A short proof of a theorem of Falmagne

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Abstract

Falmagne (1978) proved the sufficiency of the Block-Marschak inequalities (Block and Marschak, 1960) and normalization equalities for a complete system of choice probabilities to be induced by rankings. Here, we give a considerably shorter proof of this result. Our approach combines Möbius inversion and network flows.

Key words: Choice probabilities. Block-Marschak inequalities. Möbius inversion. Network flows.

To my wife Agathe.

1 Introduction

Let S be a finite set of *alternatives*, and let n denote the cardinality of S. We assume that $n \geq 2$. In order to explain the choices made by an individual in a choice experiment, the *random utility model* assumes the existence of n random variables $\mathbf{U}_s$, one for each alternative $s \in S$, on the same probability space with probability measure Pr. The random variable $\mathbf{U}_s$ models the internal value that the individual attaches to alternative s. In the random utility model, the probability $\mathfrak{p}(s, T)$ that alternative s is chosen in a certain subset T of the alternatives containing s is given by

$$\mathfrak{p}(s, T) = \Pr(\mathbf{U}_s = \max\{\mathbf{U}_t : t \in T\}). \tag{1}$$

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For convenience, let $\mathfrak{I}$ denote the set of all pairs (s, T) with $s \in T \subseteq S$. The quantities $\mathfrak{p}(s, T)$ for $(s, T) \in \mathfrak{I}$ form a *complete system of choice probabilities* $\mathfrak{p}$, which we regard as a vector in $\mathbb{R}^{\mathfrak{I}}$. The question we address here is the characterization of the complete systems of choice probabilities arising from the random utility model. The aim is to find testable necessary and sufficient conditions on experimental choice data, i.e., the complete system of choice probabilities, ensuring that the data satisfies the random utility model. This question is central in probabilistic preference theory (Koppen, 2001).

A *ranking* of (the elements of) S is a bijection ρ from $\{1, 2, \ldots, n\}$ to S. Identifying ρ with its vector of images, the convention we adopt here is that $\rho = (\rho(1), \ldots, \rho(n))$ ranks the elements of S by decreasing order of preference. We write $s\,\rho\,t$ for $s, t \in S$ whenever $\rho^{-1}(s) \leq \rho^{-1}(t)$. Thus $s\,\rho\,t$ means that s is equal or preferred to t in ρ. The collection $\{\mathbf{U}_s\}_{s \in S}$ is said to be *noncoincident* if $\Pr(\mathbf{U}_s = \mathbf{U}_t) = 0$ for $s \neq t$. For each ranking ρ of S, we let $\lambda(\rho) = \Pr(\mathbf{U}_{\rho(1)} > \mathbf{U}_{\rho(2)} > \ldots > \mathbf{U}_{\rho(n)})$. Then, under the noncoincidence assumption, Equation (1) is equivalent to

$$\mathfrak{p}(s, T) = \sum \{\lambda(\rho) : s\,\rho\,t \text{ for all } t \in T\}, \tag{2}$$

where the summation is made over all rankings of S which rank s at the top of T (Falmagne, 1978). A complete system of choice probabilities $\mathfrak{p}$ is said to be *induced by rankings* if there exists a probability distribution λ on the set of rankings of S satisfying (2) for all $(s, T) \in \mathfrak{I}$. Under the noncoincidence assumption, a complete system of choice probabilities arises from the random utility model if and only if it is induced by rankings. More than twenty five years ago, Falmagne proved the following result.

Theorem 1 (Falmagne (1978)) *A complete system of choice probabilities $\mathfrak{p} \in \mathbb{R}^{\mathfrak{I}}$ is induced by rankings iff*

$$\sum_{U : T \subseteq U \subseteq S} (-1)^{|U \setminus T|} \mathfrak{p}(s, U) \geq 0 \qquad \forall (s, T) \in \mathfrak{I}, \tag{3}$$

$$\sum_{s \in T} \mathfrak{p}(s, T) = 1 \qquad \forall T \subseteq S, T \neq \varnothing. \tag{4}$$

Inequalities (3) are known as the *Block-Marschak conditions* (Block and Marschak, 1960). Here we call them Block-Marschak *inequalities* and refer to Equalities (4) as the *normalization equalities*. More than twenty years ago, Falmagne proved that (3) and (4) are also sufficient for $\mathfrak{p}$ to be induced by rankings.

2 Möbius inversion

Let $\mathfrak{p}$ be a complete system of choice probabilities induced by a probability distribution λ on the rankings of S. Consider the vector $\mathfrak{q}$ in $\mathbb{R}^{\mathfrak{I}}$ defined by

$$\mathfrak{q}(s,T) = \sum\{\lambda(\rho) : s\,\rho\,t \text{ for all } t \in T \text{ and } u\,\rho\,s \text{ for all } u \in S \setminus T\}. \tag{5}$$

The quantity $\mathfrak{q}(s,T)$ is to be interpreted as the probability that s is ranked at the top of T and of no proper superset of T. Then $\mathfrak{p}$ and $\mathfrak{q}$ are related by

$$\mathfrak{p}(s,T) = \sum_{U:T\subseteq U\subseteq S} \mathfrak{q}(s,U). \tag{6}$$

For each fixed alternative $s \in S$, $\mathfrak{p}(s,.)$ and $\mathfrak{q}(s,.)$ are two functions defined on the interval $[\{s\},S]$ of the power set of S. By applying Möbius inversion (van Lint and Wilson, 2001) to these two functions, we infer from (6) the following expression of $\mathfrak{q}$ in terms of $\mathfrak{p}$:

$$\mathfrak{q}(s,T) = \sum_{U:T\subseteq U\subseteq S} (-1)^{|U\setminus T|}\mathfrak{p}(s,U). \tag{7}$$

In view of (5), $\mathfrak{q}$ is nonnegative so the right-hand side of (7) has to be nonnegative for each $(s,T) \in \mathfrak{I}$. This proves the necessity of the Block-Marschak inequalities. (The necessity of the normalization equalities is obvious.)

3 Polyhedral combinatorics

From now on, we take the point of view of polyhedral combinatorics (Schrijver, 1995). To each ranking ρ on S we associate the vector $\mathfrak{p}^\rho$ in $\mathbb{R}^{\mathfrak{I}}$ defined by

$$\mathfrak{p}^\rho(s,T) = \begin{cases} 1 \text{ if } s\,\rho\,t \text{ for all } t \in T \\ 0 \text{ otherwise.} \end{cases}$$

The convex hull of all such vectors $\mathfrak{p}^\rho$, for ρ ranking of S, is known as the *multiple choice polytope*. A complete system of choice probabilites $\mathfrak{p}$ is induced by rankings if and only if it belongs to the multiple choice polytope. Falmagne's theorem, Theorem 1, gives a complete linear description of the multiple choice polytope.

Now let $D = (N,A)$ be a simple, acyclic digraph, and let p and q be two distinguished nodes of D. Furthermore, assume that D has at least one arc and that each arc of D is contained in some p–q dipath. We encode each p–q dipath Π in D by means of the 0/1 characteristic vector in $\mathbb{R}^A$ of its arc set, which we denote $\mathfrak{r}^\Pi$. The convex hull of the vectors $\mathfrak{r}^\Pi$, for Π a p–q dipath in D, is referred to as the *p–q dipath polytope of D*. Whenever v is a node of D, we let $\delta^-(v) = \{(w,v) : w \in N, (w,v) \in A\}$ and $\delta^+(v) = \{(v,w) : w \in N, (v,w) \in A\}$. For $B \subseteq A$, we let $\mathfrak{r}(B) = \sum\{\mathfrak{r}(v,w) : (v,w) \in B\}$. A matrix is said to be *totally unimodular* if the determinant of each of its square submatrices is in $\{0, 1, -1\}$. Let M denote a matrix whose rows are indexed by nodes of D and columns are indexed by arcs of D such that the entry corresponding to node v and arc a equals 1 if a enters v, -1 if a leaves v and 0 else. It is well known that M is totally unimodular (Schrijver, 1986). This implies that the polyhedron $\{\mathfrak{r} \in \mathbb{R}^A : M\mathfrak{r} = d, \mathfrak{r} \geq 0\}$ has all its vertices integral for every integral vector d in $\mathbb{R}^A$. The next result easily follows from these basic facts. Note that our assumptions on D force $\delta^-(p) = \delta^+(q) = \varnothing$ and that equation $-\mathfrak{r}(\delta^+(p)) = -1$ is implied by (8)–(9).

Theorem 2 *A point $\mathfrak{r} \in \mathbb{R}^A$ belongs to the p–q dipath polytope of D iff*

$$\mathfrak{r}(\delta^-(v)) - \mathfrak{r}(\delta^+(v)) = 0 \qquad \forall v \in N \setminus \{p,q\}, \tag{8}$$

$$\mathfrak{r}(\delta^-(q)) = 1, \tag{9}$$

$$\mathfrak{r}(v,w) \geq 0 \qquad \forall (v,w) \in A. \tag{10}$$

In the language of network flows (Ahuja et al., 1993), Equations (8)–(10) define a *flow* of *value* 1 in the *network* $D = (N, A)$, with *source node* p and *sink node* q.

4 The proof

Our proof simply reduces Theorem 1 to Theorem 2. Let $D = (N, A)$ be the digraph defined as follows. The node set of D is the power set of S. The arc set of D is $\{(T \setminus \{s\}, T) : (s,T) \in \mathfrak{I}\}$, that is, the covering relation of the power set of S partially ordered by inclusion. We distinguish two special nodes in D, namely, $p = \varnothing$ and $q = S$. Digraph D is referred to as the *power set digraph* of S. Each ranking ρ on S determines a p–q dipath in D, namely, the dipath with node sequence

$$\varnothing, \{\rho(n)\}, \{\rho(n-1), \rho(n)\}, \{\rho(n-2), \rho(n-1), \rho(n)\}, \ldots, S.$$

We denote it by $\Pi(\rho)$.

Now define a linear mapping f from $\mathbb{R}^{\mathfrak{I}}$ to $\mathbb{R}^A$ by $\mathfrak{p} \mapsto f(\mathfrak{p}) = \mathfrak{r}$ with $\mathfrak{r}(T \setminus \{s\}, T) = \mathfrak{q}(s,T)$, where $\mathfrak{q}$ is defined by (7). So $\mathfrak{r}$ is simply $\mathfrak{q}$ viewed in another vector space of the same dimension. It is easily verified from (5) that for any ranking ρ on S, we have $f(\mathfrak{p}^\rho) = \mathfrak{r}^{\Pi(\rho)}$.

PROOF (of Theorem 1) Let P denote the multiple choice polytope and let Q denote the $\varnothing$–S dipath polytope of the power set digraph of S. By what precedes, we have $f(P) = Q$. Because f is bijective, we have $\mathfrak{p} \in P$ iff $\mathfrak{r} = f(\mathfrak{p}) \in Q$. By Theorem 2, the latter holds iff $\mathfrak{r}$ satisfies (8)–(10). Given (7), $\mathfrak{r}$ satisfies (10) iff $\mathfrak{p}$ satisfies (3). Finally, it is easy to check, by a straightforward dimension argument, that $\mathfrak{r}$ satisfies (8)–(9) iff $\mathfrak{p}$ satisfies (4). $\square$

5 Closing comments

We mention that two other proofs of Theorem 1 appear in Barbera and Pattanaik (1986) and in Monderer (1992). The first one is essentially identical to Falmagne's original proof, and the second one relies on game-theoretic techniques. Moreover, Möbius inversion has been used before by Colonius (1984) to simplify part of the proof of Falmagne's theorem.

As in the previous section, let P, D and Q respectively denote the multiple choice polytope, the power set digraph of S and the $\varnothing$–S dipath polytope of D. Assume that $n \geq 3$. Then it is easily seen that Inequalities (10) with (v, w) of the form $(v, w) = (S \setminus \{s\}, S)$ or $(v, w) = (\varnothing, \{s\})$ for some $s \in S$ can be derived from the other conditions defining Q. Furthermore all other inequalities in (10) are nonredundant, hence facet-defining, for the following reason. Let T be a proper subset of S with at least two elements, let $s \in T$, and let a denote the arc $(T \setminus \{s\}, T)$. Now consider the dipath Π_1 with arc set $\{a\}$, any $\varnothing$–T dipath Π_2, any $T \setminus \{s\}$–S dipath Π_3 and any $\varnothing$–S dipath Π_4. Moreover, assume that Π_2, Π_3 and Π_4 all avoid arc a. For each $\varepsilon \in]0, 1[$, the vector

$$\mathfrak{r} = -\varepsilon\mathfrak{r}^{\Pi_1} + \varepsilon\mathfrak{r}^{\Pi_2} + \varepsilon\mathfrak{r}^{\Pi_3} + (1 - \varepsilon)\mathfrak{r}^{\Pi_4}$$

satisfies all conditions in (8)–(10) except $\mathfrak{r}(T \setminus \{s\}, T) \geq 0$. The next result follows. It was obtained by Suck (2003) by different techniques.

Proposition 3 *Provided $n \geq 3$, the Block-Marschak inequality (3) is facet-defining for P if and only if $1 < |T| < n$.*

Interestingly, the connection between P and Q can also be exploited to find the coefficients $\lambda(\rho)$ given a complete system of choice probabilities which is induced by rankings. Indeed, we can resort to standard techniques for decomposing a flow into path flows (Ahuja et al., 1993).

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