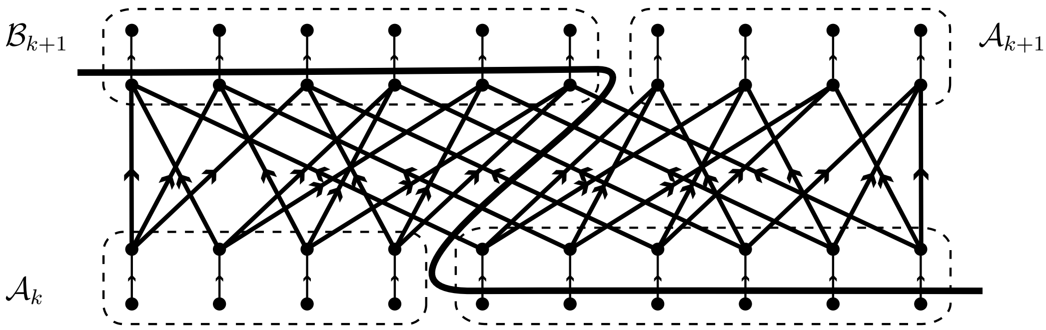

Figure 2. A sketch of subsets $\mathcal{A}_{k+1}$, $\mathcal{B}_{k+1}$ and $\mathcal{A}_k$ from the proof of Proposition 1.

As the cut value is not infinite, $\mathcal{A}_k \cup \mathcal{A}_{k+1}$ is an antichain. Our assumption $x(\mathcal{A}_k \cup \mathcal{A}_{k+1}) \leq 1$ together with Equation (2) implies $x(\mathcal{A}_k) \leq x(\mathcal{B}_{k+1})$. From Γ, construct a new cut Γ' by setting

$$\Gamma' \quad = \quad \Gamma \setminus (\Gamma_{k+1} \cup \{X^+ \,|\, X \in \mathcal{A}_k\}).$$

This cut Γ' has a value $v(\delta^+(\Gamma'))$ not larger than $v(\delta^+(\Gamma))$, and moreover no set Y with $|Y| = k+1$ satisfies $Y^- \in \Gamma$ and $Y^+ \notin \Gamma$. We have thus shown that the minimum value of a cut of the network G equals 1.

Consequently, there exists a flow f on G from $\varnothing^-$ to S^+ with value 1. Notice that this flow necessarily saturates all arcs of the first type, that is $f(Y^-, Y^+) = v(Y^-, Y^+)$. The proof can now be completed by applying the Flow Decomposition Theorem (see e.g. [13], page 157). Any path in G from the source $\varnothing^-$ to the sink S^+ univocally corresponds to a linear ordering of S; moreover, G has no circuit. Hence, there exists for each π in Π a nonnegative real number $\lambda(\pi)$ in such a way that

$$\sum_{\pi \in \Pi} \lambda(\pi) \quad = \quad 1 \tag{5}$$

(remember that 1 is the flow value) and

$$\sum_{\pi \in \Pi_X} \lambda(\pi) \quad = \quad x_X, \qquad \forall X \in \mathcal{P}(S) \tag{6}$$

(because the arc (X^-, X^+) is saturated). The latter equation amounts to

$$x_X \quad = \quad \sum_{\pi \in \Pi} M_{X,\pi}\, \lambda(\pi), \tag{7}$$

so that x is a convex combination of the columns of M. This completes the proof. □

Notice that the following explicit value can be given for the coefficients $\lambda(\pi)$ appearing in the above proof. Seeing any π from Π as a chain of $n+1$ subsets

$$\varnothing \subset \{\pi(1)\} \subset \{\pi(1), \pi(2)\} \subset \ldots \subset \{\pi(1), \pi(2), \ldots, \pi(n)\},$$

we write $X_i = \{\pi(1), \pi(2), \ldots, \pi(i)\}$ and then set

$$\lambda(\pi) \quad = \quad \prod_{i=0}^{n-1} \frac{f(X_i^+, X_{i+1}^-)}{x_{X_i}}$$

with the convention $\lambda(\pi) = 0$ if some denominator x_{X_i} vanishes. As the flow satisfies for all A in $\mathcal{P}(S,k)$ and B in $\mathcal{P}(S,k+1)$

$$\sum \{f(A,Y) \,|\, Y \in \mathcal{P}(S,k+1), A \sqsubset Y\} \quad = \quad x_A, \tag{8}$$

$$\sum \{f(X,B) \,|\, X \in \mathcal{P}(S,k), X \sqsubset B\} \quad = \quad x_B, \tag{9}$$

it can be directly checked that Equations (5) and (6) hold. Thus the proof can also be completed without calling the Flow Decomposition Theorem.

3. THE FACETS OF THE APPROVAL-VOTING POLYTOPE

The system in Proposition 1 is generally redundant. Doignon and Fiorini [6] provide an irredundant system. As for any polytope, the inequalities in such a system are exactly the facet-defining inequalities for the approval voting polytope P^n_{AV}. To list the facet-defining inequalities of P^n_{AV}, we need more terminology about antichains. Two distinct elements A, B of $\mathcal{P}(S,k)$ are *adjacent* when the following four equivalent conditions hold:

$$\begin{aligned} A \sim B \quad &\Longleftrightarrow \quad \exists X \in \mathcal{P}(S,k-1) : X \subset A \text{ and } X \subset B \\ &\Longleftrightarrow \quad \exists Y \in \mathcal{P}(S,k+1) : A \subset Y \text{ and } B \subset Y \\ &\Longleftrightarrow \quad |A \cap B| = k-1 \\ &\Longleftrightarrow \quad |A \cup B| = k+1. \end{aligned}$$

The graph $(\mathcal{P}(S,k), \sim)$ is the *Johnson graph* $\mathrm{J}(S,k)$ (as for instance in [2]).

A *bilayer antichain with parameter* k is an antichain in $\mathcal{P}(S)$ of the form $\mathcal{A}_k \cup \mathcal{A}_{k+1}$ with $\varnothing \neq \mathcal{A}_k \subset \mathcal{P}(S,k)$, and $\varnothing \neq \mathcal{A}_{k+1} \subset \mathcal{P}(S,k+1)$ for some k with $0 < k < n-1$, which moreover satisfies

(C1) the graph induced on $\mathcal{A}_k$ by $\mathrm{J}(S,k)$ is connected,

(C2) the graph induced on $\mathcal{A}_{k+1}$ by $\mathrm{J}(S,k+1)$ is connected,

(C3) $\mathcal{A}_k = \{X \in \mathcal{P}(S,k) \,|\, \forall Y \in \mathcal{A}_{k+1} : X \not\subset Y\}$ and

(C4) $\mathcal{A}_{k+1} = \{Z \in \mathcal{P}(S,k+1) \,|\, \forall T \in \mathcal{A}_k : T \not\subset Z\}$.

Here is the main result of Doignon and Fiorini [6].

PROPOSITION 2. *The approval-voting polytope P^n_{AV} consists exactly of the solutions to the following irredundant system of (in)equalities:*

$$\begin{aligned} x(\mathcal{P}(S,k)) &= 1, && \textit{for } k = 0,1,\dots,n; && (10)\\ x_Y &\geq 0, && \textit{for } Y \in \mathcal{P}(S) \setminus \{\varnothing, S\}, && (11)\\ x(\mathcal{A}) &\leq 1, && \textit{for any bilayer antichain } \mathcal{A}. && (12) \end{aligned}$$

It is possible to strengthen the arguments from previous section in order to show that (In)equalities (10)-(12) still characterize P^n_{AV}. But proving that none of these inequalities is redundant requires another proof. Thus the simplest way to establish Proposition 2 is to derive it from Proposition 1 along the combinatorial approach followed in [6].

An open problem is to better understand the bilayer antichains: for instance, how many are there? how to algorithmically produce them? We proceed to give some (partial) answers.

4. COMBINATORIAL INTERPRETATION OF THE FACETS

As before, let $|S| = n$ and $\mathcal{A}_k \cup \mathcal{A}_{k+1}$ be a bilayer antichain with parameter k.

We consider first the case $k = 1$. Set then $T = \{i \in S \mid \{i\} \notin \mathcal{A}_1\}$. By Condition (C4), $\mathcal{A}_2$ consists exactly of all unordered pairs of elements from T. Consequently, the bilayer antichains with parameter $k = 1$ bijectively correspond to subsets T of S with size 2, 3, …, $n-1$; we need to assume $n \geq 3$. There are in all $2^n - (n+2)$ of them.

Next, we study bilayer antichains with parameter 2. Here, $\mathcal{A}_2$ can be seen as the set of edges of a graph on S; this graph will be denoted as $\Phi = (S, \mathcal{A}_2)$. Condition (C4) exactly says that $\mathcal{A}_3$ consists of all stable subsets of Φ with size 3, and does not impose any restriction on the graph Φ. In turn, Condition (C3) states that any pair of elements of S which do not form an edge of Φ is included in at least one stable subset of size 3. Call a graph $G = (S, E)$ *almost connected* if exactly one of its connected components has more than one vertex. Condition (C1) amounts to the almost connectedness of Φ (notice in passing that $(\mathcal{A}_2, \sim)$ is the line graph of Φ). Unfortunately, translating in a useful way Condition (C2) seems to be more difficult.

A graph $G = (S, E)$ with vertex set S is *facet-defining* when $\mathcal{A}_2 \cup \mathcal{A}_3$ is a bilayer antichain (with parameter $k = 2$), where

$$\begin{aligned} \mathcal{A}_2 &= E, && (13)\\ \mathcal{A}_3 &= \{Y \in \mathcal{P}(S) \mid |Y| = 3 \text{ and } Y \text{ is a stable subset of } G\}. && (14) \end{aligned}$$

As we are not able to fully characterize the facet-defining graphs, we collect a bunch of conditions, either necessary or sufficient.

PROPOSITION 3. *Let $G = (S, E)$ be a (simple) graph with $n = |S| \geq 3$.*

(i) Assume G has at least two isolated vertices. Then G is facet-defining iff G is almost connected.

(ii) Assume G has exactly one isolated vertex v. Then G is facet-defining iff G is almost connected and moreover the complement of the graph $G \setminus \{v\}$ is connected.

(iii) Assume now that G has no isolated vertex. If G is facet-defining, then G is connected without being a complete graph, any two nonadjacent vertices in G are contained in at least one stable subset of size 3 of G and the complement graph $\bar{G}$ is almost connected. Conversely, if G satisfies the three latter conditions and moreover G has no induced $K_1 + C_4$, then G is facet-defining.

Proof. We know that Condition (C4) always holds, because of Equation (14), and that Condition (C1) is equivalent to G being almost connected.

(i) If G has isolated vertices v and w, Condition (C3) is true. Moreover, any stable subset of size 3 is easily seen to be at distance at most 3 in the graph $(\mathcal{A}_3, \sim)$ from any stable subset of the form $\{v, w, u\}$. There follows the connectedness of $(\mathcal{A}_3, \sim)$, that is Condition (C2).

(ii) Assume G has one isolated vertex v. Let G be facet-defining. By Condition (C1), the graph $G \setminus \{v\}$ is connected and by Conditions (C2) and (C3) its complement is connected. The converse is also straightforward.

(iii) Finally, assume G has no isolated vertex. The proof of the first assertion is routine. To prove the second assertion, there remains to establish Condition (C2). Let $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ be two elements of $\mathcal{A}_3$. By our assumptions, there is a shortest path $u_0, u_1, \ldots, u_m$ from v_1 to w_1 in $\bar{G}$, and any two successive vertices u_i, u_{i+1} along this path belong to some element $\{u_i, u_{i+1}, t_i\}$ of $\mathcal{A}_3$. If $t_i = u_{i+2}$ and $t_{i+1} = u_i$, we have $\{u_i, u_{i+1}, t_i\} = \{u_{i+1}, u_{i+2}, t_{i+1}\}$. If exactly one of the two equalities $t_i = u_{i+1}$ and $t_{i+1} = u_i$ holds, then $\{u_i, u_{i+1}, t_i\} \sim \{u_{i+1}, u_{i+2}, t_{i+1}\}$. Assume now $t_i \neq u_{i+1}$ and $t_{i+1} \neq u_i$. If $t_i = t_{i+1}$, we again have $\{u_i, u_{i+1}, t_i\} \sim \{u_{i+1}, u_{i+2}, t_{i+1}\}$. Finally, if $t_i \neq t_{i+1}$ (the generic case), $\{u_i, u_{i+1}, t_i\}$ and $\{u_{i+1}, u_{i+2}, t_{i+1}\}$ are at distance 2 in $(\mathcal{A}_3, \sim)$ except if the graph induced by G on $\{u_i, u_{i+1}, t_i, u_{i+2}, t_{i+1}\}$ is isomorphic to $K_1 + C_4$. As the latter cannot occur, we deduce the existence of a path from $\{v_1, v_2, v_3\}$ to $\{w_1, w_2, w_3\}$ in $(\mathcal{A}_3, \sim)$. □

Among the graphs G with at least one isolated vertex, we thus have characterized those which are facet-defining. Notice also that a facet-defining graph G on n vertices cannot have a vertex of degree $n - 2$; moreover, if G has a vertex v of degree $n - 3$, the two vertices nonadjacent to v must be nonadjacent. Moreover, facet-defining trees can be easily characterized.

PROPOSITION 4. *Let G be a tree on n vertices, with $n \geq 3$. Then G is facet-defining iff G is* not *isomorphic to any of the trees depicted in Figure 3.*

Proof. Condition (C1) holds for any tree, and Condition (C4) always holds. Condition (C3) reads: any two nonadjacent vertices a and b belong to a common stable set of size 3. If some vertices a, b do not satisfy this requirement, then any other vertex of G is adjacent to a or b. It is then easily seen that Condition (C3) holds for a tree G iff G is not isomorphic to any of the trees appearing in Figure 3. This

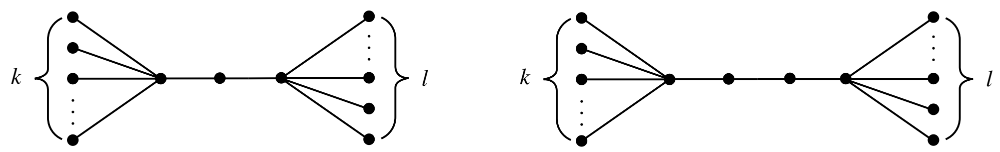

Figure 3. The trees mentioned in Proposition 4. Here, $k \geq 0$, $l \geq 0$ with $k + l = n - 3$ on the left and $k + l = n - 4$ on the right.

establishes the implication from left to right. The converse follows now directly from the very last assertion in Proposition 3. □

Bilayer antichains $\mathcal{A}_k \cup \mathcal{A}_{k+1}$ with parameter $k > 2$ can be combinatorially interpreted as uniform hypergraphs $(S, \mathcal{A}_k)$, with $\mathcal{A}_{k+1}$ the collection of all stable subsets of size $k + 1$. We leave their study for later work.

5. THE CASES WITH FEW ALTERNATIVES

Relying on our previous results, we will now derive a minimum system of (in)equalities for the approval-voting polytope when $n \leq 6$. In other words, we will list all facets, which essentially means all bilayer antichains.

We first explain that in general we need only investigate the bilayer antichains with parameter $2 \leq k \leq (n-1)/2$. Indeed, bilayer antichains with $k = 1$ were easily listed (at the beginning of previous section). Moreover, the linear permutation sending the point x onto the point y with $y_X = x_{S \setminus X}$ for any $X \in \mathcal{P}(S)$ stabilizes P^n_{AV}, and thus induces an affine automorphism of P^n_{AV} (cf. [8]). This automorphism maps a facet corresponding to a bilayer antichain with parameter k onto a facet corresponding to a bilayer antichain with parameter $n - k - 1$.

Let us first consider the cases $2 \leq n \leq 5$ (which are completely solved in Doignon and Regenwetter [8] by using the `Porta` software [3]). For $n = 2$, the polytope has only two vertices and is a segment. For $n = 3$, the bilayer antichains have necessarily parameter $k = 1$, and are thus known. Together with the 6 inequalities from Equation (11), they deliver 9 facet-defining inequalities for P^3_{AV}. When $n = 4$, we need only count the bilayer antichains with $k = 1$ (because those with $k = 2$ follow from them by applying an automorphism). There are $2^4 - 2 = 14$ facets defined by Inequalities (11), $2^4 - (4 + 2) = 10$ facets coming from bilayer antichains with parameter $k = 1$, and also 10 facets coming from bilayer antichains with parameter $k = 2$. This gives a total of 34 facets for P^4_{AV}, as obtained in [8]. For $n = 5$, there are 30 Inequalities (11). Moreover, we have $2^5 - (5 + 2) = 25$ bilayer antichains with parameter $k = 1$, and the same number with $k = 3$. Further work is necessary to find the bilayer antichains with $k = 2$. Applying the previous section, we list all facet-defining graphs with 5 vertices. The result of our search (up to isomorphism) is given below, with graph designations taken from Read and Wilson [15].

(i) G20, G21, G23;

(ii) G25;

(iii) G29, G34, G46.

Notice that each of these graphs can appear a certain number of times (equal to 5! divided by the number of its automorphisms). All counting done, we obtain 235 facets, exactly as in the output of `Porta` reported in [8].

We now turn to the case $n = 6$. Here again, only bilayer antichains with $k = 2$ require thorough investigation, which rely on Propositions 3 and 4. The resulting graphs (up to isomorphism) are the following ones:

(i) G54, G56, G57, G58, G59, G62, G63, G71, G86;

(ii) G65, G66, G73, G74, G75, G76, G88, G90;

(iii) G77, G79, G80, G92, G94, G98, G111, G117, G133, G136, G161, G179, G201.

The summary of our computation is as follows: there are 62 Inequalities (11), and the respective numbers of bilayer antichains equal 56 for $k = 1$; 5,068 for $k = 2$; 5,068 for $k = 3$; and 56 for $k = 4$. Summing up, we get:

PROPOSITION 5. *For $n = 6$, the approval-voting polytope has* 10,310 *facets.*

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