Facets of Linear Signed Order Polytopes

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Abstract

Self-reflecting signed orders have been proposed to aid assessment of preferences between subsets of an n-item set $\{1, 2, \ldots, n\}$ by considering desirabilities of excluding as well as including items in a set. A *linear signed order* for n is a linear order $\succ$ on the $2n$-element set $\{1, \ldots, n\} \cup \{1^*, \ldots, n^*\}$, where $(x^*)^* = x$, which satisfies the self-reflection property $x \succ y \Leftrightarrow y^* \succ x^*$. The linear signed order polytope $\mathbf{Q}_n$ for n is defined in a standard way as a polytope in $[0, 1]^{2n(2n-1)}$. It has dimension n^2. We note a complete equation system for $\mathbf{Q}_n$ and specify all facet defining inequalities for $n \leq 4$. Additional classes of facets for larger n that are not induced by a lifting lemma are identified. Comparisons to linear ordering polytopes are included.

Key words: Signed order, Polytope, Facet, Facet defining inequalities

1 Introduction

Self-reflecting signed orders were introduced in [6] to facilitate assessment of preferences between subsets in subset evaluation and choice problems. A study of their geometric properties was initiated in [9]. The present paper extends the latter work by examining linear signed order polytopes and their facets in greater depth. We begin with motivating comments on subset choice problems and then outline the paper's central concepts and results.

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Many decisions require the choice of a subset of items from a set of n candidates. Examples are people to serve on a committee, capabilities to include in a software package, types of aircraft in a fleet, research proposals to fund, journals to subscribe to, and cities for a series of annual meetings. An important input to many subset decision processes is information on preference among subsets from those responsible for the decision. For example, each person might be asked to (a) identify a subset of most desired items, or (b) rank all items from most to least preferred, or (c) rank all feasible subsets in order of preference. An aggregation procedure and decision rule could then be used to select a winning subset or identify a small number of subsets for further consideration.

Although (c) provides detailed information for subset comparisons, it is often impractical when n is large. Both (a) and (b) have been used in practice as data for subset comparisons, but each has significant shortcomings. The data of (a) give no information on preferences within each person's most desired subset, or within its complement. The data of (b) say nothing about whether an item is desirable or undesirable *per se*. And neither accounts for important interdependencies among items considered jointly, such as complementarities, substitutabilities, and diversity.

Signed orders alleviate these shortcomings with a modest informational increase that accounts for relative desirabilities of excluding items from, as well as including items in, a subset. Let $S = \{1, 2, \dots, n\}$ denote the set of items, and let $S^* = \{1^*, 2^*, \dots, n^*\}$ be a disjoint copy of S. For every $x \in S \cup S^*$, we refer to x^* as x's *anti-item*, and take $(x^*)^* = x$ so that $S \cup S^*$ is closed under the $*$ operation. A *signed order* for S is a binary relation $\succsim$ on $S \cup S^*$ that is

transitive: $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$, for all $x, y, z \in S \cup S^*$;

complete: $x \succsim y$ or $y \succsim x$, for all $x, y \in S \cup S^*$;

self-reflecting: $x \succsim y \Leftrightarrow y^* \succsim x^*$, for all $x, y \in S \cup S^*$.

According to self-reflection, one element is preferred or indifferent to a second if and only if the anti-item of the second is preferred or indifferent to the anti-item of the first. Methods for inducing subset preferences from signed orders and other types of partial preference information are described in [2,3,5,6,10,16].

We refer to the non-identity relation $\succ$ of a signed order for which ($x \succsim y$ and $y \succsim x$) $\Rightarrow x = y$ as a *linear signed order*. An example for $n \geq 3$ is

$$1 \succ 2^* \succ 3 \succ \cdots \succ 3^* \succ 2 \succ 1^*,$$

with 1 preferred to 2^*, 2^* preferred to 3, and so forth. When a committee is to be chosen from S, this order represents the preferences of a person whose first priority is to have 1 on the committee, whose second priority is to exclude 2 from the committee, and so forth.

We focus henceforth on linear signed orders and let L_n denote the set of such orders on $\{1,2,\ldots,n\} \cup \{1^*,2^*,\ldots,n^*\}$. Each $\succ$ in L_n is a chain $r_1 \succ r_2 \succ \cdots \succ r_{2n}$ with $\{r_1,r_2,\ldots,r_{2n}\} = \{1,2,\ldots,n\} \cup \{1^*,2^*,\ldots,n^*\}$ which, by self-reflection, satisfies

$$r_j^* = r_{2n+1-j} \quad \text{for} \quad j = 1,\ldots,2n.$$

It follows that $\succ$ is uniquely determined by $r_1 \succ \cdots \succ r_n$ and that $|L_n| = 2^n(n!)$. We sometimes abbreviate the linear signed order $r_1 \succ r_2 \succ \cdots \succ r_n \succ r_n^* \succ \cdots \succ r_2^* \succ r_1^*$ as $r_1 r_2 \cdots r_n | r_n^* \cdots r_2^* r_1^*$. We note also that each $\succ$ in L_n is represented by a utility function $u : S \cup S^* \to \mathbb{R}$ that satisfies $x \succ y \Leftrightarrow u(x) > u(y)$, with $u(x) + u(x^*) = 0$ for all $x \in S \cup S^*$. If $u(i) > 0$, then $i \succ i^*$, which suggests that i's inclusion in a subset is desirable; if $u(i) < 0$, then $i^* \succ i$, which suggests that i's exclusion from a subset is desirable.

To define the linear signed order polytope for L_n, we index the coordinates of $\mathbb{R}^{2n(2n-1)}$ by the $2n(2n-1)$ members of

$$\mathcal{A}_n = \{(x,y) : x,y \in \{1,\ldots,n\} \cup \{1^*,\ldots,n^*\}, x \neq y\}$$

and denote by $p^{\succ}$ the incidence vector of $\succ$ in $\mathbb{R}^{2n(2n-1)}$:

$$p^{\succ}(x,y) = \begin{cases} 1 \text{ if } x \succ y \\ 0 \text{ otherwise} \end{cases} \quad \text{for all} \quad (x,y) \in \mathcal{A}_n.$$

The *linear signed order polytope* $\mathbf{Q}_n$ for L_n is the convex hull of $\{p^{\succ} : \succ \in L_n\}$. Each $p^{\succ}$ is a *vertex* of $\mathbf{Q}_n$.

An equivalent characterization of $\mathbf{Q}_n$ is that it is the set of every $p \in \mathbb{R}^{2n(2n-1)}$ for which there is a probability distribution P on L_n such that

$$p(x,y) = \Sigma\{P(\succ) : \succ \in L_n \text{ and } x \succ y\} \text{ for all } (x,y) \in \mathcal{A}_n.$$

This is of interest in random utility theory [8,15,18], where the $p(x,y)$ are binary choice probabilities.

For comparison to $\mathbf{Q}_n$ we define the *linear ordering polytope* $\mathbf{P}_n$ as the convex hull of the $p^{\succ'}$ for all linear orders $\succ'$ on $\{1,2,\ldots,n\}$, where $p^{\succ'}$ is the incidence vector of $\succ'$ in $\mathbb{R}^{n(n-1)}$ defined by $p^{\succ'}(x,y) = 1$ if $x \succ' y$ and 0 otherwise under an indexing of the coordinates of $\mathbb{R}^{n(n-1)}$ by the $n(n-1)$ pairs (x, y) of distinct elements in $\{1, 2, \ldots, n\}$.

The next section of the paper recalls basic aspects of the $\mathbf{Q}_n$ from [9], including a complete equation system for $\mathbf{Q}_n$, a lifting lemma that takes facet defining inequalities for $\mathbf{Q}_n$ into $\mathbf{Q}_{n+1}$, and the fact that $\mathbf{Q}_n$ has dimension n^2. We note also that $\mathbf{Q}_2$ is a four-dimensional cross-polytope. Section 3 describes complete sets of facet defining inequalities for $\mathbf{Q}_3$ and $\mathbf{Q}_4$ that were generated by the PORTA computer program [4]. A fundamental classification of facet defining inequalities for $\mathbf{Q}_n$ in terms of pales, pickets, and slats for directed graphs that represent the inequalities is included in the section. Section 4 relates $\mathbf{Q}_n$ to $\mathbf{P}_n$. More specifically, we note that $\mathbf{P}_n$ appears as a face of $\mathbf{Q}_n$, and that each face of $\mathbf{P}_n$ can be lifted to a face of $\mathbf{Q}_n$ of the same codimension. Section 5 develops an additional class of facet defining inequalities for $\mathbf{Q}_n$ whose number increases as n increases.

2 Linear Signed Order Polytopes

We begin with facts about linear equations in $\mathbb{R}^{2n(2n-1)}$ that hold for all $p \in \mathbf{Q}_n$, and then consider linear inequalities for $\mathbf{Q}_n$. Throughout, $N = 2n(2n-1)$ and $\langle a, p\rangle = \sum_{\mathcal{A}_n} a(x, y)p(x, y)$, the inner product of $a, p \in \mathbb{R}^N$.

A fundamental theorem of polytopes [12, p. 244] implies that $\mathbf{Q}_n$ equals the set of all $p \in \mathbb{R}^N$ that satisfy a finite system of equations of the form $\langle a, p\rangle = b$ and a finite system of inequalities of the form $\langle a, p\rangle \leq b$ that define half-spaces of $\mathbb{R}^N$. We refer to a system $\mathcal{S}$ of equations that hold for all $p \in \mathbf{Q}_n$ as *complete* if every equation that holds for all $p \in \mathbf{Q}_n$ is implied by the equations in $\mathcal{S}$. The following result is Lemma 4.2 in [9].

Lemma 1 *The equation system*

$$p(x, y) + p(y, x) = 1 \quad \text{for all} \quad (x, y) \in \mathcal{A}_n \tag{1}$$
$$p(x, y) = p(y^*, x^*) \quad \text{for all} \quad (x, y) \in \mathcal{A}_n \tag{2}$$

is complete for $\mathbf{Q}_n$.

The equations of (1) and (2) partition $\mathcal{A}_n$ into n 2-element subsets

$$\{(i, i^*), (i^*, i)\} \quad i = 1, \ldots, n$$

and $2\binom{n}{2}$ 4-element subsets

$$\{(i,j),(j,i),(i^*,j^*),(j^*,i^*)\}\ 1 \le i < j \le n$$
$$\{(i,j^*),(j^*,i),(i^*,j),(j,i^*)\}\ 1 \le i < j \le n,$$

with one degree of affine independence within each subset. The partition has $n + 2\binom{n}{2} = n^2$ members, and this is the dimension of $\mathbf{Q}_n$ [9, Lemma 4.1].

Lemma 2 *The dimension of* $\mathbf{Q}_n$ *is* n^2*.*

Because $\mathbf{Q}_n$ is not full-dimensional, a given halfspace of aff $\mathbf{Q}_n$, the affine hull of $\mathbf{Q}_n$, can be defined by several linear inequalities in $\mathbb{R}^N$ that are not necessarily positive multiples of each other. For example, the inequalities $p(x,y) \le 1$, $-p(y,x) \le 0$, $-p(x^*,y^*) \le 0$ and $p(y^*,x^*) \le 1$ define the same halfspace of aff $\mathbf{Q}_n$. The following definitions are meant to cope with this problem. A linear inequality $\langle a,p\rangle \le b$ (respectively, a linear equation $\langle a,p\rangle = b$) in $\mathbb{R}^N$ is said to be in *internal normal form* if $a(y,x) = -a(x,y)$ and $a(y^*,x^*) = a(x,y)$ for all x, y distinct in $S \cup S^*$. In geometrical terms, $\langle a,p\rangle \le b$ is in internal normal form if and only if a belongs to the unique vector subspace of $\mathbb{R}^N$ that is a translate of aff $\mathbf{Q}_n$. Because they are elementary, we skip the proofs of the two following lemmas, which are not noted in [9].

Lemma 3 *Every halfspace of* aff $\mathbf{Q}_n$ *is defined by a linear inequality in internal normal form that is unique up to multiplication by a positive scalar.*

The internal normal form and the next lemma will prove to be very useful in the proofs of Theorems 12 and 13.

Lemma 4 *Let F be any nonempty subset of* aff $\mathbf{Q}_n$*, let $\langle a^1,p\rangle = b^1,\ldots,\langle a^K,p\rangle = b^K$ be K independent equations satisfied by all points of F, in internal normal form, with $1 \le K \le n^2$. If c is a linear combination of $a^1,\ldots,a^K$ for every equation $\langle c,x\rangle = d$ that is satisfied by all points of F and is in internal normal form, then*

$$\begin{aligned} \operatorname{aff} F &= \{p \in \operatorname{aff} \mathbf{Q}_n : \langle a^k,p\rangle = b^k \textit{ for } k = 1,\ldots,K\} \textit{ and} \\ \dim F &= n^2 - K. \end{aligned}$$

We now consider inequalities that hold for all $p \in \mathbf{Q}_n$ and whose equality parts define facets of $\mathbf{Q}_n$, i.e., faces of the polytope of dimension $n^2 - 1$. The inequalities are *facet defining inequalities*, or FDIs for short. The following lifting lemma [9, Lemma 4.3] notes that FDIs for n extend trivially to higher dimensions.

Lemma 5 *Suppose $\langle a, p\rangle \leq b$ is an FDI for $\mathbf{Q}_n$. Define a' on $\mathcal{A}_{n+1}$ by $a' \equiv a$ on $\mathcal{A}_n$ and $a' \equiv 0$ on $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$. Then $\langle a', p\rangle \leq b$ is an FDI for $\mathbf{Q}_{n+1}$.*

The only independent FDIs for $n = 1$ are $p(1, 1^*) \leq 1$ and $p(1^*, 1) \leq 1$, which in conjunction with $p(1, 1^*) + p(1^*, 1) = 1$ completely characterize the one-dimensional $\mathbf{Q}_1$. We assume henceforth that $n \geq 2$.

The following three inequality types have independent and nontrivial realizations for all $n \geq 2$:

$$
\begin{aligned}
p(x,y) &\leq 1 \quad \text{for all} \quad (x,y) \in \mathcal{A}_n && (3)\\
p(x,y)+p(y,z)+p(z,x) &\leq 2 \quad \text{for all distinct} \quad x,y,z \in S \cup S^* && (4)\\
p(x,y)+p(x^*,x)+p(y,y^*) &\leq 2 \quad \text{for all} \quad (x,y) \in \mathcal{A}_n. && (5)
\end{aligned}
$$

The triangle inequality (4) is familiar from the study of linear ordering polytopes [7,11,12,17]. Inequality (5) has no counterpart for linear ordering polytopes, but appears in somewhat different form as an FDI for the four-dimensional cross-polytope when $n = 2$ [12, p. 245]. Note that (1)–(4) already formulate the vertex set of $\mathbf{Q}_n$ in the sense that the set of integral solutions to (1)–(4) coincides with the vertex set of $\mathbf{Q}_n$. The following results for (3)–(5) are Theorems 4.4 and 4.5 in [9]. Lemma 5 is used in the proof of the first result after basic facts are verified for $n = 2$ and, with respect to (4), for $n = 3$.

Lemma 6 *Each of* (3), (4) *and* (5) *defines a family of* FDIs *for $\mathbf{Q}_n$, $n \geq 2$.*

Theorem 7 *For all $p \in \mathbb{R}^{12}$, $p \in \mathbf{Q}_2$ if and only if p satisfies equation system* (1) *and* (2) *along with* (3), (4) *and* (5).

Because $\mathbf{Q}_2$ is a centrally symmetric four-dimensional polytope with eight vertices, it is a four-dimensional cross-polytope. That is, $\mathbf{Q}_2$ is affinely equivalent to the *standard four-dimensional cross-polytope* C_4^Δ [12, p. 245] in $\mathbb{R}^4$, defined as follows. Let e^i denote the i^{th} unit vector in $\mathbb{R}^4$. Then

$$
\begin{aligned}
C_4^\Delta &= \text{convex hull}\,\{\pm e^1, \pm e^2, \pm e^3, \pm e^4\}\\
&= \left\{ x \in \mathbb{R}^4 : \sum_{i=1}^{4} |x_i| \leq 1 \right\},
\end{aligned}
$$

so C_4^Δ is a centrally symmetric four-dimensional polytope with eight vertices.

3 Facets of $\mathbf{Q}_3$ and $\mathbf{Q}_4$

Two more types of FDIs were verified in [9] for $n \geq 3$. They are, for all $x, y, z \in S \cup S^*$ for which $x \in \{i, i^*\}$, $y \in \{j, j^*\}$ and $z \in \{k, k^*\}$ with $|\{i, j, k\}| = 3$:

$$p(x,y) + p(y,z) + p(z,x) + p(x,y^*) + p(y^*,z) + p(z,z^*) \leq 4, \tag{6}$$

$$p(x,z) + p(y,z) + p(x,y^*) + p(x^*,x) + p(y^*,y) + p(z,z^*) \leq 4. \tag{7}$$

It is easily checked that at most four of the p terms in (6), or in (7), can equal 1 for a vertex of $\mathbf{Q}_n$, $n \geq 3$.

Lemma 8 *Inequalities* (6) *and* (7) *with x, y, z as noted are FDIs for $\mathbf{Q}_n$, $n \geq 3$.*

It was conjectured in [9] that Lemmas 6 and 8 identify all FDIs for $\mathbf{Q}_3$. We have now verified this with the aid of the PORTA program.

Theorem 9 *For all $p \in \mathbb{R}^{30}$, $p \in \mathbf{Q}_3$ if and only if p satisfies equation system* (1) *and* (2) *along with* (3)–(5), (6) *and* (7).

The PORTA program was also used to generate all FDIs for $\mathbf{Q}_4$. We summarize the result in the following theorem and then explain the classes of FDIs depicted by the bipartite weighted directed graphs of Figure 1.

Theorem 10 *For all $p \in \mathbb{R}^{56}$, $p \in \mathbf{Q}_4$ if and only if p satisfies equation system* (1) *and* (2) *along with the* 43 *classes of FDIs described by Figure 1.*

Each graph of Figure 1 generates a *representative FDI* $\langle a, p \rangle \leq b$ as follows:

(i) label the top-row points by $1, 2, \ldots$ left to right, then label the bottom-row point directly beneath i (if any) by i^*;
(ii) for all labeled points x and y, set $a(x,y) = 1$ if there is a thin-line arrow from x to y, $a(x,y) = 2$ if there is a thick-line double arrow from x to y, $a(x,y) = 0$ otherwise;
(iii) set b equal to the circled integer immediately to the right of the drawing.

All other FDIs in the representative's class are obtained from the representative $\langle a, p \rangle \leq b$ by permutations on $\{1, 2, 3, 4\}$ and interchanges of i and i^* for any subset of the i in $\{1, 2, 3, 4\}$.

We refer to a line in an FDI's graph as a *slat* if it is between points in the same row, a *picket* if it is between i and j^* with $i \neq j$, and a *pale* if it is between i and i^*. The number of pales for the FDIs in a class is fixed, but the

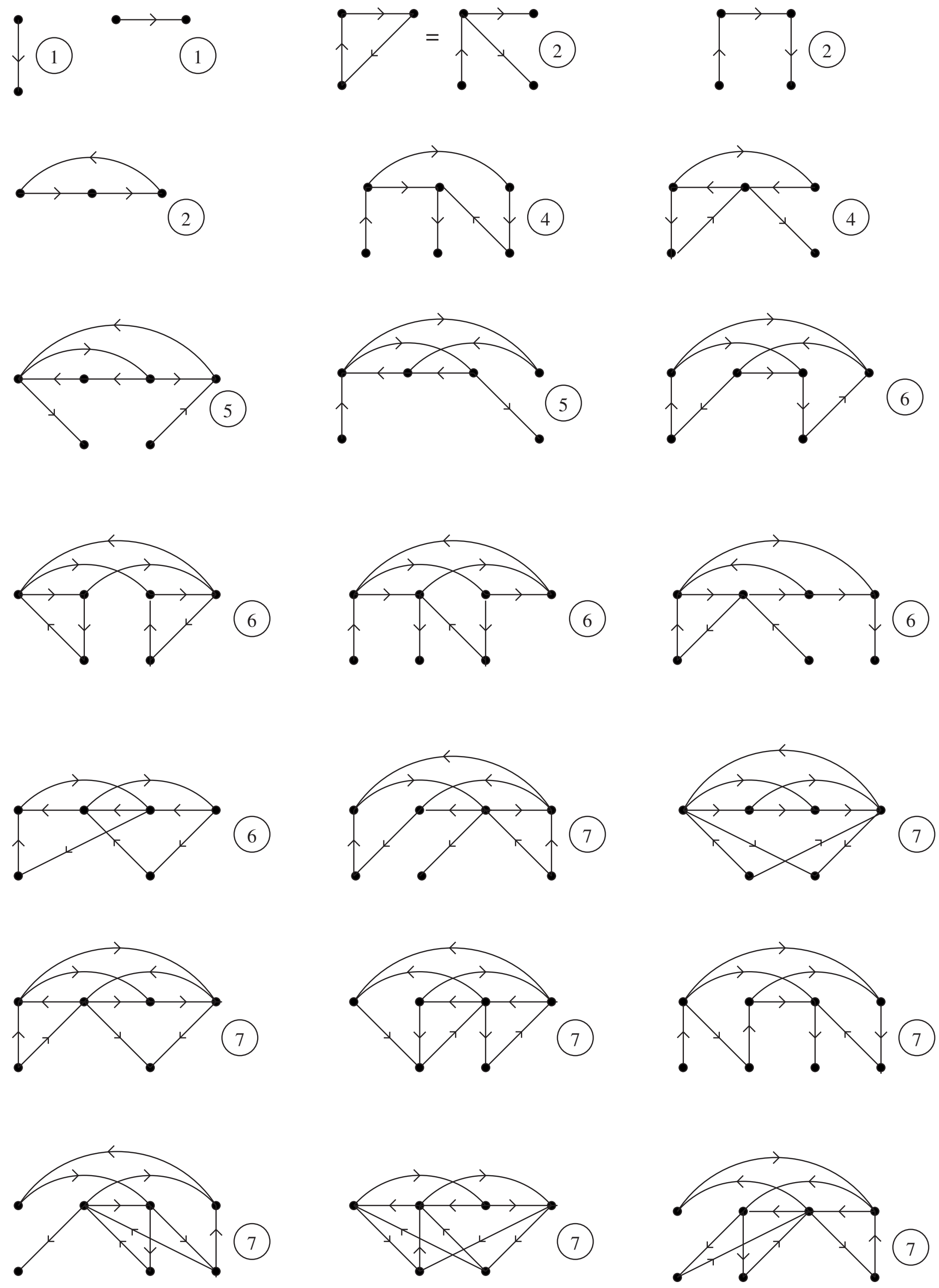

Fig. 1. (part 1)

number of slats and pickets can vary (by interchanges of i and i^*) while their total remains fixed.

The equation system of Lemma 1 implies that an FDI $\langle a, p\rangle \leq b$ can always be written with $a \geq 0$, and we have done this on Figure 1 with the $a(x, y)$ and b in smallest-integer format. Moreover, because $p(i^*, j^*) = p(j, i)$, $p(i, j) + p(j^*, i^*) = 2p(i, j)$, and $p(i, j) + p(i^*, j^*) = p(i, j) + p(j, i) = 1$, every graph of a representative FDI can have *all* slats in its top row. This is done on Figure 1, where we have then maximized the weighted sum of the in-degrees and

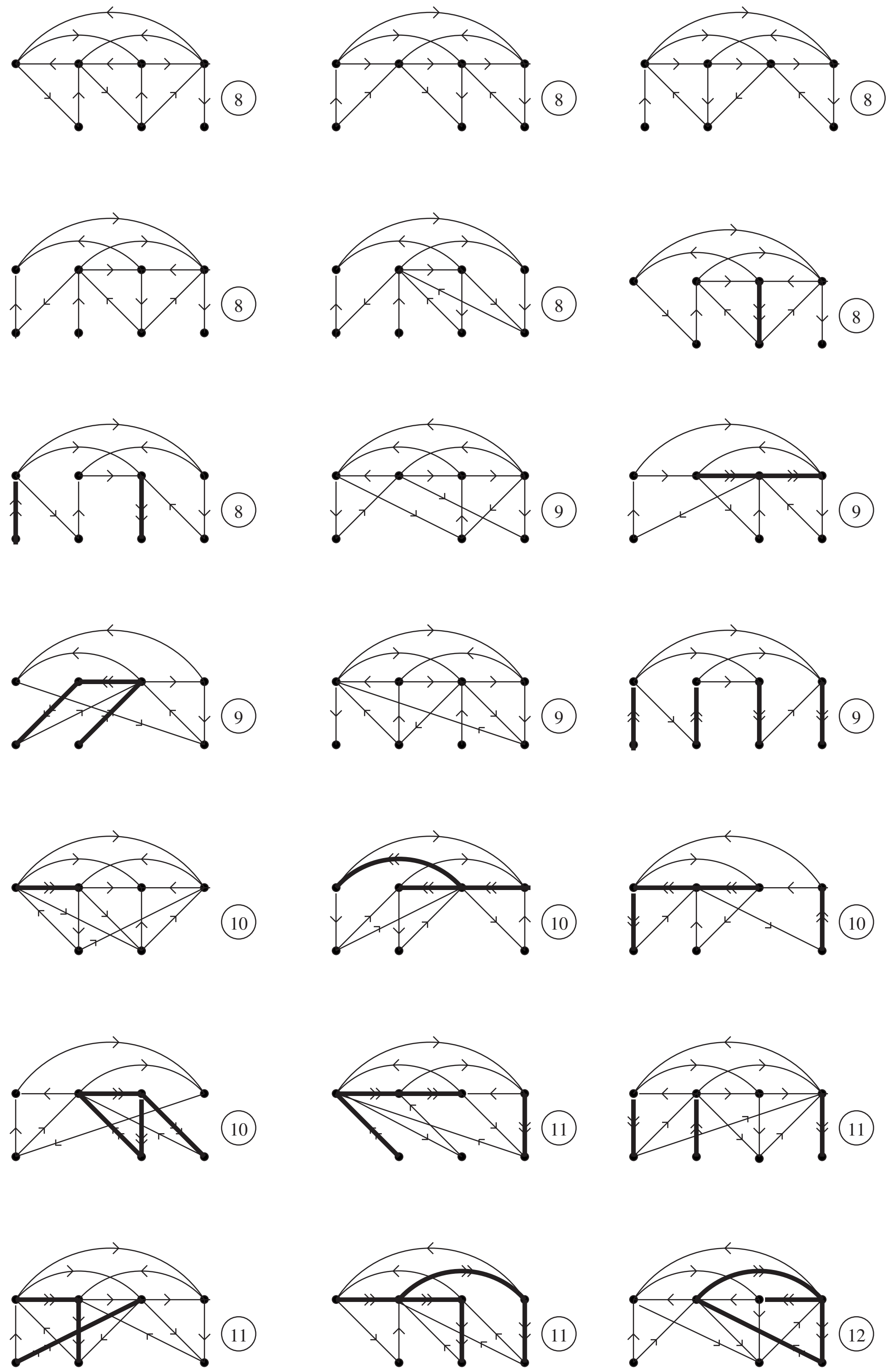

Fig. 1. (part 2)

out-degrees of the points in the top row.

Among the 43 classes of FDIs for $\mathbf{Q}_4$, we note that four have no pales, one has four weight-2 pales, and all b between 1 and 12 are present except for $b = 3$. Moreover, all FDIs for $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ appear on Figure 1. For example, the

	number of vertices	dimension	number of FDI classes	total number of facets
$\mathbf{Q}_1$	2	1	1	2
$\mathbf{Q}_2$	8	4	4	16
$\mathbf{Q}_3$	48	9	7	82
$\mathbf{Q}_4$	384	16	43	8480

Table 1
Parameters for $\mathbf{Q}_n$ $(n = 1, 2, 3, 4)$

first row applies to $\mathbf{Q}_2$ and the first two rows provide representative FDIs for $\mathbf{Q}_3$. Table 1 summarizes salient parameters for $\mathbf{Q}_1$ through $\mathbf{Q}_4$.

4 $\mathbf{Q}_n$ versus $\mathbf{P}_n$

As is easily seen, $\mathbf{P}_n$ appears as a face of $\mathbf{Q}_n$. For instance, $\mathbf{P}_n$ is affinely equivalent to the intersection of the n facets of $\mathbf{Q}_n$ defined by the inequalities $p(i, i^*) \leq 1$ for $i = 1, \dots, n$. Moreover, $\mathbf{P}_n$ is the image of $\mathbf{Q}_n$ by an affine mapping as we now explain. Let π be the affine mapping from $\mathbb{R}^{2n(2n-1)}$ to $\mathbb{R}^{n(n-1)}$ mapping each point p to the point $q = \pi(p)$ such that $q(i, j) = p(i, j)$ for all i, j distinct in S. Because $\pi(\mathbf{Q}_n) = \mathbf{P}_n$, the affine mapping π is a projection of polytopes, of $\mathbf{Q}_n$ onto $\mathbf{P}_n$. Let $\dot{\pi}$ denote the restriction of π to $\mathbf{P}_n$.

It is well-known that any projection of polytopes determines an inclusion-preserving map from the face lattice of the image to the face lattice of the domain [19, p. 196]. In our case, with π as above, each nonempty face F of $\mathbf{P}_n$ determines the face $\dot{\pi}^{-1}(F) = \{p \in \mathbf{Q}_n : \pi(p) \in F\}$ of $\mathbf{Q}_n$. The latter face is called the *preimage* of F. If G is defined by the inequality $\langle c, q \rangle \leq d$, then its preimage is defined by the inequality $\langle c, \pi(p) \rangle \leq d$, which is said to be *lifted* from inequality $\langle c, q \rangle \leq d$. A natural question is to determine whether every FDI for $\mathbf{P}_n$ is lifted to an FDI for $\mathbf{Q}_n$. It turns out that this is the case: see Theorem 12. In fact, Theorem 12 states even more, namely that the codimension of the preimage of F is equal to the codimension of F. Theorem 12 relies entirely on the following lemma.

Lemma 11 *For all v in the vertex set of $\mathbf{P}_n$, we have*

$$|\operatorname{vert} \dot{\pi}^{-1}(v)| = 2^n \quad \textit{and} \quad \dim \dot{\pi}^{-1}(v) = n^2 - \binom{n}{2}.$$

PROOF. Let F denote the preimage of v in $\mathbf{Q}_n$, that is, let $F = \dot{\pi}^{-1}(v)$. It is not restrictive to assume that v is the characteristic vector of the linear order $1 \succ 2 \succ \cdots \succ n$. Thus, the vertices of F exactly correspond to the signed linear orders whose restriction to S is $1 \succ 2 \succ \cdots \succ n$ and, consequently, whose restriction to S^* is $n^* \succ \cdots \succ 2^* \succ 1^*$. It follows that the number of vertices of F is 2^n. This is easily seen by considering the bijection from vert F to $\{-1, +1\}^n$ mapping $w = p^{\succ}$ to the vector $s = s(w)$ with $s_i = -1$ if the element of rank i in $\succ$ belongs to S^* and $s_i = +1$ otherwise, for $i = 1, \ldots, n$.

The vertices of F satisfy $\binom{n}{2}$ independent linear equations in internal normal form, namely the equations

$$p(i,j) - p(i^*, j^*) - p(j,i) + p(j^*, i^*) = 2 \tag{8}$$

for all i, j in S with $i > j$. Let $\langle c, p\rangle = d$ be an equation in internal normal form that is satisfied by all points of F. First consider the signed linear orders $\succ_1$ and $\succ_2$ respectively of the forms

$$n^*(n-1)^* \cdots (i+1)^* 12 \cdots (i-1)\underline{i|i^*}(i-1)^* \cdots 2^* 1^* (i+1) \cdots (n-1)n$$

and

$$n^*(n-1)^* \cdots (i+1)^* 12 \cdots (i-1)\underline{i^*|i}(i-1)^* \cdots 2^* 1^* (i+1) \cdots (n-1)n,$$

with $i \in S$. By construction, $p^{\succ 1}$ and $p^{\succ 2}$ belong to F. From $\langle c, p^{\succ 1}\rangle = \langle c, p^{\succ 2}\rangle$, we get $c(i, i^*) = c(i^*, i)$, hence $c(i, i^*) = c(i^*, i) = 0$ because equation $\langle c, p\rangle = d$ is in internal normal form. Now consider the signed linear orders $\succ_3$ and $\succ_4$ respectively of the forms

$$\begin{aligned} &n^*(n-1)^* \cdots (j+1)^* 12 \cdots (i-1)\underline{ij^*}(j-1)^* \cdots (i+1)^*| \\ &\qquad (i+1) \cdots (j-1)\underline{ji^*}(i-1)^* \cdots 2^* 1^* (j+1) \cdots (n-1)n, \end{aligned}$$

and

$$\begin{aligned} &n^*(n-1)^* \cdots (j+1)^* 12 \cdots (i-1)\underline{j^*i}(j-1)^* \cdots (i+1)^*| \\ &\qquad (i+1) \cdots (j-1)\underline{i^*j}(i-1)^* \cdots 2^* 1^* (j+1) \cdots (n-1)n, \end{aligned}$$

with i, j in S and $i < j$. Clearly, $p^{\succ 3}$ and $p^{\succ 4}$ are vertices of F. It follows from $\langle c, p^{\succ 3}\rangle = \langle c, p^{\succ 4}\rangle$ that $c(i, j^*) + c(j, i^*) = c(j^*, i) + c(i^*, j)$. Since equation $\langle c, p\rangle = d$ is in internal normal form, we infer that $c(i, j^*) = c(j, i^*) = c(j^*, i) = c(i^*, j) = 0$. In conclusion, the left-hand side of equation $\langle c, p\rangle = d$ is a linear combination of the left-hand sides of equations (4.1). By Lemma 4, we have $\dim F = n^2 - \binom{n}{2}$. $\square$

The next theorem follows immediately from Lemma 11.

Theorem 12 *If F is any nonempty face of $\mathbf{P}_n$, then*

$$\dim \dot{\pi}^{-1}(F) = \dim F + n^2 - \binom{n}{2}.$$

The preceding theorem is particularly important because it allows one to trivially infer FDI's for $\mathbf{Q}_n$ from FDI's for $\mathbf{P}_n$. For instance, the *k-fence inequalities* of Grötschel, Jünger and Reinelt [11], the *Möbius ladder inequalities* of Reinelt [17], the *t-reinforced k-fence inequalities* of Leung and Lee [14], the *α-critical fence inequalities* of Koppen [13], and all inequalities deduced from these by symmetries of the linear ordering polytopes (see Bolotashvili, Kovalev and Girlich [1]) also define facets of the linear signed order polytopes.

5 New facets

We define the *span* of a linear inequality $\langle a, p\rangle \leq b$ in $\mathbb{R}^N$ as

$$|\{i \in S : a(x,y) \neq 0 \quad \text{for some} \quad (x,y) \quad \text{with} \quad \{x,y\} \cap \{i,i^*\} \neq \phi\}|,$$

and the *span* of a facet of $\mathbf{Q}_n$ as the span of any inequality in internal normal form defining the facet. Similarly, the *span* of a maximal class of facets of $\mathbf{Q}_n$ whose members are equivalent under permutations of S and interchanges of i and i^* is the span of each member of the class. Figure 1 shows that $\mathbf{Q}_2$ has three classes of span 2, $\mathbf{Q}_3$ has three classes of span 3, and $\mathbf{Q}_4$ has 36 classes of span 4. We expect the number of classes of span n to increase rapidly as n increases, but have no proof of this. We will, however, identify a family of facet classes in which the number of classes of span n increases linearly in n and the number of facets of span n increases exponentially in n.

The family we have in mind is based on graphs of FDIs that have no slats. Although only the first graph of Figure 1 has no slats, seven others have slatless class members. Figure 2 shows four of these. Each slatless graph on Figure 2 has equally-weighted downward-directed pales and one weight-one upward-directed picket for each involved $\{i,j\}$ with $i \neq j$.

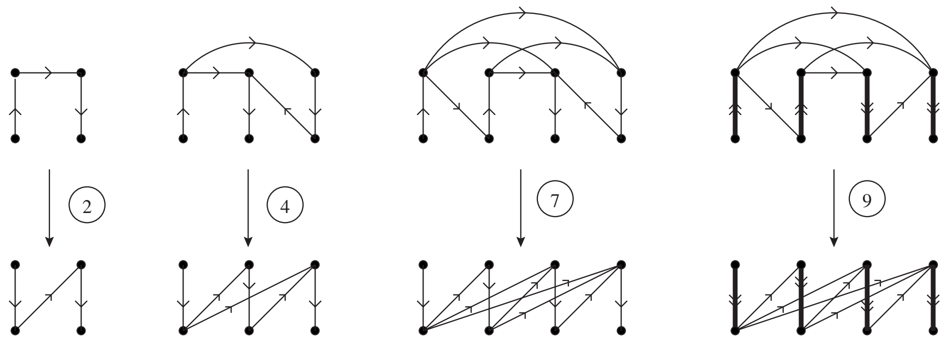

Fig. 2. Slateless FDIs

For positive integers k and μ, we refer to the inequality

$$\mu \sum_{i=1}^{k} p(i, i^*) + \sum_{1 \le i < j \le k} p(i^*, j) \le \binom{k-\mu}{2} + k\mu \tag{9}$$

as the *μ-reinforced signed k-fence.* The lower row of Figure 2 shows the graphs of these fences for (k, μ) equal to $(2, 1)$, $(3, 1)$, $(4, 1)$ and $(4, 2)$, respectively. We will prove the following theorem after we comment on its implications for numbers of FDIs.

Theorem 13 *The μ-reinforced signed k-fence is an* FDI *for* $\mathbf{Q}_n$ *if* $1 = \mu \le k \le n$ *or if* $\mu \ge 2$ *and* $\mu + 2 \le k \le n$.

It follows that for each $n \ge 3$ there are $n - 2$ μ-reinforced signed n-fences with span n that are FDIs for $\mathbf{Q}_n$. Each of these gives rise to 2^n different FDIs for $\mathbf{Q}_n$ according to the possible combinations of interchanges of i and i^*, so the theorem generates $(n-2)2^n$ span-n facets for $\mathbf{Q}_n$.

PROOF (of Theorem 13) We abbreviate "μ-reinforced signed k-fence" as "(k, μ) signed fence" in this proof. We know that $(1, 1)$ and $(2, 1)$ signed fences are FDIs for $n \ge k = 1, 2$, so we assume henceforth that $k \ge \mu + 2$ and $\mu \ge 1$. We will prove that the (k, μ) signed fence is an FDI for $\mathbf{Q}_k$. The lifting lemma, Lemma 5, then completes the proof for $n > k$.

Our proof that the (k, μ) signed fence is an FDI for $\mathbf{Q}_k$ has two parts. The first part shows that (9) is a valid inequality for $\mathbf{Q}_k$ and that equality in (9) defines a nonempty face of $\mathbf{Q}_k$. The second part shows that the face is a facet.

We begin the first part by multiplying (9) by 2 and using (2) to obtain the more symmetric inequality

$$2\mu \sum_{i=1}^{k} p(i, i^*) + \sum_{\substack{i,j=1 \\ i \ne j}}^{k} p(i^*, j) \le k(k-1) + \mu(\mu+1). \tag{10}$$

Let $\{A, B\}$ be a partition of $\{1, \ldots, k\}$ with $|A| = m$ and $|B| = k - m$. Let $\succ$ in L_k have $i \succ i^*$ for $i \in A$ and $i^* \succ i$ for $i \in B$. With $A^* = \{i^* : i \in A\}$ and $B^* = \{i^* : i \in B\}$, we denote all such $\succ$ by $\{A, B^*\}|\{A^*, B\}$. Given m, the number of $p(i^*, j) = 1$ for $i \neq j$ is maximized when $\succ$ is arranged as $B^*A|A^*B$. Given this arrangement, B^*A and B^*B have $(k-m)(k-1)$ instances of $p(i^*, j) = 1$, and A^*B has $m(k-m)$ such instances, for a total of $(k-m)(m+k-1)$. When these are added to 2μ times the number of i for which $p(i, i^*) = 1$, the left-hand side of (10) is

$$2\mu m + (k-m)(m+k-1) = k(k-1) + (2\mu+1)m - m^2.$$

Differentiation shows that this quantity is maximized if and only if $m \in \{\mu, \mu+1\}$, where its maximum value is $k(k-1) + \mu(\mu+1)$. We conclude that (10) holds for all $p \in \mathbf{Q}_k$, and that equality in (10), intersected with $\mathbf{Q}_k$, identifies a nonempty face of $\mathbf{Q}_k$ because equality holds for some vertices of $\mathbf{Q}_k$.

To prove that the face is a facet, we use Lemma 4. Let $\langle a, p\rangle \leq b$ be any inequality in internal normal from defining the same halfspace of aff $\mathbf{Q}_n$ as (9), say, for definiteness,

$$2\mu \sum_{i=1}^{k} [p(i, i^*) - p(i^*, i)] + \sum_{i,j=1, i\neq j}^{k} [p(i^*, j) - p(j, i^*)]$$
$$\leq k(k-1) + 2\mu(\mu - k + 1).$$

Let $\langle c, p\rangle = d$ be any inequality in internal normal form such that $\langle a, p^{\succ}\rangle = b$ implies $\langle c, p^{\succ}\rangle = d$ for all $\succ \in L_n$. According to the preceding paragraph,

$$\langle c, p^{\succ}\rangle = d \text{ whenever } \{A, B\} \text{ partitions } \{1, \ldots, k\},\ |A| \in \{\mu, \mu+1\},$$
$$\text{and } \succ \text{ is arranged as } B^*A|A^*B.$$

We now use this to prove that c is a multiple of a. The proof proceeds in three steps.

STEP 1. Take $r \in \{\mu, \mu+1\}$ with $r \geq 2$. Let $\succ_1$ and $\succ_2$ be arranged as follows with $A = \{1, 2, \ldots, r\}$:

$\succ_1$: $B^* r \cdots 21 | 1^* 2^* \cdots r^* B$

$\succ_2$: $B^* r \cdots 12 | 2^* 1^* \cdots r^* B$.

Then $\langle c, p^{\succ_j}\rangle = d$ for $j = 1, 2$, so $\langle c, p^{\succ_1}\rangle = \langle c, p^{\succ_2}\rangle$ and therefore

$$c(2,1) + c(1^*, 2^*) = c(1,2) + c(2^*, 1^*).$$

Because $c(1,2) = c(2^*,1^*) = -c(1^*,2^*) = -c(2,1)$, the preceding equation reduces to $4c(1,2) = 0$. Hence $c(1,2) = 0$. Similarly, $c(i,j) = 0$ and $c(i^*,j^*) = 0$ whenever $i \neq j$.

STEP 2. Let $A = \{1, 2, \ldots, \mu + 1\}$, and let $\succ_3$ and $\succ_4$ be arranged as follows:

$$\succ_3\colon B^* 12 \cdots \mu + 1 | (\mu+1)^* \cdots 2^* 1^* B$$

$$\succ_4\colon B^* 1^* 2 \cdots \mu + 1 | (\mu+1)^* \cdots 2^* 1 B.$$

Then $\langle c, p^{\succ_3} \rangle = \langle c, p^{\succ_4} \rangle$ which, in view of the conclusion of Step 1, gives

$$\sum_{i=2}^{\mu+1} c(1,i^*) + c(1,1^*) + \sum_{i=2}^{\mu+1} c(i,1^*)$$
$$= \sum_{i=2}^{\mu+1} c(1^*,i) + c(1^*,1) + \sum_{i=2}^{\mu+1} c(i^*,1).$$

Because $\langle c, p \rangle = d$ is in internal normal form, this reduces to

$$c(1,1^*) = 2 \sum_{i=2}^{\mu+1} c(i^*,1).$$

Similarly, when $|C| = \mu$, $C \subseteq \{1, \ldots, k\}$ and $i \notin C$,

$$c(i,i^*) = 2 \sum_{j \in C} c(j^*,i).$$

STEP 3. Because $k \geq \mu + 2$, we can take $C = \{2, \ldots, \mu\} \cup \{\mu + 2\}$ to get

$$c(1,1^*) = 2 \left[\sum_{i=2}^{\mu} c(i^*,1) + c((\mu+2)^*,1) \right],$$

and it follows that $c((\mu+1)^*,1) = c((\mu+2)^*,1)$. Similarly, whenever i, j and h are three members of $\{1, \ldots, k\}$, $c(j^*,i) = c(h^*,i)$, and we obtain

$$c(j^*,i) = c(h^*,i) = c(i^*,j) = c(i^*,h).$$

It follows that there is a number λ such that

$$c(i^*,j) = \lambda \quad \text{for all distinct } i \text{ and } j \text{ in } \{1, \ldots, k\}.$$

By $c(x,y) = -c(y,x)$, this gives

$$c(i, j^*) = -\lambda \quad \text{for all distinct } i \text{ and } j \text{ in } \{1, \ldots, k\}.$$

The final equality of the preceding step then gives

$$c(i, i^*) = 2\mu\lambda, \quad c(i^*, i) = -2\mu\lambda \quad \text{for all} \quad i \in \{1, \ldots, k\}.$$

In conclusion, c is a multiple of a. $\square$

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