Extendability of cyclic orders

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Abstract

A cyclic order is a ternary relation that satisfies ternary transitivity and asymmetry conditions. Such a ternary relation is *extendable* if it is included in a complete cyclic order on the same ground set. Unlike the case of linear extensions of partial orders, a cyclic order need not be extendable. The *extension problem* for cyclic orders is to determine if a cyclic order is extendable. This problem is known to be NP-complete. We introduce a class of cyclic orders in which the extension problem can be solved in polynomial time. The class provides many explicit examples of nonextendable cyclic orders that were not previously known, including a nonextendable cyclic order on seven points. Let μ be the minimum cardinality of a ground set on which all cyclic orders are extendable. It has been shown that $\mu \leq 9$. We prove that $\mu = 6$. This answers a question of Novák. In addition, we characterize the nonextendable cyclic orders on seven and eight points. Our results are intimately related to irreducible partially ordered of order dimension three, and to fractional vertices of generalized transitive tournament polytopes. As by-products, we obtain a characterization of cyclically ordered sets of dimension two, and a new proof of a theorem of Dridi on small linear ordering polytopes.

1 Introduction

Let $\mathcal{Y}$ be a finite subset of an oriented circle $\mathcal{C}$ in the Euclidean plane. Then $\mathcal{Y}$ determines a ternary relation $\mathcal{T}$ on $\mathcal{Y}$ with $(x,y,z) \in \mathcal{T}$ if and only if y lies in the open circular arc that goes from x to z in the given orientation of $\mathcal{C}$ (see Figure 1). By definition, $(x,y,z) \in \mathcal{T}$ holds only if x, y and z are distinct points of $\mathcal{Y}$, and implies $(y,z,x) \in \mathcal{T}$ and $(z,x,y) \in \mathcal{T}$.

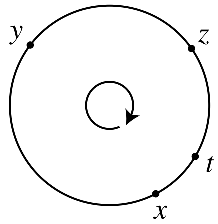


Figure 1: Some points of oriented circle $\mathcal{C}$

A *tricycle* on $\mathcal{Y}$ is a set $\{(x,y,z),(y,z,x),(z,x,y)\}$, where x, y, z are distinct points in $\mathcal{Y}$. We abbreviate the tricycle as xyz; its oppositely oriented tricycle is zyx. It is clear that $\mathcal{T}$ is a disjoint union of tricycles. By abuse of notation, we consider $\mathcal{T}$ as a set of tricycles. Obviously, $\mathcal{T}$ satisfies the following conditions.

$$\begin{aligned} \textit{transitivity}:&\ \ xyz \in \mathcal{T}, xzt \in \mathcal{T} \text{ and } y \neq t \text{ imply } xyt \in \mathcal{T}; && (1)\\ \textit{asymmetry}:&\ \ xyz \in \mathcal{T} \text{ implies } zyx \notin \mathcal{T}; && (2)\\ \textit{completeness}:&\ \ xyz \in \mathcal{T} \text{ or } zyx \in \mathcal{T} \text{ for all distinct } x, y \text{ and } z \text{ in } \mathcal{Y}. && (3)\end{aligned}$$

Conditions (1), (2) and (3), which are contained in Huntington [20, 21], exactly characterize the sets of tricycles that encode the relative positions of points on an oriented circle. Not surprisingly, these conditions also appear in texts on projective geometry (see, e.g., Pickert [30] and Heyting [18]). We now switch to a more general setting.

Let Y be a finite set, and let T be a set of tricycles on Y. Then $\mathbf{T} = (Y,T)$ is a *trigraph* with *ground set* Y. We say that T is *transitive* if it satisfies condition (1), *asymmetric* if it satisfies condition (2), and *complete* if it satisfies condition (3). When T is asymmetric and transitive, we refer to T as a *cyclic order* and to $\mathbf{T} = (Y,T)$ as a *cyclically ordered set.* If C is a complete cyclic order on Y, then $\mathbf{C} = (Y,C)$ is a *ternary cycle*, or simply a *cycle* when no confusion can occur. It is customary to denote by $\{x_1x_2\cdots x_m\}$ the cyclic order on $Y = \{x_1,x_2,\ldots,x_m\}$ defined by $T = \{x_ix_jx_k : i,j,k \in \{1,2,\ldots,m\}$ and $i<j<k\}$.

If T is a set of tricycles on Y and if there exists a complete cyclic order C on Y that contains T, then T is said to be *extendable*, and C is referred to as a *complete extension* of T. The extension problem for sets of tricycles is to decide if a set of tricycles is extendable. The problem was shown by Galil and Megiddo [14] to be NP-complete. Deciding whether a set of tricycles is extendable is not easier if it is transitive and asymmetric, for the two following reasons. Define the *transitive closure* of set of tricycles T as the intersection of all transitive sets of tricycles containing T. We denote the transitive closure of T by T^+. The first reason is that a set of tricycles is extendable if and only if its transitive closure is extendable. Note that if T^+ is not asymmetric, then it is not extendable. The second reason is that the transitive closure of a set of tricycles can be computed in polynomial time. In conclusion, the extension problem for sets of tricycles and the extension problem for cyclic orders are polynomially equivalent. A surprising fact illustrating the hardness of these problems is that, contrary to the binary case where all partial orders have at least one linear extension, a cyclic order is not necessarily extendable, as is shown in the following example.

Example 1.1 Let $Y = \{a, b, c, a', b', c'\} \cup \{\infty\}$ and $T^* = \{\infty ab', \infty ac', \infty ba', \infty bc', \infty ca', \infty cb', ab'b, ab'a', ac'c, ac'a', ba'a, ba'b', bc'c, bc'b', ca'a, ca'c', cb'b, cb'c'\}$. Then $\mathbf{T}^* = (Y, T^*)$ is a cyclically ordered set whose cyclic order is nonextendable (see Section 3 for a general construction and a proof).

In Section 2, we rephrase the extension problem for cyclic orders as a problem on linear extensions of partial orders. In Section 3, we restrict the discussion to a certain class of cyclically ordered sets: the augmented cones. We show that the extension problem is solvable in polynomial time for augmented cones. Augmented cones provide many explicit examples of nonextendable cyclic orders that were not previously known. As a by-product, we obtain a characterization of cyclically ordered sets of dimension two. In Section 4, building on the results of Sections 2 and 3, sufficient conditions for extendability are developed. These conditions are then used in Section 5 to settle a question due to Novák [28]: What is the greatest natural μ such that all cyclic orders on μ points or less are extendable ? In 1976, Megiddo [23] obtained $\mu < 13$. His bound was improved eight years later by Novák [28] to $\mu < 10$. We prove that cyclic orders on less than seven points are extendable, which together with Example 1.1 implies $\mu = 6$. In Section 6, cyclic orders on seven and eight points are considered. We present computer-assisted proofs of two facts. The first fact is that every cyclically ordered set on seven points has an extendable cyclic order or is isomorphic to $\mathbf{T}^*$ (see Example 1.1). The second fact is that there are 8 inclusion-minimal nonextendable cyclic orders on eight points, up to isomorphism and duality. In Section 7, further relations to order dimension theory and polyhedral combinatorics are discussed.

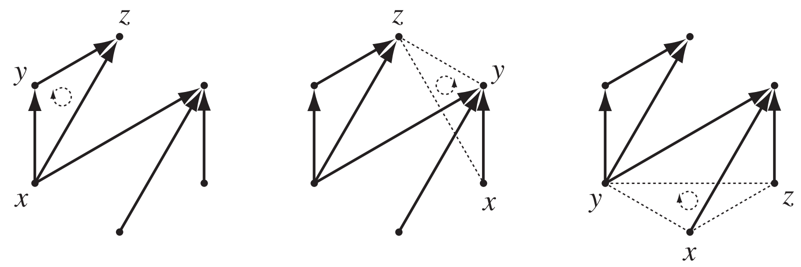
We assume that the reader is familiar with the basic terminology of partially ordered sets (posets) and graphs (and digraphs). Let $\mathbf{P} = (X, P)$ be a poset with P irreflexive and transitive, and therefore asymmetric also. By convention, we have $x \perp y$ in P if x and y are comparable in P, and $x \parallel y$ in P if $x \neq y$ and x and y are incomparable in P. The comparability graph and the incomparability graph of $\mathbf{P}$ are respectively denoted by $\mathbf{G}(\mathbf{P})$ and $\overline{\mathbf{G}}(\mathbf{P})$. In our figures, posets are represented either as digraphs or by Hasse diagrams. We denote by $\mathbf{P}^d = (X, P^d)$ the dual of $\mathbf{P}$. A *linear extension* of P is a linear order L on the same ground set such that $P \subseteq L$. The *dimension* of $\mathbf{P}$, denoted by $\dim(\mathbf{P})$ or $\dim(X, P)$, is the minimum cardinality of a collection of linear orders on X whose intersection is P. Dimension theory of posets was initiated by Dushnik and Miller [9] and is extensively covered in Trotter [33]. There exist several bounds on the dimension of a partial order, the most useful for our discussion being *Hiraguchi's inequality* [19] which states that $\dim(X, P) \leq |X|/2$ for all posets $\mathbf{P} = (X, P)$ with $|X| \geq 4$. Hiraguchi's inequality is tight, as shown by poset $\mathbf{S}_n$ ($n \geq 3$), the *standard example* of an n-dimensional poset with $2n$ points (see, e.g., Trotter [33], page 12). Let $\mathbf{G} = (X, E)$ be a graph and P be a partial order on X. Then P is a *transitive orientation* of $\mathbf{G}$ if the comparability graph of poset (X, P) is $\mathbf{G}$. Quite naturally, a graph is said to be *transitively orientable* if it has a transitive orientation. A classic result of Dushnik and Miller [9] is that a poset has dimension at most two if and only if its incomparability graph is transitively orientable.

The notions of *dual* trigraph, of *induced* (sub)trigraph of a trigraph on a subset of its ground set, of *restriction* of a set of tricycles to a subset of its ground set, and of *isomorphism* of two trigraphs are easily defined by adapting the definitions of the corresponding notions for digraphs. The dual of trigraph $\mathbf{T} = (Y, T)$ is denoted by $\mathbf{T}^d = (Y, T^d)$. References on cyclically ordered sets that are not cited elsewhere in this article include Alles [1], Chajda and Novák [6], Novák [27], and Quillot [31].

2 Rephrasing the extension problem

Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set. Fix a point ∞ in Y. We say that a tricycle xyz is *through* ∞ when $\infty \in \{x, y, z\}$. Because T satisfies the ternary transitivity condition (1) and the ternary asymmetry condition (2), the tricycles of T through ∞ determine a partial order P on $X = Y \setminus \{\infty\}$, with $x < y$ in P if and only if $\infty xy \in T$. The poset $\mathrm{Res}(\mathbf{T}, \infty) = (X, P)$ is called the *residue* of $\mathbf{T}$ at ∞. Determining if there exists a complete cyclic order containing T amounts to determining if there exists a linear extension of P that satisfies certain extra conditions. The next few paragraphs develop this equivalence.

There are three possible placements of a tricycle xyz of T not through ∞ with respect to the residue of $\mathbf{T}$ at ∞, as follows. Let $\mathbf{P} = (X, P) = \mathrm{Res}(\mathbf{T}, \infty)$. The induced poset of $\mathbf{P}$ on $\{x, y, z\}$ is either: (i) a chain, which is necessarily one of $x < y < z$, $y < z < x$ and $z < x < y$, or (ii) a **2** + **1** poset, whose partial order must contain exactly one directed edge in $\{xy, yz, zx\}$, or (iii) an antichain. We say that the tricycle xyz is *implied* by $\mathbf{P}$ if (i) occurs, *touches* $\mathbf{P}$ if (ii) occurs, and is *disjoint* from $\mathbf{P}$ if (iii) occurs. This is illustrated in Figure 2.



(i) (ii) (iii)

Figure 2: Placements of tricycle xyz w.r.t. poset $\mathbf{P}$

We denote by $\mathrm{Impl}(\mathbf{P})$, $\mathrm{Touch}(\mathbf{P})$ and $\mathrm{Disj}(\mathbf{P})$ the set of all tricycles on X implied by $\mathbf{P}$, touching $\mathbf{P}$ and disjoint from $\mathbf{P}$, respectively. By definition, we have

$$\begin{aligned}
\mathrm{Impl}(\mathbf{P}) &= \{xyz : x < y < z \text{ in } P\}, \\
\mathrm{Touch}(\mathbf{P}) &= \{xyz : x < y,\ x \parallel z \text{ and } y \parallel z \text{ in } P\}, \\
\mathrm{Disj}(\mathbf{P}) &= \{xyz : x \parallel y,\ x \parallel z \text{ and } y \parallel z \text{ in } P\}.
\end{aligned}$$

Clearly, the preceding definitions still make sense when $\mathbf{P} = (X, P)$ is any poset. Let T_∞ denote the set of tricycles of T through ∞. Let T_{impl}, T_{touch} and T_{disj} respectively denote the set of tricycles of T implied by $\mathbf{P}$, touching $\mathbf{P}$ and disjoint from $\mathbf{P}$. Thus, we have $T_{\mathrm{impl}} = T \cap \mathrm{Impl}(\mathbf{P})$, $T_{\mathrm{touch}} = T \cap \mathrm{Touch}(\mathbf{P})$ and $T_{\mathrm{disj}} = T \cap \mathrm{Disj}(\mathbf{P})$. The cyclic order T is now split into four disjoint parts

$$T = T_\infty \cup T_{\mathrm{impl}} \cup T_{\mathrm{touch}} \cup T_{\mathrm{disj}}. \tag{4}$$

Note that every tricycle implied by $\mathbf{P}$ is in T because $\mathbf{P}$ is the residue of $\mathbf{T}$ at ∞, while there could exist a tricycle touching $\mathbf{P}$ that is not in T. That is, we always have $T_{\mathrm{impl}} = \mathrm{Impl}(\mathbf{P})$, but the inclusion $T_{\mathrm{touch}} \subseteq \mathrm{Touch}(\mathbf{P})$ can be proper. Let $\mathbf{C} = (Y, C)$ be a ternary cycle. The residue of $\mathbf{C}$ at ∞ is a chain, say $\mathbf{L} = (X, L)$. It should be clear from the definition of a residue and from Equation (4) that $T \subseteq C$ if and only if $P \subseteq L$ and L contains exactly two directed edges in $\{xy, yz, zx\}$ for each tricycle xyz in $T_{\mathrm{touch}} \cup T_{\mathrm{disj}}$. This is stated in a slightly different way in Proposition 2.1.

Let S be a subset of $\mathrm{Touch}(\mathbf{P})$. A linear extension L of P is said to be *separating for* S if there exist elements x, y and z such that $x < y$ in P, $xyz \in S$ and $x < z < y$ in L. If S coincides with $\mathrm{Touch}(\mathbf{P})$ and L is separating for S, we simply say, as in Dushnik and Miller [9], that L is *separating*.

Proposition 2.1 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, let $\mathbf{P} = (X, P)$ be the residue of $\mathbf{T}$ at ∞, let T_{touch} be the set of tricycles of T touching $\mathbf{P}$ and let T_{disj} be the set of tricycles of T disjoint from $\mathbf{P}$. Then T is extendable if and only if P has a linear extension L such that L is nonseparating for T_{touch}, and such that $x < y < z$ or $y < z < x$ or $z < x < y$ in L for all $xyz \in T_{\mathrm{disj}}$.* ■

Suppose that $T_{\mathrm{touch}} = \mathrm{Touch}(\mathbf{P})$. By Proposition 2.1, if T is extendable, then P has a nonseparating linear extension. Dushnik and Miller [9] proved that a poset is of dimension at most two if and only if its partial order has a nonseparating linear extension. Therefore, under the assumption $T_{\mathrm{touch}} = \mathrm{Touch}(\mathbf{P})$, cyclic order T is extendable only if the dimension of $\mathbf{P}$ is at most two. Moreover, under the assumptions $T_{\mathrm{touch}} = \mathrm{Touch}(\mathbf{P})$ and $T_{\mathrm{disj}} = \varnothing$, cyclic order T is extendable if and only if the dimension of $\mathbf{P}$ is at most two, by Proposition 2.1. In the latter case, a cyclically ordered set is entirely determined by its residue at ∞. The next section focuses on such cyclically ordered sets.

3 Augmented cones

Let $\mathbf{P} = (X, P)$ be a poset with $\infty \notin X$. The *augmented cone of* $\mathbf{P}$ *at* ∞ is the trigraph $\mathrm{AugCone}(\mathbf{P}, \infty) = (Y, T)$ defined as follows:

$$Y = X \cup \{\infty\}, \text{ and}$$
$$T = \{\infty xy : x < y \text{ in } P\} \cup \mathrm{Impl}(\mathbf{P}) \cup \mathrm{Touch}(\mathbf{P})$$

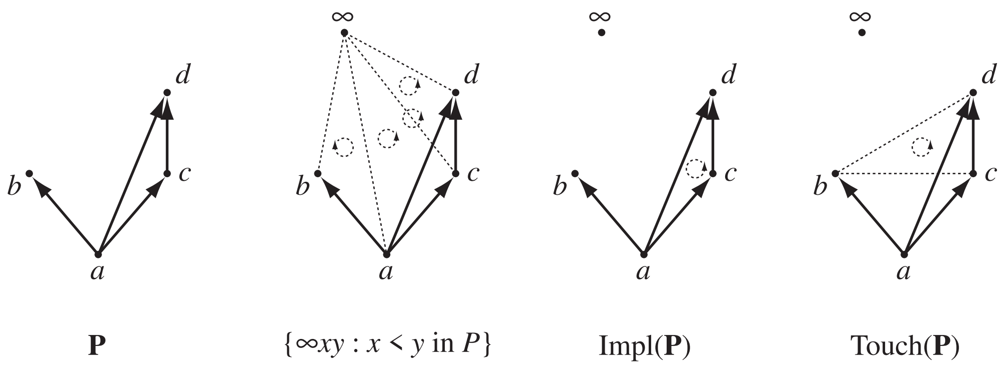


Figure 3: The augmented cone construction

Example 3.1 If $\mathbf{P}$ is the poset represented in Figure 3, then the augmented cone of $\mathbf{P}$ at ∞ has $Y = \{a, b, c, d\} \cup \{\infty\}$ as ground set, and $T = \{\infty ab, \infty ac, \infty ad, \infty cd\} \cup \{acd\} \cup \{cdb\}$ as its set of tricycles.

The next proposition is a straightforward consequence of the definitions.

Proposition 3.2 *If* $\mathbf{P} = (X, P)$ *is a poset with* $\infty \notin X$*, then the augmented cone of* $\mathbf{P}$ *at* ∞ *is a cyclically ordered set.* ∎

Quite naturally, a cyclically ordered set $\mathbf{T} = (Y, T)$ is said to be an *augmented cone* if $\mathbf{T} = \mathrm{AugCone}(\mathbf{P}, \infty)$ for some ∞ in Y and some poset $\mathbf{P}$ with ground set $Y \setminus \{\infty\}$. Proposition 3.3 gives three other equivalent characterizations of augmented cones, which are easily established.

Proposition 3.3 *Let* $\mathbf{T} = (Y, T)$ *be a cyclically ordered set. Then the following four assertions are equivalent.*

(i) $\mathbf{T}$ *is an augmented cone;*

(ii) $\mathbf{T} = \mathrm{AugCone}(\mathrm{Res}(\mathbf{T}, y_0), y_0)$ *for all* $y_0 \in Y$*;*

(iii) *there exists* $z_0 \in Y$ *such that the induced cyclically ordered set of* $\mathbf{T}$ *on any subset* S *of* Y *with* $|S| = 4$ *and* $z_0 \in S$ *has an even number of tricycles;*

(iv) *the induced cyclically ordered set of* $\mathbf{T}$ *on any subset* S *of* Y *with* $|S| = 4$ *has an even number of tricycles.* ■

Proposition 3.4 is a specialization of Proposition 2.1 to augmented cones. The main argument for Proposition 3.4 is given in the remark that follows Proposition 2.1.

Proposition 3.4 *Let* $\mathbf{P} = (X, P)$ *be a poset with* $\infty \notin X$*. Then the cyclic order of* $\mathrm{AugCone}(\mathbf{P}, \infty)$ *is extendable if and only if* $\dim(\mathbf{P}) \leq 2$*.* ■

The latter proposition shows that the extension problem for augmented cones is solvable in polynomial time, because recognizing posets of dimension at most two can be done in polynomial time (see, e.g., Trotter [33], page 97). Alles, Nešetřil and Poljak [2], and independently Fishburn and Woodall [13] defined the *(intersection) dimension* of a cyclically ordered set $\mathbf{T} = (Y, T)$, hereafter denoted by $\dim(\mathbf{T})$, as the minimum cardinality of a collection of complete cyclic orders whose intersection is T. If T is not an intersection of complete cyclic orders, we set $\dim(\mathbf{T}) = +\infty$ by convention. Propositions 3.3 and 3.4 provide a characterization of cyclically ordered sets of dimension at most two that we state in Proposition 3.5. To our knowledge, no such characterization was previously known. Note that Propositions 3.3 and 3.5 imply that cyclically ordered sets of dimension at most two can be recognized in polynomial time.

Proposition 3.5 *A cyclically ordered set is of dimension at most two if and only if it is an augmented cone whose cyclic order is extendable.* ■

Proposition 3.4 also provides a large family of nonextendable cyclic orders, which in particular contains the cyclic order T^* defined in Example 1.1. In fact, $\mathbf{T}^*$ is the augmented cone of standard example $\mathbf{S}_3$ at ∞ (see Figure 6 for a picture of $\mathbf{S}_3$). Other general constructions of nonextendable cyclic orders are available in the literature, as for example the one of Alles, Nešetřil and Poljak [2]. However, for our purposes, the augmented cone construction has two advantages over the other constructions. First, its basic structure admits many explicit examples because each poset of dimension at least three determines a nonextendable cyclic order. The most interesting cases of this occur perhaps for irreducible dimension-3 posets: see Section 7. Second, the augmented cone construction generates a variety of nonextendable cyclic orders on few points, including the smallest known nonextendable cyclic order. Further relations between the augmented cone construction, order dimension theory and polyhedral combinatorics are discussed in Section 7.

4 Extensions by glueing

Let $\mathbf{P} = (X, P)$ be any poset. It is well known that L is a nonseparating linear extension of P if and only if $L = P \cup Q$ for some transitive orientation Q of the incomparability graph of $\mathbf{P}$. Trivially, a nonseparating linear extension is also nonseparating for any subset of $\mathrm{Touch}(\mathbf{P})$. Consequently, Proposition 2.1 implies the following proposition.

Proposition 4.1 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let $\mathbf{P}$ be the residue of $\mathbf{T}$ at ∞. Suppose there exists a transitive orientation Q of the incomparability graph of $\mathbf{P}$ such that for every tricycle xyz of $\mathbf{T}$ disjoint from $\mathbf{P}$, we have $x < y < z$, $y < z < x$ or $z < x < y$ in Q. Then T is extendable.* ∎

This proposition explains our main strategy for finding a complete cyclic order extending a given cyclic order. Given a cyclically ordered set $\mathbf{T} = (Y, T)$ with a distinguished point $\infty \in Y$, we split its set of tricycles into four disjoint parts according to Equation (4). Let $\mathbf{P} = (X, P)$ be the residue of $\mathbf{T}$ at ∞. We extend T_∞, T_{impl} and T_{touch} by the cyclic order of $\mathrm{AugCone}(\mathbf{P}, \infty)$. Then we look for a transitive orientation Q of the incomparability graph of $\mathbf{P}$ such that all tricycles of T_{disj} are implied by poset $\mathbf{Q} = (X, Q)$. Proposition 4.1 asserts that the union of the cyclic orders of $\mathrm{AugCone}(\mathbf{P}, \infty)$ and $\mathrm{AugCone}(\mathbf{Q}, \infty)$ is a complete cyclic order containing T.

We now formalize the condition that Proposition 4.1 imposes on the incomparability graph of $\mathbf{P}$. Let $\mathbf{H} = (X, E)$ be a graph. A set Δ of tricycles on X is a *(ternary) constraint* on $\mathbf{H}$ if all tricycles xyz of Δ are such that the induced subgraph of $\mathbf{H}$ on $\{x, y, z\}$ is a triangle. A transitive orientation

Q of $\mathbf{H}$ is *consistent* with constraint Δ if we have $x < y < z$, $y < z < x$ or $z < x < y$ in Q for all tricycles xyz in Δ. Now let $\mathbf{T} = (Y, T)$ be a cyclically ordered set, and let $\mathbf{P}$ be the residue of $\mathbf{T}$ at ∞. Proposition 4.1 states that if $\overline{\mathbf{G}}(\mathbf{P})$ has a transitive orientation consistent with the set of tricycles of $\mathbf{T}$ disjoint from $\mathbf{P}$, then T is extendable. For the sake of simplicity in the rest of the paper, we use Δ instead of T_{disj} to denote the set of tricycles of $\mathbf{T}$ disjoint from $\mathbf{P}$. When the incomparability graph of $\mathbf{P}$ admits such an orientation, we say that T can be extended *by glueing at* ∞. Every cyclic order which is extendable by glueing at ∞ is extendable. But there exist extendable cyclic orders that are not extendable by glueing at ∞ (see Proposition 5.1). The following corollaries give sufficient conditions for extendability by glueing.

Corollary 4.2 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let $\mathbf{P}$ be the residue of $\mathbf{T}$ at ∞. If the dimension of $\mathbf{P}$ is at most 2, and if at most one tricycle of $\mathbf{T}$ is disjoint from $\mathbf{P}$, then T can be extended by glueing at ∞.*

Proof Because $\dim(\mathbf{P}) \leq 2$, there exists a transitive orientation Q of $\overline{\mathbf{G}}(\mathbf{P})$. Because at most one tricycle of $\mathbf{T}$ is disjoint from $\mathbf{P}$, either Q or its dual Q^d is consistent with Δ. In conclusion, T is extendable by glueing at ∞. ■

Lemma 4.3 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let $\mathbf{P} = (X, P)$ be the residue of $\mathbf{T}$ at ∞. If there exist distinct elements x, y, z and t of X such that the induced subgraph of $\overline{\mathbf{G}}(\mathbf{P})$ on $\{x, y, z, t\}$ is a complete graph minus edge $\{y, t\}$, then $xyz \in T$ implies $xzt \notin T$, and $zyx \in T$ implies $tzx \notin T$.*

Proof We prove only the first implication: the other follows easily by duality. Suppose that xyz and xzt are in T. So xyt and yzt belong to T because T is transitive. By hypothesis, y and t are comparable in P, so ∞yt or ∞ty belongs to T. If $\infty yt \in T$, then $\infty yt \in T$ and $zty \in T$ imply $\infty yz \in T$ by transitivity, which contradicts the fact that y and z are incomparable in P. If $\infty ty \in T$, then $\infty ty \in T$ and $xyt \in T$ imply ∞xy by transitivity, which contradicts the fact that x and y are incomparable in P. We conclude that $xyz \in T$ implies $xzt \notin T$. ■

Corollary 4.4 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let $\mathbf{P}$ be the residue of $\mathbf{T}$ at ∞. If the dimension of $\mathbf{P}$ is at most two and* *if all triangles of* $\overline{\mathbf{G}}(\mathbf{P})$ *share a common edge, then* T *can be extended by glueing at* ∞*.*

Proof Let k be the number of triangles in $\overline{\mathbf{G}}(\mathbf{P})$. If $k = 0$, then the result is trivial. Because of the hypotheses, there exist vertices a, b, c_1, ..., c_k of $\overline{\mathbf{G}}(\mathbf{P})$ such that the induced subgraph of $\overline{\mathbf{G}}(\mathbf{P})$ on $\{a, b\} \cup \{c_1, \ldots, c_k\}$ is a complete tripartite graph with classes $\{a\}$, $\{b\}$ and $\{c_1, \ldots, c_k\}$.

Let Δ be the set of tricycles in T disjoint from $\mathbf{P}$. Because we assume that $\dim(\mathbf{P}) \leq 2$, graph $\overline{\mathbf{G}}(\mathbf{P})$ has a transitive orientation, say Q. In particular, we have $c_i \parallel c_j$ in Q for $i \neq j$. Suppose that Q is not consistent with Δ. Without loss of generality, we can assume that $abc_1 \in \Delta$ and $c_1 < b < a$ in Q (the other cases are similar). Then $c_i < b < a$ in Q for $i = 1, \ldots, k$ because otherwise we would have $b < c_i$ in Q for some $i \neq 1$, so $c_1 < c_i$ in Q by transitivity, a contradiction. Also, $c_i ba \notin \Delta$ for $i = 1, \ldots, k$ by Lemma 4.3. It follows that $a < b < c_i$ in Q^d for $i = 1, \ldots, k$ and that $\Delta \subseteq \{abc_i : i = 1, \ldots, k\}$. Consequently, Q^d is consistent with Δ. In conclusion, T is extendable by glueing at ∞. ∎

Corollary 4.5 *Let* $\mathbf{T} = (Y, T)$ *be a cyclically ordered set with* $\infty \in Y$*, let* $\mathbf{P}$ *be the residue of* $\mathbf{T}$ *at* ∞*, and let* Δ *be the set of tricycles of* $\mathbf{T}$ *disjoint from* $\mathbf{P}$*. Suppose that* $\overline{\mathbf{G}}(\mathbf{P})$ *is a complete graph. If* Δ *is extendable, then* T *can be extended by glueing at* ∞*.*

Proof Let $X = \{x_1, \ldots, x_n\}$. If Δ is extendable, then there exists a complete cyclic order containing Δ, say $\{x_1 x_2 \cdots x_n\}$. Define L as the linear order on X such that $x_1 < x_2 < \ldots < x_n$ in L. By construction, L is a transitive orientation of $\overline{\mathbf{G}}(\mathbf{P})$ consistent with Δ. Therefore, T is extendable by glueing at ∞. ∎

5 Cyclic orders on at most six points

Thanks to the sufficient conditions for extendability developed in Section 4, we can now prove that all cyclic orders on at most six points are extendable: see Theorem 5.2. Most of the job is done in the following proposition, whose special graphs appear in Figure 4.

Proposition 5.1 *Let* Y *be a set with* $\infty \in Y$ *and* $|Y| \leq 6$*, let* $\mathbf{T} = (Y, T)$ *be a cyclically ordered set with ground set* Y*, let* $\mathbf{P}$ *be the residue of* $\mathbf{T}$

at ∞, and let Δ be the set of tricycles of $\mathbf{T}$ *disjoint from* $\mathbf{P}$. *Then either* T *can be extended by glueing at* ∞, *or* $|Y| = 6$ *and* T *satisfies one of the following conditions up to duality, and up to isomorphism (with* ∞ *fixed): (A)* $\overline{\mathbf{G}}(\mathbf{P}) = \mathbf{H}_3$ *and* $\Delta = \{eab, edc\}$*; (B)* $\overline{\mathbf{G}}(\mathbf{P}) = \mathbf{H}_4$ *and* $\Delta = \{eac, edb\}$*; (C)* $\overline{\mathbf{G}}(\mathbf{P}) = \mathbf{H}_5$ *and* $\Delta = \{adce\}$.

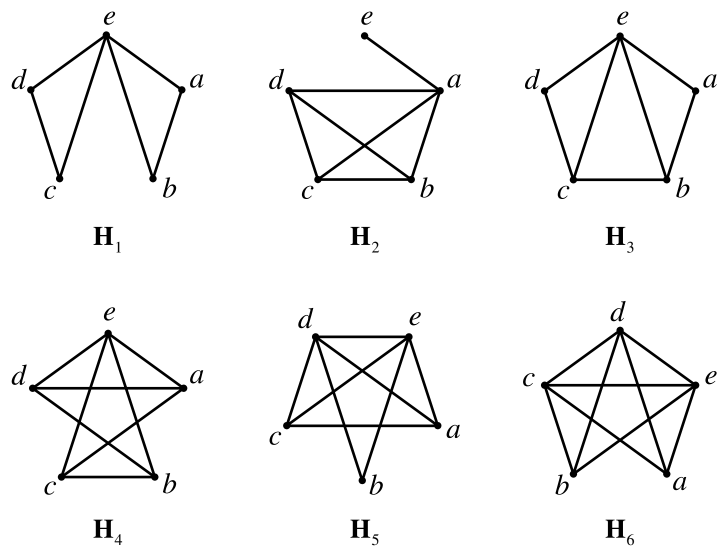


Figure 4: Six graphs with five vertices

Proof Let $X = Y \setminus \{\infty\}$ be the ground set of $\mathbf{P}$, and let $\mathbf{H}$ be the incomparability graph of $\mathbf{P}$. Because $|X| = |Y| - 1 \leq 5$ by hypothesis, the dimension of $\mathbf{P}$ is at most two. If all triangles of $\mathbf{H}$ share a common edge, then T is extendable by glueing at ∞ by Corollary 4.4. Therefore, the proposition trivially holds when $|X| \leq 3$. If $|X| = 4$, then the only case that needs to be considered is the case where $\mathbf{H}$ is a complete graph on four vertices, and it is easily settled using Corollary 4.5.

Now suppose $|X| = 5$. If $\mathbf{H}$ is not connected, then we easily construct a transitive orientation of $\mathbf{H}$ consistent with Δ by treating each component separately, so T is extendable by glueing at ∞. If all triangles of $\mathbf{H}$ share a common edge, then Corollary 4.4 implies that T is extendable by glueing at ∞. So assume that $\mathbf{H}$ is connected and that no edge of $\mathbf{H}$ is incident to all triangles of $\mathbf{H}$. This leaves only seven possible graphs up to isomorphism. All except the complete graph on five vertices are shown in Figure 4.

(I) If $\mathbf{H} = \mathbf{H}_1$, then $\mathbf{H}$ obviously has a transitive orientation consistent with Δ.

(II) If $\mathbf{H} = \mathbf{H}_2$, then a transitive orientation of $\mathbf{H}$ consistent with Δ is constructed as follows. First extend the restriction of T to $\{a, b, c, d\}$ by a complete cyclic order. We assume without loss of generality that it is $\{abcd\}$. Then poset $\mathbf{Q}_1$ (see Figure 5) defines a transitive orientation of $\mathbf{H}$ consistent with Δ.

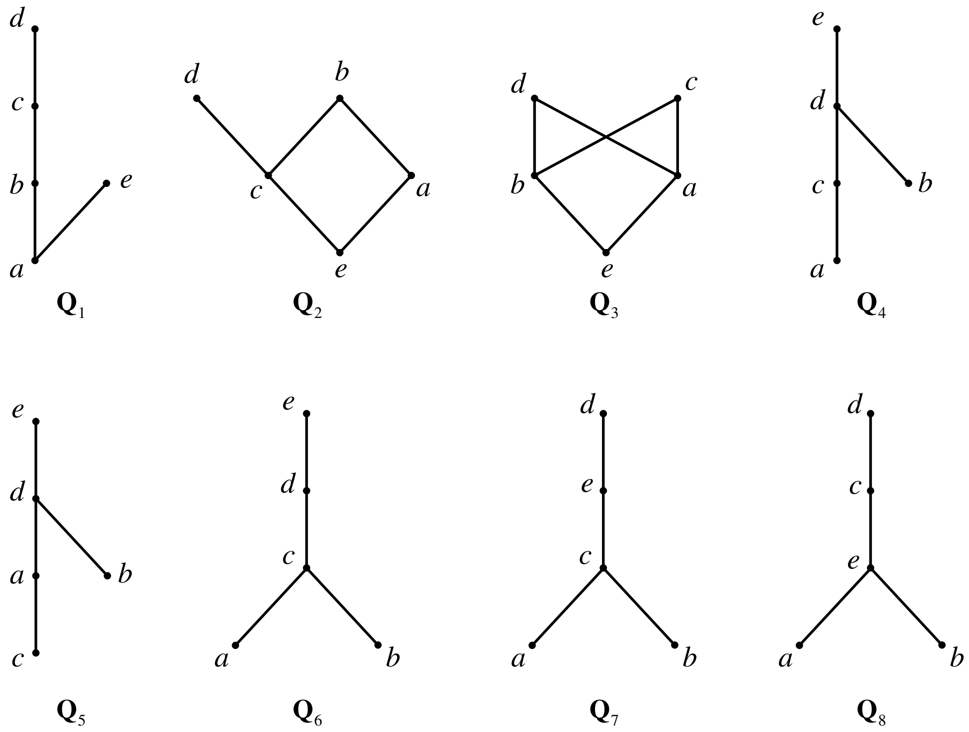


Figure 5: Eight posets with five points

(III) If $\mathbf{H} = \mathbf{H}_3$, suppose that $\mathbf{H}$ has no transitive orientation consistent with Δ. By Corollary 4.2 and because $\mathbf{H}$ has exactly three triangles, we have $2 \leq |\Delta| \leq 3$. Then Δ or its dual Δ^d is necessarily equal to $\{eab, edc\}$, because otherwise $\Delta \subseteq \{eab, ecb, ecd\}$ or $\Delta \subseteq \{eba, ebc, edc\}$, by Lemma 4.3, and poset $\mathbf{Q}_2$ (see Figure 5) or its dual $\mathbf{Q}_2^d$ defines a transitive orientation of $\mathbf{H}$ consistent with Δ, a contradiction.

(IV) If $\mathbf{H} = \mathbf{H}_4$, suppose that $\mathbf{H}$ does not have a transitive orientation consistent with Δ. By Corollary 4.2, Δ contains at least two tricycles. Up to isomorphism, we can assume that ace belongs to Δ. By Lemma 4.3, neither aed nor bec is in Δ. If edb does not belong to Δ, then $\Delta \subseteq \{eac, ead, ebd, ebc\}$ and the partial order of $\mathbf{Q}_3$ (see Figure 5) is a transitive orientation of $\mathbf{H}$ consistent with Δ, a contradiction. Therefore, edb belongs to Δ. Another application of Lemma 4.3 shows that neither ade nor bce is in Δ. In conclusion, we have $\Delta = \{eac, edb\}$, up to isomorphism.

(V) If $\mathbf{H} = \mathbf{H}_5$, suppose that $\mathbf{H}$ has no transitive orientation consistent with Δ. We consider two subcases: either Δ contains bde or bed; or Δ contains neither bde nor bed. In the first subcase, we have $bde \in \Delta$, up to isomorphism. By Lemma 4.3, neither aed nor ced is in Δ. From this, we infer that $\Delta \subseteq \{bde, eac, dac, ade, cde\}$ or $\Delta \subseteq \{bde, eca, dca, ade, cde\}$. Therefore the partial order of $\mathbf{Q}_4$ or $\mathbf{Q}_5$ (see Figure 5) is a transitive orientation of $\mathbf{H}$ consistent with Δ, a contradiction.

In the second subcase, we have $bde, bed \notin \Delta$. Consider the restriction of Δ to $\{a, c, d, e\}$. It is a cyclic order on 4 points, so it is contained in some complete cyclic order. Up to isomorphism and duality, only two extensions need to be considered: $\{acde\}$ and $\{adce\}$. If $\{acde\}$ is an extension of Δ, then poset $\mathbf{Q}_4$ defines a transitive orientation of $\mathbf{H}$ consistent with Δ, a contradiction. We conclude that $\Delta = \{adce\}$, up to isomorphism.

(VI) If $\mathbf{H} = \mathbf{H}_6$, then we can assume without loss of generality that $n_a \leq n_b$, where n_a and n_b respectively denote the number of tricycles of Δ through a and through b. If $n_b = 0$, then $n_a = 0$, and Δ has at most one tricycle, which is cde or edc. Therefore, $\mathbf{H}$ has a transitive orientation consistent with Δ, by Corollary 4.2. From now on, we assume that there is at least one tricycle of Δ through b, say bcd without loss of generality.

Let Δ_a and Δ_b respectively denote the restriction of Δ to $\{a, c, d, e\}$ and to $\{b, c, d, e\}$. Clearly, we have $\Delta = \Delta_a \cup \Delta_b$ by definition, and $|\Delta_a| \leq |\Delta_b|$ by assumption. Now suppose that Δ_b is complete. It is not restrictive to assume that $\Delta_b = \{bcde\}$. Lemma 4.3 implies that adc, aec and aed do not belong to Δ. Therefore, $\Delta_a \subseteq \{acde\}$ and the partial order of $\mathbf{Q}_6$ (see Figure 5) is a transitive orientation of $\mathbf{H}$ consistent with Δ. We assume henceforth that Δ_b is not complete, which together with inequality $|\Delta_a| \leq |\Delta_b|$ implies that Δ_a is not complete.

Because $bcd \in \Delta$ and because Δ_b is not complete by assumption, tricycles bde, cbe and dce do not belong to Δ. Moreover, Lemma 4.3 implies $adc \notin \Delta$. We consider two subcases: either $cde \in \Delta$, or $cde \notin \Delta$. In the first subcase,

$cde \in \Delta$, so bed, bce, ace and aed do not belong to Δ, because otherwise Δ_a or Δ_b would be complete, a contradiction. For the same reason, Δ contains at most one member of $\{aec, acd, ade\}$. In conclusion, $\Delta = \{bcd, cde\} \cup \Delta'$ where Δ' is a subset of $\{aec, acd, ade\}$ of cardinality at most one. Therefore, the partial order $\mathbf{Q}_6$ or $\mathbf{Q}_8$ (see Figure 5) is a transitive orientation of $\mathbf{H}$ consistent with Δ. In the second subcase ($cde \notin \Delta$), we leave it to the reader to check by inspection that $\mathbf{Q}_6$, $\mathbf{Q}_7$ or $\mathbf{Q}_8$ defines a transitive orientation of $\mathbf{H}$ consistent with Δ.

(VII) If $\mathbf{H}$ is a complete graph on five vertices, then T is extendable by glueing at ∞ by Corollary 4.5 because all cyclic orders on five points are extendable, as shown above. ∎

Theorem 5.2 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $|Y| \leq 6$. Then T is extendable.*

Proof Take $\infty \in Y$. If T is extendable by glueing at ∞, then T is extendable. So it suffices to treat cases (A), (B) and (C) of Proposition 5.1 because in all other cases, T is extendable by glueing at ∞. As in Proposition 5.1, let $\mathbf{P} = (X, P)$ be the residue of $\mathbf{T}$ at ∞. Define T_{touch} as the set of tricycles of $\mathbf{T}$ touching $\mathbf{P}$.

(A) If $a < c$ in P, then $T = \{\infty ac, \infty ad, \infty bd, eab, edc\} \cup T_{\text{touch}}$. By definition, $T_{\text{touch}} \subseteq \{acb, ace, ade, bdc, bde\}$. By transitivity, $acb \in T$ implies $ecb \in T$ because $eab \in T$, a contradiction; $ade \in T$ implies $adc \in T$ because $edc \in T$, a contradiction; and $bde \in T$ implies $abd \in T$ because $eab \in T$, a contradiction. In conclusion, $T_{\text{touch}} \subseteq \{ace, bdc\}$, and complete cyclic order $\{\infty abdce\}$ is a complete extension of T. If $c < a$ in P, then by similar arguments we see that complete cyclic order $\{\infty cedab\}$ extends T.

(B) Set $T_1 = \{\infty ab, \infty cd\}$, $T_2 = \{\infty ab, \infty dc\}$, $T_3 = \{\infty ba, \infty cd\}$ and $T_4 = \{\infty ba, \infty dc\}$. Now $T = T_i \cup \{eac, edb\} \cup T_{\text{touch}}$ for some $i \in \{1, 2, 3, 4\}$. By transitivity, $abc \in T$ implies $ebc \in T$ because $eac \in T$, a contradiction; $abd \in T$ implies $eda \in T$ because $edb \in T$, a contradiction; $cde \in T$ implies $ead \in T$ because $eac \in T$, a contradiction. By similar arguments, tricycles bae, dca and dcb do not belong to T. There are four possible cases, as follows. Either $T = T_1 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{abe, cda, cdb\}$, or $T = T_2 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{abe, dce\}$, or $T = T_3 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{bac, bad, cda, cdb\}$, or $T = T_4 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{bac, bad, dce\}$. Then $\{\infty acdbe\}$, $\{\infty eadbc\}$, $\{\infty cedba\}$ and $\{\infty beadc\}$ respectively extend T.

(C) Up to duality, $T = \{\infty bc, \infty ba, adce\} \cup T_{\text{touch}}$ where $T_{\text{touch}} \subseteq \{bad, bce\}$ because $bae \in T$ implies $deb \in T$, a contradiction, and $bcd \in T$ implies $dbe \in T$, another contradiction. Therefore, complete cyclic order $\{\infty badce\}$ extends T. ∎

6 Cyclic orders on seven and eight points

A *closure operator* on a set $\mathfrak{X}$ is a map $\mathfrak{C} : A \mapsto A^{\mathfrak{C}}$ from the power set of $\mathfrak{X}$ to itself that is idempotent ($(A^{\mathfrak{C}})^{\mathfrak{C}} = A^{\mathfrak{C}}$), expanding ($A \subseteq A^{\mathfrak{C}}$) and monotone ($A \subseteq B$ implies $A^{\mathfrak{C}} \subseteq B^{\mathfrak{C}}$). A subset A of $\mathfrak{X}$ is said to be *closed* if $A^{\mathfrak{C}} = A$. In the sequel, $\mathfrak{X}$ is the set of all tricycles on a certain finite ground set Y, and $\mathfrak{C}$ is defined from the transitive closure operator for sets of tricycles, as follows. For each subset T of $\mathfrak{X}$, we set $T^{\mathfrak{C}} = T^+$ if T^+ is asymmetric, and $T^{\mathfrak{C}} = \mathfrak{X}$ otherwise. It should be clear that a proper subset of $\mathfrak{X}$ is closed if and only if it is a cyclic order on Y.

The *next-closure* algorithm of Ganter and Reuter [16] enumerates all closed sets of certain operators which generalize closure operators, in increasing lexicographic order. With $\mathfrak{C}$ as above, the next-closure algorithm applied to $\mathbf{C}$ enumerates all cyclic orders on Y (after $\mathfrak{X}$, which is a closed set but not a cyclic order, is removed from the output). By composing $\mathfrak{C}$ with the operator mapping a set of tricycles to the lexicographic maximum of its duality-isomorphism class, we define a new operator $\mathfrak{C}^*$. The next-closure algorithm applied to $\mathfrak{C}^*$ yields all cyclic orders on Y, up to duality and isomorphism. By adapting an idea of Ganter [15], we designed an improved version of the next-closure algorithm for operator $\mathfrak{C}^*$, which we then implemented.

m	$f(m)$
3	2
4	4
5	16
6	178
7	14 169
8	9 903 256

Table 1: The values of $f(m)$ for $3 \leq m \leq 8$

Let m be the cardinality of Y, and let $f(m)$ denote the number of cyclic orders on Y, up to duality and isomorphism. The values we obtained for $f(m)$ by the improved next-closure algorithm are given in Table 1. We tested each enumerated cyclic order for extendability with a simple backtracking algorithm. For $m \leq 6$, all passed the extendability test (see Theorem 5.2). For $m = 7$, only one enumerated cyclic order was found to be nonextendable. This proves the next theorem. Note that $\mathbf{T}^*$ is self-dual.

Theorem 6.1 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $|Y| = 7$. Then either T is extendable, or $\mathbf{T}$ is isomorphic to cyclically ordered set $\mathbf{T}^*$ defined in Example 1.1.*

For $m = 8$, our program found a list $\mathcal{L}$ of 274 nonextendable cyclic orders on Y. Set $Y = \{a, b, c, d, e, f, g, h\}$ for definiteness. Let $\mathcal{M}$ be the subset of $\mathcal{L}$ consisting of the elements of $\mathcal{L}$ that do not properly contain any nonextendable cyclic order. The 8 elements of $\mathcal{M}$ are given in Table 2. It turns out that only 11 cyclic orders of $\mathcal{L}$ define augmented cones on Y, and that none of them belongs to $\mathcal{M}$. Note that $(Y \setminus \{h\}, T_6)$ is isomorphic to $\mathbf{T}^*$.

Theorem 6.2 *Let $\mathbf{T} = (Y, T)$ be a cyclically ordered set with $|Y| = 8$. Then either T is extendable, or there exist a one-to-one map α from $\{a, \ldots, h\}$ to Y and an integer $i \in \{1, \ldots, 8\}$ such that T contains $\alpha(T_i)$ or its dual $\alpha(T_i)^d$.*

7 Further remarks

This section discusses further relations between nonextendable cyclically ordered sets, order dimension theory and polyhedral combinatorics. Because its contents are facultative for proving the main results of this paper, we will be brief: proofs and most details are omitted. In Subsection 7.1, two aspects of augmented cones relevant to order dimension theory are investigated. In Subsection 7.2, we study relations between nonextendable cyclic orders and two families of polytopes: the linear ordering polytopes and their relaxations, the generalized transitive tournament polytopes.

$T_1 = \{abc, abd, abe, acd, afd, afe, bcd, bcf, beg, bhf, bhg, cde, cfe, deg, dhf, dhg\}$
$T_2 = \{abc, abd, abe, acd, afd, afe, agh, bcd, bcf, beh, bgf, bgh, cde, cfh, cge, deh, dgf, dgh\}$
$T_3 = \{abc, abd, abe, abf, agc, agd, ahe, ahf, bch, bdh, beg, bfg, cde, cfd, cfe, cfg, che, def, deg, dhf\}$
$T_4 = \{abc, abd, abe, abf, acd, aef, agd, ahf, bcd, bch, bef, beg, cde, cdf, cef, cgf, def, deh\}$
$T_5 = \{abc, abd, abe, abf, acd, aef, agd, agf, bcd, bcg, bef, bhg, cdf, cdh, cgf, chf, deg, dhf\}$
$T_6 = \{abc, abd, abe, abf, acd, aef, agd, agf, bcd, bcg, bef, beg, cde, cdf, cef, cgf, def, deg\}$
$T_7 = \{abc, abd, abe, abf, acd, aef, agd, agf, agh, ahd, bcd, bcg, bef, beg, cdf, cgf, dfh, dgh, efh, egh\}$
$T_8 = \{abc, abd, abe, abf, acd, aef, agd, agf, agh, ahd, bcd, bcg, bch, bef, beh, bgh, cde, cdf, cef, cgf, cgh, def, deh, dgh\}$

Table 2: The inclusion-minimal nonextendable cyclic orders on 8 points

7.1 Order dimension theory

In view of Proposition 3.3 (ii), an augmented cone is determined by any of its residues, that is, by the tricycles through any of its points. The residues of a given augmented cone need not be isomorphic. For example, $\mathbf{T}^*$ (defined in Example 1.1) has one residue isomorphic to the standard example $\mathbf{S}_3$, three residues isomorphic to the chevron $\mathbf{D}$, and three residues isomorphic to the dual of the chevron (see Figure 6). This leads to the following definition.

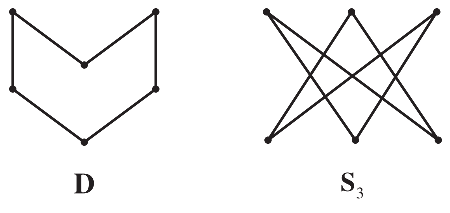


Figure 6: Chevron $\mathbf{D}$ and standard example $\mathbf{S}_3$

Let $\mathbf{P} = (X, P)$ be a poset with $\infty \notin X$, and let r be a point in X. Define $E_r(\mathbf{P})$ as the poset obtained from the residue of $\mathrm{AugCone}(\mathbf{P}, \infty)$ at r by relabeling ∞ as r. By construction, the ground set of $E_r(\mathbf{P})$ is X. We consider E_r as a map from the collection of all posets with ground set X to itself, and refer to E_r as the *exchange mapping*. An immediate consequence of the definition is that $\mathrm{AugCone}(E_r(\mathbf{P}), \infty)$ is equal to $\mathrm{AugCone}(\mathbf{P}, \infty)$ with ∞ relabeled as r, and vice versa. Therefore, $E_r(E_r(\mathbf{P})) = \mathbf{P}$ for all posets $\mathbf{P}$ with ground set X.

From preceding observations and from Proposition 3.4, we infer that $\dim(E_r(\mathbf{P})) = \dim(\mathbf{P})$ whenever $\dim(\mathbf{P}) \leq 2$. However, the exchange mapping does not always preserve the dimension. For example, $E_r(\mathbf{S}_4)$ has dimension three for any r in the ground set, while standard example $\mathbf{S}_4$ has dimension four. We defined the exchange mapping E_r by a detour via augmented cones. It is in fact not hard to describe directly how E_r transforms poset $\mathbf{P}$, but we do not elaborate on this here.

Hereafter, a cyclic order is said to be *critically nonextendable* if it is nonextendable and all of its proper restrictions are extendable. The critically nonextendable cyclic orders that are obtained by the augmented cone construction are intimately related to 3-irreducible posets, which are defined as posets of dimension three such that the removal of any point leaves a poset of dimension at most two. The 3-irreducible posets have been independently classified by Trotter and Moore [34], and Kelly [22]. They established a list $\mathcal{L}$ of 3-irreducible posets with the following properties. If $\mathbf{Q}$ and $\mathbf{Q}'$ are distinct members of $\mathcal{L}$, then neither $\mathbf{Q}$ nor its dual $\mathbf{Q}^d$ is isomorphic to $\mathbf{Q}'$. Moreover, if $\mathbf{P}$ is a 3-irreducible poset, then there exists a poset $\mathbf{Q}$ of $\mathcal{L}$ such that $\mathbf{Q}$ or $\mathbf{Q}^d$ is isomorphic to $\mathbf{P}$.

Let $\mathbf{P} = (X, P)$ be a poset with $\infty \notin X$, let A be a subset of X, and let $\mathbf{Q}$ be the induced poset of $\mathbf{P}$ on A. Then the induced cyclically ordered set of the augmented cone of $\mathbf{P}$ at ∞ to $A \cup \{\infty\}$ is equal to the augmented cone of $\mathbf{Q}$ at ∞. It follows from Proposition 3.3 (iv) that the induced cyclically ordered set of an augmented cone on any of its nonempty subsets is also an augmented cone. It is well known that a poset of dimension at least three contains an induced 3-irreducible poset. Therefore, if the cyclic order of the augmented cone of $\mathbf{P}$ at ∞ is critically nonextendable, then $\mathbf{P}$ is necessarily 3-irreducible. But the converse is not true, for there exist 3-irreducible posets $\mathbf{P}$ such that the removal of point ∞ to the cyclic order of the augmented cone of $\mathbf{P}$ at ∞ leaves a nonextendable cyclic order (take, e.g., $\mathbf{P} = \mathbf{C}$ in Kelly's list [22]). It follows from preceding observations that the resulting cyclic order is always critically nonextendable. It is straightforward, though tedious, to determine which posets in $\mathcal{L}$ give critically nonextendable cyclic orders by the augmented cone construction, and which pairs of such posets have isomorphic or dual-isomorphic augmented cones. A characterization of critically nonextendable cyclic orders of augmented cones is thus obtained.

7.2 Polyhedral combinatorics

Throughout this subsection, $X = \{1, \ldots, n\}$, $n \geq 2$, each coordinate of $\mathbb{R}^{n(n-1)}$ is identified with a distinct member of $\{ij : i, j \in X,\ i \neq j\}$, and x_{ij} denotes the coordinate of $x \in \mathbb{R}^{n(n-1)}$ corresponding to pair ij. A $\{\lambda_1, \ldots, \lambda_k\}$-*point* in $\mathbb{R}^{n(n-1)}$ is a point whose coordinates are in $\{\lambda_1, \ldots, \lambda_k\}$, and a *fractional point* is a point which has some nonintegral coordinate. When R is a reflexive relation on X, the *characteristic vector* of R is a point of $\mathbb{R}^{n(n-1)}$ defined by $\chi(R)_{ij} = 1$ if $ij \in R$ and $\chi(R)_{ij} = 0$ otherwise. The *linear ordering polytope* $\mathbf{P}_n$ is the convex hull of the set $\{\chi(L) : L$ is a linear order on $X\}$. Linear ordering polytopes have numerous applications in combinatorial optimization and probabilistic choice theory, and are widely studied in the literature: see Fishburn [12] for a survey. By definition, all points of $\mathbf{P}_n$ satisfy

$$
\begin{aligned}
x_{ij} + x_{ji} &= 1 && (5)\\
x_{ij} &\leq 1 && (6)\\
x_{ij} + x_{jk} + x_{ki} &\leq 2 && (7)
\end{aligned}
$$

for all distinct elements i, j and k in X. The polytope defined by (5), (6) and (7) is referred to as the *generalized transitive tournament polytope* and is denoted by $\mathbf{C}_n$ hereafter. As Megiddo [24] noted, the integral points of $\mathbf{C}_n$ coincide with the vertices of $\mathbf{P}_n$. The generalized transitive tournament polytopes play an important role in cutting plane algorithms for solving linear programs on the linear ordering polytopes: see, e.g., Reinelt [32], Goemans and Hall [17], and Mitchell and Borchers [25]. Prime open problems for $\mathbf{P}_n$ and $\mathbf{C}_n$ are the determination of the facet-defining inequalities of $\mathbf{P}_n$ and of the vertices of $\mathbf{C}_n$. In the sequel, we focus on the problem of determining the vertices of $\mathbf{C}_n$ and its relation to nonextendable cyclic orders.

By definition, the linear ordering polytope $\mathbf{P}_n$ is included in the generalized transitive tournament polytope $\mathbf{C}_n$. Moreover, all vertices of $\mathbf{P}_n$ are vertices of $\mathbf{C}_n$. By the following theorem, the converse is true if and only if $2 \leq n \leq 5$. That is, $\mathbf{C}_n$ has fractional vertices if and only if $n \geq 6$.

Theorem 7.1 (Dridi [8]) $\mathbf{P}_n = \mathbf{C}_n$ *for* $2 \leq n \leq 5$, *and* $\mathbf{P}_n \neq \mathbf{C}_n$ *for* $n \geq 6$.

A proof of the first part of Theorem 7.1 was independently given in Fishburn [10], with completion in Fishburn [11]. To our knowledge, the first example of a fractional vertex of $\mathbf{C}_6$, a $\{0, \frac{1}{2}, 1\}$-point, was found by Cruse [7] apparently thanks to an idea of Brualdi. Independently, Dridi [8] obtained a general construction that in particular yields this $\{0, \frac{1}{2}, 1\}$-point. He observed that there exists a bijection ϕ between the partial orders on X and the $\{0, \frac{1}{2}, 1\}$-points of $\mathbf{C}_n$. When P is a partial order on X, following Dridi, we define point $\phi(P) \in \mathbb{R}^{n(n-1)}$ as follows

$$\phi(P)_{ij} = \begin{cases} 1 & \text{if } i < j \text{ in } P \\ \frac{1}{2} & \text{if } i \parallel j \text{ in } P \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition characterizes which of the $\{0, \frac{1}{2}, 1\}$-points of the generalized transitive tournament polytope $\mathbf{C}_n$ belong to the linear ordering polytope $\mathbf{P}_n$.

Proposition 7.2 (Dridi [8]) $\phi(P) \in \mathbf{P}_n$ *if and only if* $\dim(X, P) \leq 2$.

Because there exist posets of dimension three with n points for $n \geq 6$, there exists a $\{0, \frac{1}{2}, 1\}$-point in $\mathbf{C}_n$ that does not belong to $\mathbf{P}_n$ for each $n \geq 6$. So Proposition 7.2 implies the second part of Theorem 7.1. The fractional point of $\mathbf{C}_6$ found by Cruse [7] is in fact equal to $\phi(D)$ where D is the partial order of the chevron (see Figure 6).

By a previous remark, each $\{0, \frac{1}{2}, 1\}$-point of $\mathbf{C}_n$ that corresponds to a poset of dimension one (i.e., to a chain) is a vertex of $\mathbf{C}_n$. By Proposition 7.2, the $\{0, \frac{1}{2}, 1\}$-points of $\mathbf{C}_n$ that correspond to posets of dimension two are not vertices of $\mathbf{C}_n$. Note that $\dim(X, P) \geq 3$ is not a sufficient condition for $\phi(P)$ to be a vertex of $\mathbf{C}_n$. By extending results of Brualdi and Hwang [5], Borobia [3] characterized the $\{0, \frac{1}{2}, 1\}$-points of $\mathbf{C}_n$ that are extreme in $\mathbf{C}_n$. Note also that not all vertices of $\mathbf{C}_n$ are necessarily $\{0, \frac{1}{2}, 1\}$-points. Nutov and Penn [29] showed that $\mathbf{C}_n$ has non-$\{0, \frac{1}{2}, 1\}$ vertices for $n \geq 8$. They asked if all vertices of $\mathbf{C}_n$ for $6 \leq n \leq 7$ are $\{0, \frac{1}{2}, 1\}$-points. A partial answer for $n = 6$ was found by Borobia and Chumillas [4]. To our knowledge, the complete answer for $n = 6$ and 7 is due to Nalivaiko [26] (page 31). Using a convex hull algorithm, he proved the following theorem.

Theorem 7.3 *For* $6 \leq n \leq 7$, *all vertices of* $\mathbf{C}_n$ *are* $\{0, \frac{1}{2}, 1\}$*-points.*

We now explain how the augmented cone construction, Proposition 3.4 and Theorems 5.2, 6.1 and 6.2 relate to the aforementioned results. Define the *incidence* of a point $\tilde{x}$ in $\mathbf{C}_n$ as the trigraph $\mathrm{Incid}(\tilde{x}) = (Y, T)$ with

$$\begin{aligned} Y &= X \cup \{\infty\}, \text{ and} \\ T &= \{\infty ij : \tilde{x}_{ij} = 1\} \cup \{ijk : \tilde{x}_{ij} + \tilde{x}_{jk} + \tilde{x}_{ki} = 2\}, \end{aligned}$$

where i, j and k are distinct elements of X. It can be easily shown that the incidence of a point in $\mathbf{C}_n$ is a cyclically ordered set. By definition, the ground set of $\mathrm{Incid}(\tilde{x})$ has $n+1$ points, for each $\tilde{x}$ in $\mathbf{C}_n$. Megiddo [24] implicitly used a similar construction to show that $\mathbf{P}_n = \mathbf{C}_n$ for $2 \leq n \leq 4$ and that $\mathbf{P}_{13} \neq \mathbf{C}_{13}$, except that in his case the cyclically ordered sets have n points in their ground sets. This is due to the fact that he did not use the extra point ∞ which is needed to take equalities $x_{ij} = 1$ into account. Observe that if $\tilde{x}$ is a $\{0, \frac{1}{2}, 1\}$-point in $\mathbf{C}_n$, then $\tilde{x} = \phi(P)$ for some partial order P on X and the incidence of $\tilde{x}$ is equal to the augmented cone of (X, P) at ∞.

Each point $\tilde{x}$ in $\mathbf{C}_n$ determines a face $F(\tilde{x})$ of $\mathbf{C}_n$ which is the intersection of $\mathbf{C}_n$ and of the affine subspace obtained by transforming into equalities all inequalities of the form (6) and (7) that are satisfied up to equality at $\tilde{x}$. In fact, $F(\tilde{x})$ is the inclusion-minimal face of $\mathbf{C}_n$ containing $\tilde{x}$. Conversely, if F is any face of $\mathbf{C}_n$, then there exists a point $\tilde{x}$ in $\mathbf{C}_n$ such that $F = F(\tilde{x})$. Any point $\tilde{x}$ in the relative interior of F will do. Therefore, each face of $\mathbf{C}_n$ determines a unique cyclically ordered set.

Because linear ordering polytope $\mathbf{P}_n$ is included in generalized transitive tournament polytope $\mathbf{C}_n$, each face F of $\mathbf{C}_n$ determines a face of $\mathbf{P}_n$ which is the intersection of F and $\mathbf{P}_n$. It can be verified that $F(\tilde{x})$ determines a nonempty face of $\mathbf{P}_n$ if and only if the cyclic order of $\mathrm{Incid}(\tilde{x})$ is extendable. In particular, if $\tilde{x}$ is a vertex of $\mathbf{C}_n$, then $F(\tilde{x}) = \{\tilde{x}\}$ and the cyclic order of $\mathrm{Incid}(\tilde{x})$ is extendable if and only if $\tilde{x}$ is a vertex of $\mathbf{P}_n$. In other words, each fractional vertex of $\mathbf{C}_n$ yields a nonextendable cyclic order. A similar observation was made by Megiddo [24]. We conclude the present section by discussing some consequences.

Theorem 5.2 proves that every vertex of $\mathbf{C}_n$ is a vertex of $\mathbf{P}_n$ for $2 \leq n \leq 5$. Therefore, Theorem 5.2 implies the first part of Theorem 7.1. Note that Proposition 3.4 and Proposition 7.2 are provably equivalent. Theorem 6.1 implies Theorem 7.3 for $n = 6$. Similarly, Theorem 6.2 can be used to show Theorem 7.3 for $n = 7$. This requires some limited amount of extra work.

We do not think that the first part of Theorem 7.1 trivially implies Theorem 5.2, or that Theorem 7.3 implies Theorems 6.1 and 6.2. A priori, there could exist nonextendable cyclic orders which are not obtained from a face of $\mathbf{C}_n$ disjoint from $\mathbf{P}_n$.

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