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A Combinatorial Study of Partial Order Polytopes

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Abstract

To each finite set with at least two elements, there corresponds a partial order polytope. It is defined as the convex hull of the characteristic vectors of all partial orders which have that set as ground set. This 0/1-polytope contains the linear ordering polytope as a proper face. The present article deals with the facial structure of partial order polytopes. Our main results are: (i) a proof that the nonadjacency problem on partial order polytopes is $\mathcal{NP}$-complete; (ii) a characterization of the polytopes that are affinely equivalent to a face of some partial order polytope.

1. Introduction

Let n be an integer with $n \geq 2$, and let $\mathcal{A}_n$ denote the set of all pairs of distinct elements in $[n] = \{1, 2, \ldots, n\}$. We refer to the elements of $\mathcal{A}_n$ as *arcs*. The real vector space $\mathbb{R}^{\mathcal{A}_n}$ has one coordinate per arc. In this vector space, the coordinate corresponding to arc $(i, j) \in \mathcal{A}_n$ is denoted by $x_{(i,j)}$. The *characteristic vector* of a relation R on $[n]$ is the point x^R of $\mathbb{R}^{\mathcal{A}_n}$ defined by $x^R_{(i,j)} = 1$ if $(i, j) \in R$ and $x^R_{(i,j)} = 0$ otherwise. Thus, to each relation on $[n]$ there corresponds a 0/1-vector in $\mathbb{R}^{\mathcal{A}_n}$. The *partial order polytope* P^n_{PO}, see Arditti [1], Gurgel [14], Müller [18, 19], Müller and Schulz [20], and Fiorini [7], is the convex hull of the characteristic vectors of all partial orders on $[n]$:

$$P^n_{\mathrm{PO}} = \mathrm{conv}\,\{x^P \in \mathbb{R}^{\mathcal{A}_n} : P \text{ is a partial order on } [n]\}.$$

(A *partial order* is a reflexive, antisymmetric and transitive relation.) The polytope P^n_{PO} has as vertices all points x^P with P partial order on $[n]$. Its dimension

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is $n(n-1)$ because the origin and the $n(n-1)$ vectors of the canonical basis of $\mathbb{R}^{\mathcal{A}_n}$ belong to its vertex set.

polytope	f-vector
P^2_{PO}	(1,3,3)
P^3_{PO}	(1,19,96,193,183,84,17)
P^4_{PO}	(1,219,5791,43995,157306,320597,407638,337787,185280,66796,15300,2049,128)

Table 1: The f-vector of the partial order polytope P^n_{PO} for $2 \leq n \leq 4$.

When speaking of the partial order polytopes, one can hardly avoid to mention their connection to the linear ordering polytopes, which is as follows. The face of P^n_{PO} defined by the valid inequality $\sum x_{(i,j)} \leq n(n-1)/2$ is the *linear ordering polytope* P^n_{LO}, see Fishburn [10] for a survey, Koppen [15], Bolotashvili, Kovalev and Girlich [3], Borndörfer and Weismantel [4], and Fiorini [6] for recent results. A drawing of P^3_{LO} is given in Figure 1. The vertices of P^n_{LO} are exactly the points x^L with L linear order on $[n]$, and its dimension is $n(n-1)/2$. (A *linear order* is a complete partial order.) We have $\dim(P^n_{\mathrm{LO}}) \leq n(n-1)/2$ because each of its vertices, hence each of its points, satisfies the equality $x_{(i,j)} + x_{(j,i)} = 1$ for all pairs $\{i,j\}$ contained in $[n]$. The converse inequality is easily proved, see e.g., [13]. Linear ordering polytopes play an important role in probabilistic preference theory, under the name *binary choice polytopes*, and in combinatorial optimization, where they are used as a tool to solve instances of the 'linear ordering problem'. Given a complete digraph on n vertices with nonnegative weights on the arcs, the *linear ordering problem* is to find a linear ordering of the n vertices whose total weight is maximum. The polyhedral approach of the linear ordering problem was initiated by Grötschel, Jünger and Reinelt [12].

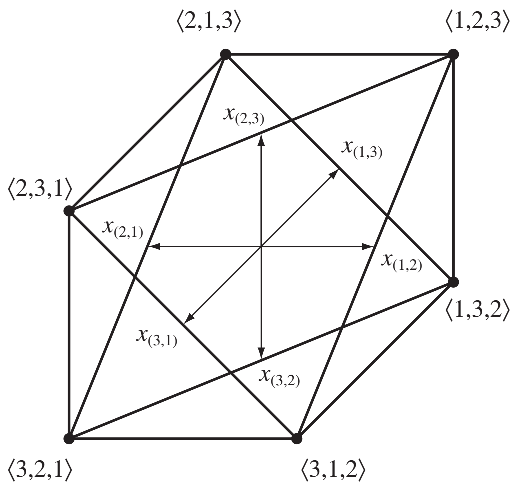

Figure 1: The linear ordering polytope P^3_{LO}.

Our motivation for studying partial order polytopes is that they naturally generalize linear ordering polytopes. From an order-theoretical point of view, the partial order polytope P^n_{PO} is obtained from the linear ordering polytope P^n_{LO} by dropping the completeness axiom. We mention that some other generalizations of the linear ordering polytopes are considered in Doignon and Fiorini [5], and Fiorini and Fishburn [8, 9].

This article deals with the questions of characterizing the edges and of determining the face lattice of the partial order polytopes. Our results on these questions are mainly (but not exclusively) of negative type. Before stating them, we give some related known results on the linear ordering polytope P^n_{LO} and the *asymmetric traveling salesperson polytope* P^n_{ATSP}. The latter polytope is defined as the convex hull in $\mathbb{R}^{\mathcal{A}_n}$ of the $(n-1)!$ characteristic vectors of all n-dicycles on $[n]$, see, e.g., Lawler *et al.* [16].

The edges of the linear ordering polytopes have been characterized by Young [23]. Adjacency on P^n_{LO} can be efficiently tested in $O(n^2)$ time (see [17], and [22] for an application to the determination of the facets of P^n_{LO}). Billera and Sarangarajan [2] have showed that every 0/1-polytope in $\mathbb{R}^d$ with $2^d - k$ vertices is affinely equivalent to a face of the asymmetric traveling salesperson polytope P^n_{ATSP} for $n \geq (4k+1)d$. So, determining the face lattice of P^n_{ATSP} for all n is at least as difficult as determining the face lattice of every 0/1-polytope, which is seemingly unfeasible. We now state our two main results and give an outline of the article (k-SAT polytopes are defined in Subsection 2.2).

THEOREM 1.1: *Determining whether two distinct vertices of partial order polytope P^n_{PO} are nonadjacent or not is a $\mathcal{NP}$-complete problem.*

THEOREM 1.2: *Let P be any (convex) polytope. Then P is affinely equivalent to a face of P^n_{PO} for some $n \geq 2$ if and only if P is affinely equivalent to a face of some 3-SAT polytope.*

Section 2 provides some facts, definitions and notations that are used globally in the paper. Section 3 deals with adjacency on partial order polytopes. We prove that every 3-SAT polytope is affinely equivalent to a face of a partial order polytope whose parameter is linear in the number of variables and the number of clauses of the Boolean formula defining the 3-SAT polytope. We then use this result to prove Theorem 1.1. Section 4 deals with the face lattices of partial order polytopes. After proving Theorem 1.2, we explicitly construct k-SAT polytopes in $\mathbb{R}^k$ (with $k \geq 2$) which cannot be affinely embedded as a face of any ℓ-SAT polytope for all $\ell < k$, and discuss some consequences.

2. Preliminaries

2.1. Known facets

The basic valid inequalities for partial order polytopes are the *trivial*, *antisymmetry* and *transitivity* inequalities which respectively read

$$\begin{aligned} x_{(i,j)} &\geq 0, && (1)\\ x_{(i,j)} + x_{(j,i)} &\leq 1 \quad \text{and} && (2)\\ x_{(i,j)} + x_{(j,k)} - x_{(i,k)} &\leq 1, && (3) \end{aligned}$$

where i, j and k are distinct elements of $[n]$. If X is any subset of $\mathbb{Z}^d$, then a *formulation* of X is a polyhedron K in $\mathbb{R}^d$ (often defined by an explicit system of linear inequalities) such that $K \cap \mathbb{Z}^d = X$. Inequalities (1), (2) and (3) obviously define a formulation of the vertex set of partial order polytope P^n_{PO}.

Every facet-defining inequality for P^n_{PO}, seen as an inequality in $\mathbb{R}^{\mathcal{A}_{n+1}}$, is facet-defining for P^{n+1}_{PO} (this is a socalled 'trivial lifting lemma', see [14, 18]). As a consequence, the trivial and antisymmetry inequalities are facet-defining for all $n \geq 2$, and the transitivity inequalities are facet-defining for all $n \geq 3$. By what precedes, to prove this, it suffices to check that inequalities (1) and (2) are facet-defining for $n = 2$ and that inequality (3) is facet-defining for $n = 3$. Seemingly, the only known facet-defining inequalities for partial order polytopes are the trivial, antisymmetry and transitivity inequalities, the *odd closed walk inequalities* of Müller [19], and the inequalities given in Fiorini [7].

2.2. Boolean formulas, SAT polytopes and k-SAT polytopes

In ensuing sections, we will resort to Boolean formulas in conjunctive normal form. In accordance with [11], we regard these as pairs (U, C) where U is a set of variables and C is a set of clauses over U (see below for definitions). Let U be a finite set. We refer to elements of U as *variables*. To each variable $u \in U$, we associate a distinct *negated variable* $\overline{u} \notin U$. We set $\overline{U} = \{\overline{u} : u \in U\}$ and $\overline{\overline{u}} = u$ for each negated variable $\overline{u} \in \overline{U}$. A *literal* is an element of $U \cup \overline{U}$, and a *clause* is a set of literals. A *truth assignment* of the variables of U is a map from U to $\{T, F\}$. Let t be a truth assignment and ℓ be a literal. Then ℓ is *true* under t either if $\ell \in U$ and $t(\ell) = T$ or if $\ell \in \overline{U}$ and $t(\overline{\ell}) = F$; and ℓ is *false* under t otherwise. Let C be a collection of clauses over U. A truth assignment t is said to be *satisfying* for C if every clause contains at least one literal that is true under t. The problem of determining if a given set of clauses over a given set of variables has a satisfying truth assignment is the famous *satisfiability problem* (SAT). If t is any truth assignment, then we define the truth assignment $\overline{t}$ by $\overline{t}(u) = T$ if $t(u) = F$ and $\overline{t}(u) = F$ if $t(u) = T$, for all $u \in U$. We call a truth assignment t *symmetrizable for C* if each clause in C has at least one true literal and one false literal under t. In particular, symmetrizable truth assignments are satisfying truth assignments.

Each truth assignment t can be encoded by its *characteristic vector* $\chi(t)$ in

$\mathbb{R}^U$ defined by $\chi(t)_u = 1$ if $t(u) = T$ and $\chi(t)_u = 0$ if $t(u) = F$ for all u in U. The *SAT polytope* defined by the pair (U, C), denoted by $P_{\text{SAT}}(U, C)$, is the convex hull of the characteristic vectors of all satisfying truth assignments for C. Whenever the variables are linearly ordered, we may equally well regard the SAT polytope of the pair (U, C) as a polytope in $\mathbb{R}^{|U|}$ by identifying the coordinate of the i-th variable in $\mathbb{R}^U$ with the coordinate x_i in $\mathbb{R}^{|U|}$ for $1 \leq i \leq |U|$.

If each clause of C has exactly k literals for some fixed positive integer k, then $P_{\text{SAT}}(U, C)$ is referred to as a *k-SAT polytope*. By definition, every SAT polytope (and in particular, every k-SAT polytope) is a 0/1-polytope. Conversely, every 0/1-polytope in $\mathbb{R}^d$ is a d-SAT polytope (and hence a SAT polytope).

3. Edges

Some sufficient conditions and some necessary conditions for adjacency on partial order polytopes are known and can be found in Arditti [1] and Fiorini [7]. This section focuses on algorithmic complexity aspects of the adjacency relation on P^n_{PO}. We consider the following problems.

ANOTHER NOT-ALL-EQUAL 3-SAT (Π_1):
Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$, plus one truth assignment g that is symmetrizable for C.
Question: Does there exist a symmetrizable truth assignment for C that is different from g and $\overline{g}$?

PPO NONADJACENCY (Π_2):
Instance: Integer $n \geq 2$, distinct vertices p_1 and p_2 of P^n_{PO}.
Question: Are p_1 and p_2 not adjacent ?

Problem Π_1 is a variant of NOT-ALL-EQUAL 3-SAT, which is known to be NP-complete, see Garey and Johnson [11].

LEMMA 3.1: Π_1 *is $\mathcal{NP}$-complete.*

Proof: Π_1 is clearly in $\mathcal{NP}$. See Fiorini [7] for a polynomial reduction from NOT-ALL-EQUAL 3-SAT to Π_1. □

LEMMA 3.2: *Let U be a set of variables and C be a collection of clauses over U, with 3 literals per clause, and let $P = P_{\text{SAT}}(U, C) \subseteq \mathbb{R}^{|U|}$ be the 3-SAT polytope defined by the pair (U, C). Then P is affinely equivalent to a face of the partial order polytope P^n_{PO}, with $n = 3|U| + 7|C|$.*

Proof: The proof is structured as follows. First, a digraph $D = (V, A)$ with $n = 3|U| + 7|C|$ vertices is constructed from U and C. This digraph yields a point $\tilde{x} \in \mathbb{R}^{\mathcal{A}_n}$. Then, a face F of P^n_{PO} is defined as the intersection of all trivial and transitivity facets that contain $\tilde{x}$ in their affine hull. Finally, it is shown that F is affinely equivalent to P.

Set $U = \{u_1, \ldots, u_{|U|}\}$ and $C = \{c_1, \ldots, c_{|C|}\}$. We arbitrarily fix the ordering of the literals in each clause of C, index the variables by r, and the clauses by s. So $1 \leq r \leq |U|$ and $1 \leq s \leq |C|$. Digraph D has three vertices v_r, $\overline{v}_r$ and v'_r per variable u_r, and seven vertices $w_{1,s}$, $w'_{1,s}$, $w_{2,s}$, $\overline{w}_{3,s}$, a_s, b_s and d_s per clause c_s. The set $\{v_r, \overline{v}_r, v'_r\}$ is referred to as a *variable group*, and the set $\{w_{1,s}, w'_{1,s}, w_{2,s}, \overline{w}_{3,s}, a_s, b_s, d_s\}$ is referred to as a *clause group*. The induced subdigraphs of D on each variable group and on each clause group are respectively given in Figure 2.

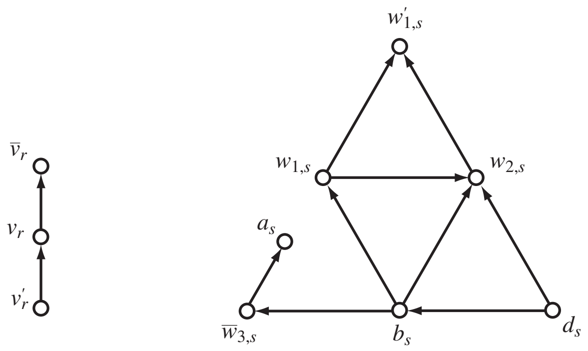

Figure 2: Induced subdigraphs on a variable group, resp. clause group

To obtain D, we add arcs linking the variable groups to the clause groups as follows. For each clause $c_s \in C$, exactly four arcs are added. Let $c_s = \{\ell_1, \ell_2, \ell_3\}$. Because ℓ_1, ℓ_2 and ℓ_3 are literals over U, there exist indices r_1, r_2 and r_3 be such that $\ell_\alpha \in \{u_{r_\alpha}, \overline{u_{r_\alpha}}\}$ for $\alpha = 1, 2, 3$. If $\ell_1 = u_{r_1}$, then the arcs $(v_{r_1}, w_{1,s})$ and $(v_{r_1}, w'_{1,s})$ are added, otherwise $\ell_1 = \overline{u_{r_1}}$ and the arcs $(\overline{v}_{r_1}, w_{1,s})$ and $(\overline{v}_{r_1}, w'_{1,s})$ are added. If $\ell_2 = u_{r_2}$, then the arc $(v_{r_2}, w_{2,s})$ is added, otherwise $\ell_2 = \overline{u_{r_2}}$ and the arc $(\overline{v}_{r_2}, w_{2,s})$ is added. Finally, if $\ell_3 = u_{r_3}$, then the arc $(\overline{v}_{r_3}, \overline{w}_{3,s})$ is added, otherwise $\ell_3 = \overline{u_{r_3}}$ and the arc $(v_{r_3}, \overline{w}_{3,s})$ is added. This completes the construction of digraph D. An example with $U = \{u_1, u_2, u_3, u_4\}$ and $C = \{\{u_1, u_3, \overline{u_2}\}, \{u_1, \overline{u_4}, u_3\}, \{u_3, u_4, \overline{u_2}\}\}$ is represented in Figure 3.

By fixing any linear ordering on the n vertices of D, we may identify each vertex of D with an element of $[n]$. Now consider the point $\tilde{x} \in \mathbb{R}^{\mathcal{A}_n}$ defined by $\tilde{x}_{(i,j)} = \frac{1}{2}$ if $(i, j) \in A$ and $\tilde{x}_{(i,j)} = 0$ otherwise. Let F be the face of P^n_{PO} that is the intersection of all trivial and transitivity facets containing $\tilde{x}$ in their affine hull. Because $\tilde{x}_{(i,j)} + \tilde{x}_{(j,k)} - \tilde{x}_{(i,k)}$ is equal to 1 if and only if $\tilde{x}_{(i,j)} = \tilde{x}_{(j,k)} = \frac{1}{2}$ and $\tilde{x}_{(i,k)} = 0$, we see that a point x of $\mathbb{R}^{\mathcal{A}_n}$ is in F if and only if it satisfies the following conditions:

(C1) $x \in P^n_{\mathrm{PO}}$,

(C2) $x_{(i,j)} = 0$ for all $i, j \in [n]$ s.t. $i \neq j$ and $(i, j) \notin A$,

(C3) $x_{(i,j)} + x_{(j,k)} = 1$ for all $i, j, k \in [n]$ s.t. $(i, j), (j, k) \in A$ and $(i, k) \notin A$.

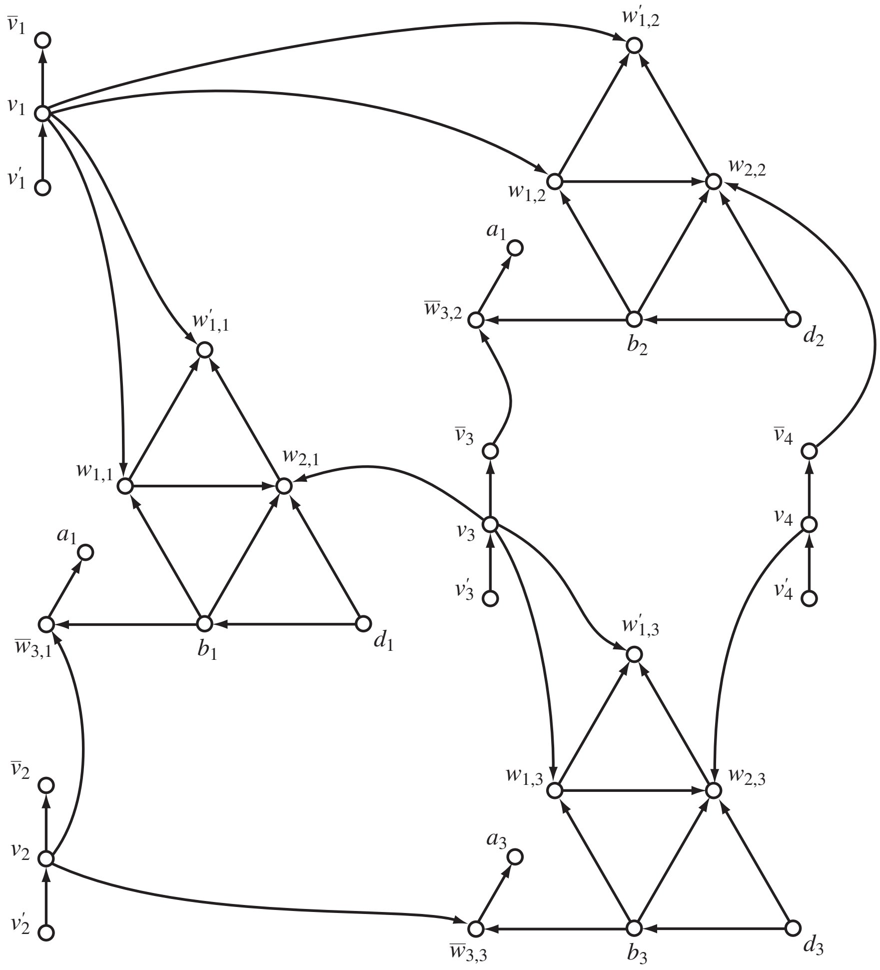

Figure 3: Example of digraph D

We now derive a combinatorial characterization of the vertices of F from conditions (C1), (C2) and (C3). Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ whose vertices are the elements of A and whose edges are the unordered pairs $\{(i,j),(k,l)\}$ of elements of A with ($j = k$ and $(i,l) \notin A$) or ($l = i$ and $(k,j) \notin A$). Let x^Q be a vertex of P^n_{PO}. Then x^Q automatically satisfies (C1), satisfies (C2) if and only if $Q \subseteq A$ and satisfies (C3) if and only if Q contains either (i,j) or (k,l) for each edge $\{(i,j),(k,l)\}$ of $\mathcal{G}$. It can be verified that $\mathcal{G}$ has no cycle, hence $\mathcal{G}$ is a forest, and that each connected component of $\mathcal{G}$ contains a unique vertex of the form $(v_r, \overline{v}_r)$. For each $r \in \{1, \ldots, |U|\}$, define A_r as the set of vertices of $\mathcal{G}$ whose distance to vertex $(v_r, \overline{v}_r)$ is finite and even, and $\overline{A}_r$ as the set of vertices of $\mathcal{G}$ whose distance to vertex $(v_r, \overline{v}_r)$ is finite and odd. The sets A_1, $\overline{A}_1$, $\ldots$, $A_{|U|}$, $\overline{A}_{|U|}$ partition A. By construction, the vertices of F exactly correspond to partial orders Q of the form $Q = \bigcup_{r=1}^{|U|} S_r$ where S_r is either equal to A_r or to $\overline{A}_r$ for $1 \leq r \leq |U|$.

Finally, we verify that the map π from $\mathbb{R}^{\mathcal{A}_n}$ to $\mathbb{R}^{|U|}$ defined by $\pi(x)_r = x_{(v_r,\overline{v}_r)}$ restricts to an affine equivalence between F and P. Let $\dot{\pi}$ be the restriction of π to F. Map $\dot{\pi}$ is one-to-one because the coordinates $x_{(v_r,\overline{v}_r)}$ of a point in F determine all its other coordinates, by construction. Now it suffices to show that $\dot{\pi}$ maps the vertex set of F onto the vertex set of P.

Let x^Q be a vertex of F. Clearly, $\pi(x^Q)$ is a 0/1-vector in $\mathbb{R}^{|U|}$, so there exists a unique truth assignment t of the variables of U such that $\chi(t) = \pi(x^Q)$. Suppose that t is not satisfying for C. Then, there exists a clause c_s in C whose literals are all false under t. Let $c_s = \{\ell_1, \ell_2, \ell_3\}$. For the sake of simplicity, we assume that all literals of c_s are nonnegated variables (leaving the other cases to the reader). Therefore, there exist indices r_α for $\alpha = 1, 2, 3$ such that $\ell_\alpha = u_{r_\alpha}$ for $\alpha = 1, 2, 3$. By hypothesis, Q contains all arcs of $\overline{A}_{r_1}$, $\overline{A}_{r_2}$ and $\overline{A}_{r_3}$ and no arcs of A_{r_1}, A_{r_2} and A_{r_3}. In particular, $(b_s, w_{1,s}) \in Q$, $(w_{1,s}, w_{2,s}) \in Q$ and $(b_s, w_{2,s}) \notin Q$. This contradicts the transitivity of Q. It follows that $\pi(\operatorname{vert} F) \subseteq \operatorname{vert} P$.

Now let t be a satisfying truth assignment for C. Set $S_r = A_r$ if u_r is true under t and $S_r = \overline{A}_r$ if u_r is false under t, for $1 \leq r \leq |U|$. Let $Q = \cup_{r=1}^{|U|} S_r$. By construction, $\pi(x^Q) = \chi(t)$. Suppose that Q is not transitive. We leave it to the reader to check that there must exist an index s such that $(b_s, w_{1,s}) \in Q$, $(w_{1,s}, w_{2,s}) \in Q$ and $(b_s, w_{2,s}) \notin Q$, and that this implies that all literals of clause c_s are false under t, a contradiction. It follows that $\operatorname{vert} P \subseteq \pi(\operatorname{vert} F)$. In conclusion, $\dot{\pi}$ is an affine equivalence between F and P. $\square$

Let P be any (convex) polytope. An (unordered) pair $\{v_1, v_2\}$ of distinct vertices of P is said to be *regular* if either v_1 and v_2 are adjacent, or there is another (unordered) pair $\{w_1, w_2\}$ of distinct vertices of P such that $\frac{1}{2}v_1 + \frac{1}{2}v_2 = \frac{1}{2}w_1 + \frac{1}{2}w_2$ (the latter condition implies that v_1 and v_2 are nonadjacent). Following Naddef and Pulleyblank [21], we call a polytope for which every pair of distinct vertices is regular a *combinatorial polytope.* It is easy to find a 3-dimensional 0/1-polytope with 5 vertices which is not combinatorial. Now it follows from Lemma 3.2 that the partial order polytope P^n_{PO} is not combinatorial for $n \geq 30$.

Proof: (of Theorem 1.1). It is not difficult to prove that Π_2 is in $\mathcal{NP}$. We polynomially reduce Π_1 to Π_2. Let U be a set of variables, C be a set of clauses over U with three literals per clause, and let $g : U \to \{T, F\}$ be a truth assignment that is symmetrizable for C. Define $\overline{C}$ as the collection of clauses obtained from C by negating each literal in each clause.

Let $P = P_{\mathrm{SAT}}(U, C \cup \overline{C})$ be the 3-SAT polytope defined by the pair $(U, C \cup \overline{C})$. By construction, the vertices of P exactly correspond to symmetrizable truth assignments for C, hence polytope P is centrally symmetric with respect to the center of the $|U|$-dimensional 0/1-cube, and $p = \chi(g)$ and $\overline{p} = \chi(\overline{g})$ are two opposite vertices of P. Moreover, p and $\overline{p}$ are nonadjacent in P if and only if P has a vertex different from p and $\overline{p}$. This occurs if and only if there exists a symmetrizable truth assignment for C that is different from g and $\overline{g}$.

By Lemma 3.2, P is affinely equivalent to a face F of P^n_{PO}, with $n = 3|U| + 7|C|$.

Let σ be an affine map from $\mathbb{R}^{|U|}$ to $\mathbb{R}^{\mathcal{A}_n}$ whose restriction to P is an affine equivalence between P and F (as the one obtained in the proof of Lemma 3.2). Thus, p and $\overline{p}$ are nonadjacent in P if and only if $\sigma(p)$ and $\sigma(\overline{p})$ are nonadjacent in F, that is, if and only if $\sigma(p)$ and $\sigma(\overline{p})$ are nonadjacent in P^n_{PO}. From what precedes, we conclude that there exists a symmetrizable truth assignment for C that is different from g and $\overline{g}$ if and only if vertices $\sigma(p)$ and $\sigma(\overline{p})$ are nonadjacent in P^n_{PO}. In this way, Π_1 can be polynomially reduced to Π_2. □

4. More on faces

Whenever P and Q are two polytopes such that P is affinely equivalent to a face of Q, we write $P \precsim Q$ and say that P can be *affinely embedded as a face* of Q. Thus $\precsim$ is a preorder on the set of all polytopes. (A *preorder* is a reflexive and transitive relation.)

LEMMA 4.1: *If $P = P_{\text{SAT}}(U, C)$ is a SAT polytope and k is a positive integer such that $|c| \le k$ for all clauses c in C, then there exists a k-SAT polytope Q such that $P \precsim Q$.*

Proof: Let $m = \min\{|c| : c \in C\}$, and let $u'_1, \ldots, u'_{k-m}$ be $k - m$ variables not in U. Define $U' = U \cup \{u'_1, \ldots, u'_{k-m}\}$ and $C' = \{c \cup \{u'_1, \ldots, u'_{k-|c|}\} : c \in C\}$. Now let Q be the k-SAT polytope defined by the pair (U', C'). Then P is affinely equivalent to the face of Q defined by the valid inequality $\sum_{i=1}^{k-m} x_{u'_i} \ge 0$. □

Proof: (of Theorem 1.2). The partial order polytope P^n_{PO} can alternatively be defined as the SAT polytope of the pair (U, C), where $U = \{u_{(i,j)} : (i,j) \in \mathcal{A}_n\}$ and $C = \{\{\overline{u_{(i,j)}}, \overline{u_{(j,i)}}\} : (i,j) \in \mathcal{A}_n\} \cup \{\{\overline{u_{(i,j)}}, \overline{u_{(j,k)}}, u_{(i,k)}\} : i, j, k \text{ distinct in } [n]\}$. By Lemma 4.1, for each $n \ge 2$, there is a 3-SAT polytope Q_n such that P^n_{PO} is affinely equivalent to a face of Q_n. The Theorem now easily follows from Lemma 3.2 and from the transitivity of $\precsim$. □

Let P be a 0/1-polytope with vertex set X, let $Ax \le b$ be a system of inequalities defining a formulation K of X, and let $a_1, \ldots, a_m$ denote the rows of A. Then formulation K is said to be *at most k-valued* if the linear form $x \mapsto a_i x$ takes at most k values on X for $i = 1, \ldots, m$.

PROPOSITION 4.1: *Let $k \ge 2$ and let P be any 0/1-polytope in $\mathbb{R}^k$ satisfying the following conditions:*

(C1) o, e_1, $e_1 + e_2$, $\ldots$, $e_1 + e_2 + \ldots + e_{k-1}$ are vertices of P;

(C2) $(e_1 + e_2 + \ldots + e_k) - e_i$ is a vertex of P for $1 \le i \le k$;

(C3) $e_1 + e_2 + \ldots + e_k$ is not a vertex of P;

where e_1, $\ldots$, e_k denote the vectors of the canonical basis of $\mathbb{R}^k$. Then P is a k-dimensional k-SAT polytope such that $P \precsim Q$ for no ℓ-SAT polytope with $\ell < k$.

Proof: By conditions (C1) and (C2), the dimension of P equals k. Moreover, P is a k-SAT polytope because it is a 0/1-polytope in $\mathbb{R}^k$. Now suppose that there is some d and some ℓ-SAT polytope Q in $\mathbb{R}^d$ such that $P \precsim Q$ and $\ell < k$. We show that this leads to a contradiction, in the following way. First we prove that no formulation of the vertex set of P is at most ℓ-valued. Then we construct an at most ℓ-valued formulation of the vertex set of P by 'lifting' an at most ℓ-valued formulation of the vertex set of Q.

By condition (C3), any system of inequalities defining a formulation of the vertex set of P must contain an inequality $cx \leq \delta$ which is violated by the point $e_1 + e_2 + \ldots + e_k$. Because $cx \leq \delta$ is valid for P, condition (C2) implies $c_i > 0$ for $i = 1, \ldots, k$. By condition (C1), the linear form $x \mapsto cx$ takes the following values on the vertex set of P: 0, c_1, $c_1 + c_2$, $\ldots$, $c_1 + c_2 + \ldots + c_{k-1}$. Because all coefficients of c are positive, these k values are distinct. In particular, as $\ell < k$, no formulation of the vertex set of P is at most ℓ-valued.

Since $P \precsim Q$ and P is full-dimensional, there exist a one-to-one affine mapping $\pi : x \mapsto y = \pi(x)$ from $\mathbb{R}^k$ to $\mathbb{R}^d$ and a face F of Q such that $\pi(P) = F$. Let $U = \{u_1, \ldots, u_d\}$ and C respectively be a set of variables and a set of clauses of size ℓ over U such that $Q = P_{\mathrm{SAT}}(U, C)$. If $c \in C$ is a clause, then we denote by $I^- = I^-(c)$ and $I^+ = I^+(c)$ the two (not necessarily disjoint) sets of indices such that $c = \{u_i : i \in I^-\} \cup \{\overline{u_i} : i \in I^+\}$. To each clause $c \in C$, we associate the *clause inequality*

$$\sum_{i \in I^+} y_i - \sum_{i \in I^-} y_i \leq |I^+| - 1.$$

Let $Ay \leq b$ be the system formed by all clause inequalities and by the trivial inequalities $-y_i \leq 0$ and $y_i \leq 1$ for $i = 1, \ldots, d$. By construction, this system of inequalities defines a formulation of the vertex set of Q. Moreover, this formulation is easily seen to be at most ℓ-valued. Consider the *lifted* system $A\pi(x) \leq b$ in $\mathbb{R}^k$. We claim it defines an at most ℓ-valued formulation of the vertex set of P. If p is a vertex of P, then $\pi(p)$ is a vertex of Q, hence we have $A\pi(p) \leq b$. Now let p be a point of $\mathbb{Z}^d$ satisfying $A\pi(p) \leq b$. By conditions (C1) and (C3), point p can be expressed as an affine combination of some vertices of P with integral coefficients. It follows that $\pi(p)$ is a point of $\mathbb{Z}^d$. Because $Ay \leq b$ defines a formulation of the vertex set of Q, point $\pi(p)$ is a vertex of Q, hence p is a vertex of P. So $A\pi(x) \leq b$ defines a formulation of the vertex set of P. We leave it to the reader to verify that this formulation is at most ℓ-valued. □

One the one hand, Lemma 3.2 allows us to affinely embed a variety of 0/1-polytopes as faces of the partial order polytopes. On the other hand, by Theorem 4.1, we can find many explicit examples of 0/1-polytopes which cannot be affinely embedded as a face of any partial order polytope. For instance, if P is a 4-dimensional 0/1-cube with one vertex removed, then there is no $n \geq 2$ such that $P \precsim P^n_{\mathrm{PO}}$. Finally, we note that, together with Billera and Sarangarajan's result, Proposition 4.1 implies that it is not possible to find Boolean formulas in conjunctive normal form that define the vertex set of the asymmetric traveling salesperson polytopes and have clauses of bounded size.

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