The Biorder Polytope

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Abstract. Biorders, also called Ferrers relations, formalize Guttman scales. Irreflexive biorders on a set are exactly the interval orders on that set. The biorder polytope is the convex hull of the characteristic matrices of biorders. Its definition is thus similar to the definition of other order polytopes, the linear ordering polytope being the proeminent example. We investigate the combinatorial structure of the biorder polytope, thus obtaining a complete linear description in a specific case, and the automorphism group in all cases. Moreover, a class of facet-defining inequalities defined from weighted graphs is thoroughly analyzed. A weighted generalization of stability-critical graphs is presented, which leads to new facets even for the well-studied linear ordering polytope.

Keywords: polytope, biorder, polyhedral combinatorics, stability-critical graph.

1. Introduction

Biorders appeared first as 'Guttman scales' [12] in psychology. In mathematics, they are also called 'Ferrers relations' after Riguet [19]; see Monjardet for a historical survey [16]. Let X and Y be two nonempty, finite sets of respective cardinalities m and n. A biorder B from X to Y is a relation from X to Y which satisfies

$$\text{for all } i,k \in X,\ j,\ell \in Y: \quad iBj \text{ and } kB\ell \text{ implies } iB\ell \text{ or } kBj.$$

Except for finiteness, no restriction is made here on the sets X and Y, which can be disjoint or equal. In the latter case, we call B also a biorder on X. The irreflexive biorders on X coincide with the interval orders on X.

This paper focuses on the biorder polytope $P_{\mathrm{Bio}}^{X\times Y}$, which is a convex polytope whose vertices correspond to biorders from X to Y. To give some motivation, let us consider the problem of finding a biorder from X to Y which is closest to a certain relation R from X to Y. Here, 'closest' refers to maximizing the number of pairs on which R and B agree, in other words to minimizing $|R\Delta B|$. Define the linear form $x \mapsto c \bullet x$ on $\mathbb{R}^{X\times Y}$ by $c_{(i,j)} = +1$ if $(i,j) \in R$, and -1 if $(i,j) \notin R$, where $\bullet$ refers to the sum of products of corresponding matrix elements. Denoting by x^B the point of $\mathbb{R}^{X\times Y}$ which is the characteristic vector or, better, matrix of B, that is

$$x^B_{(i,j)} = \begin{cases} 1 & \text{if } (i,j) \in B, \\ 0 & \text{if } (i,j) \notin B, \end{cases} \tag{1}$$

we have $c \bullet x^B = |R| - |R \triangle B|$. So the problem of finding a biorder closest to R can be formulated as a linear program on the polytope which is the convex hull of all vectors x^B, for B any biorder from X to Y. This polytope, which lies in $\mathbb{R}^{X \times Y}$, is the biorder polytope $P_{\mathrm{Bio}}^{X \times Y}$. To solve the approximation problem efficiently, one needs to know a linear description (by linear inequalities) of the polytope $P_{\mathrm{Bio}}^{X \times Y}$, or at least to be able to generate many linear inequalities on $\mathbb{R}^{X \times Y}$ valid for $P_{\mathrm{Bio}}^{X \times Y}$. As points belonging to $P_{\mathrm{Bio}}^{X \times Y}$ can be seen as 'fuzzy' or 'probabilistic biorders' (being convex combinations of the vertices), the same problem is also motivated by the search for a characterization of these 'probabilistic biorders' by a system of linear inequalities.

As easily seen the biorder polytope $P_{\mathrm{Bio}}^{X \times Y}$ has dimension $m \cdot n$, that is: $P_{\mathrm{Bio}}^{X \times Y}$ is a full-dimensional polytope in $\mathbb{R}^{X \times Y}$. (In this paper, 'dimension' is to be understood as 'affine dimension'.) Hence, there exists a unique linear description of $P_{\mathrm{Bio}}^{X \times Y}$ involving a minimum number of inequalities, where one such inequality corresponds to one facet of $P_{\mathrm{Bio}}^{X \times Y}$ (a facet being a proper, maximal face, see e.g. Ziegler [21]). The problem of determining all facets of $P_{\mathrm{Bio}}^{X \times Y}$, in other words all linear inequalities which appear in all linear descriptions of $P_{\mathrm{Bio}}^{X \times Y}$, should be considered as an impossible task. Indeed, there exist linear functionals whose maximization on $P_{\mathrm{Bio}}^{m \times n}$ is NP-hard. (We do not know whether the approximation problem mentioned above is NP-hard, but we can prove, by reduction from `MAXCLIQUE`, that it is NP-hard to find a biorder minimizing the sum of its distances to two given relations.) However, several families of those facets are described in our paper. They entail a complete minimal description of $P_{\mathrm{Bio}}^{X \times Y}$ for 'small' values of m, n, as checked with the `porta` software [3] or proved formally here for $m \leq 2$ or $n \leq 2$.

For any finite set Z, the biorder polytope $P_{\mathrm{Bio}}^{Z \times Z}$ contains as faces many other known polytopes. For instance, the interval order polytope [17] is formed by the points in $P_{\mathrm{Bio}}^{Z \times Z}$ that satisfy $x_{(i,i)} = 0$ for all $i \in Z$; its vertices correspond to interval orders on Z. The well-studied linear ordering polytope P_{LO}^{Z} (see e.g. [18], [11], [1],[2]) is obtained by imposing further $x_{(i,j)} + x_{(j,i)} = 1$ for all $i, j \in Z$. Some of our techniques for establishing facets for biorder polytopes are inspired from papers on the linear ordering polytope, especially Koppen [13]. In turn, our study led us to infer new facets of the latter polytope (these findings are left as an illustration for a paper on polytope projection, see Doignon and Fiorini [8]). A weighted generalization of stability-critical graphs is our main tool.

Another of the results presented here is the determination of the automorphism group of the biorder polytope (for the linear ordering polytope, this is done in Fiorini [10]).

2. Background and Elementary Results

Biorders can be defined as follows, with $\bar{B}$ denoting the complement $\bar{B} = (X \times Y) \setminus B$ of a relation B from X to Y, and juxtaposition of two relation symbols indicating product. A relation B from X to Y is a *biorder* when $B\,\bar{B}^{-1}\,B \subseteq B$. The complement $\bar{B}$ and the converse B^{-1} of the biorder B are again biorders, the second from Y to X. It is not difficult to check that a relation B from X to Y is a biorder exactly when there exist linear orderings L and M of respectively X and Y, such that $L\,B\,M \subseteq B$. When the latter holds, the 0/1-array built from L and M to encode B is *step-like* in the sense that all cells above or to the right of a cell containing 1 also contain 1. More results of this type can be found in the literature, see e.g. Doignon, Ducamp and Falmagne [7], which introduced the name 'biorder'. However, as far as we know, no method for determining the number of biorders from X to Y has been published. A recursive formula, plus two explicit formulas both involving Stirling numbers of the second kind, are provided in Christophe, Doignon and Fiorini [5].

Each point of $\mathbb{R}^{X\times Y}$ has one coordinate $x_{(i,j)}$ for each ordered pair (i,j) with $i \in X$, $j \in Y$; this coordinate will be abbreviated as x_{ij}. A biorder B from X to Y is represented by its characteristic matrix x^B, as in Equation (1). The convex hull of all the resulting points x^B in $\mathbb{R}^{X\times Y}$, for B any biorder from X to Y, is a 0/1-polytope that we call the *biorder polytope* $P_{\text{Bio}}^{X\times Y}$. Clearly, the structure of $P_{\text{Bio}}^{X\times Y}$ only depends on the respective cardinalities m and n of X and Y. For this reason, the biorder polytope $P_{\text{Bio}}^{X\times Y}$ is often denoted as $P_{\text{Bio}}^{m\times n}$. Similarly, the *linear ordering polytope* P_{LO}^{Z}, or P_{LO}^{q} if $q = |Z|$, is the convex hull of all characteristic vectors of linear orderings of Z (see e.g. [18],[11], [13], [1], [2]). This polytope is not full dimensional in $\mathbb{R}^{Z\times Z}$, because its dimension equals $q(q-1)/2$.

The reader is referred to Ziegler [21] for the terminology and notation about polytopes. An *affine mapping* $\mathbb{R}^d \to \mathbb{R}^e : (x_k) \mapsto (x'_k)$ specifies each coordinate x'_k as a polynomial of degree at most 1 in the x_ℓ's. We will always describe such a mapping by providing the polynomial expression for x'_k. An *affine automorphism* of a polytope P in $\mathbb{R}^d$ is an affine permutation of $\mathbb{R}^d$ that induces a permutation of P. A *combinatorial automorphism* of P is a permutation of the set $\operatorname{vert} P$ of vertices of P which, for any face F of P, transforms $\operatorname{vert} F$ into vert F' for some face F' of P (the two faces F and F' necessarily have the same dimension). Notice that each affine automorphism induces a combinatorial automorphism.

Some obvious linear inequalities valid for the biorder polytope are the *trivial inequalities* and *axiomatic inequalities*

$$-x_{ij} \leq 0, \qquad x_{ij} \leq 1, \tag{2}$$
$$x_{ij} + x_{k\ell} - x_{i\ell} - x_{kj} \leq 1, \tag{3}$$

where $i, k \in X$, $j, \ell \in Y$, and $i \neq k$, $j \neq \ell$. Indeed, any vertex x^B of $P_{\text{Bio}}^{X\times Y}$ satisfies all of these inequalities: the definition of a biorder B exactly means that the array x^B does not contain any subarray

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

Because the origin o and all unit vectors e_{ij} of $\mathbb{R}^{X\times Y}$ are characteristic matrices of biorders, there follows that $P_{\text{Bio}}^{X\times Y}$ has dimension $m \cdot n$. A similar argument shows that the trivial inequalities (2) with 0 in the right-hand side define facets of $P_{\text{Bio}}^{X\times Y}$. Complementation of a relation from X to Y induces the affine permutation of $\mathbb{R}^{X\times Y}$ defined by $x'_{ij} = 1 - x_{ij}$. This permutation is an affine automorphism of $P_{\text{Bio}}^{X\times Y}$, that we call the *complementation automorphism.* As a consequence, trivial inequalities (2) with 1 in the right-hand side are facet defining. Not much more work is required to show that also axiomatic inequalities define facets. In the last three sections, many more facets of $P_{\text{Bio}}^{X\times Y}$ will be established.

3. Automorphisms

Automorphisms of a polytope are also called symmetries. As illustrated in [9], [1] and also at the end of our paper, they can be instrumental in producing new facets of a polytope from known facets. Keeping our notation $|X| = m$ and $|Y| = n$, we determine the symmetry group of the biorder polytope. As $P_{\text{Bio}}^{m\times n}$ and $P_{\text{Bio}}^{n\times m}$ are affinely isomorphic, the following theorem settles all cases.

Theorem 1 *The group* $\mathrm{Aut}(P_{\mathrm{Bio}}^{m\times n})$ *of combinatorial automorphisms of the biorder polytope is isomorphic to*

$$\begin{cases} \mathbb{Z}_2^n \rtimes \mathrm{Sym}(n) & \text{if } m=1, \\ \mathbb{Z}_2 \times \mathrm{Sym}(4) & \text{if } m=n=2, \\ \mathbb{Z}_2^{n+1} \rtimes \mathrm{Sym}(n) & \text{if } 2=m<n, \\ \mathbb{Z}_2^2 \times \mathrm{Sym}(m) \times \mathrm{Sym}(n) & \text{if } 2<m=n, \\ \mathbb{Z}_2 \times \mathrm{Sym}(m) \times \mathrm{Sym}(n) & \text{if } 2<m<n. \end{cases}$$

In all cases, every combinatorial automorphism of $P_{\mathrm{Bio}}^{m\times n}$ *is induced by some affine automorphism.*

The semidirect product $\rtimes$ in the first case comes from the obvious equality $P_{\mathrm{Bio}}^{1\times n} = C_n$ between the biorder polytope with $m=1$ and the n-hypercube C_n, which gives $\mathrm{Aut}(P_{\mathrm{Bio}}^{1\times n}) = \mathrm{Aut}(C_n)$. Moreover, in the third case, we will in fact prove $\mathrm{Aut}(P_{\mathrm{Bio}}^{2\times n}) \simeq \mathbb{Z}_2 \times \mathrm{Aut}(C_n)$.

We delay the proof of Theorem 1 after the introduction of specific automorphisms plus the proofs of two lemmas. The complementation automorphism of $P_{\mathrm{Bio}}^{X\times Y}$ was used in Section 2. Other easy automorphisms derive from permutations or relabellings of the elements in X, resp. Y. For any permutations α of X and β of Y, the linear permutation specified by $x'_{\alpha(k)\beta(\ell)} = x_{kl}$ is an affine automorphism of $P_{\mathrm{Bio}}^{X\times Y}$. It is called the *relabelling automorphism* defined from α and β.

When $m=n$, we may (in $m!$ ways) identify Y with X. Because the converse of a biorder on X is again a biorder on X, the mapping from $\mathbb{R}^{X\times X}$ to itself with $x'_{k\ell} = x_{\ell k}$ is an affine automorphism of $P_{\mathrm{Bio}}^{X\times X}$, called a *transposition automorphism.*

When $m = 2 \leq n$, the biorder polytope $P_{\mathrm{Bio}}^{2\times n}$ has still another type of 'obvious' automorphisms. Indeed, for any fixed j in Y, consider the mapping with $x'_{k\ell} = x_{k\ell}$ when $\ell \neq j$, and $x'_{1j} = 1 - x_{2j}$, $x'_{2j} = 1 - x_{1j}$. This mapping, which is an affine automorphism of $P_{\mathrm{Bio}}^{2\times n}$, is denoted as σ_j.

For $m, n \geq 2$ and $(m,n) \neq (2,2)$, the automorphisms we just mentioned generate the group displayed in Theorem 1; thus it remains only to prove that the total number of automorphisms of $P_{\mathrm{Bio}}^{m\times n}$ does not exceed the order of this group. We denote by J the point of $\mathbb{R}^{X\times Y}$ having all coordinates equal to 1, and call *trivial* any facet defined by a trivial inequality (2).

Lemma 2 *Any facet of the biorder polytope* $P_{\mathrm{Bio}}^{m\times n}$ *contains at most half the total number of vertices. Moreover, any facet which contains exactly half the number of vertices is a trivial facet.*

Proof. The complementation automorphism of $P^{m\times n}_{\mathrm{Bio}}$ is the central symmetry with respect to the point $\frac{1}{2}J$. If a facet contained more than half the vertices, then the point $\frac{1}{2}J$ would be the midpoint of some two vertices of this facet, which is impossible. The first assertion follows. For the second one, suppose the inequality $c \bullet x \leq \delta$ defines a facet containing exactly half of the vertices. The linear form $x \mapsto c \bullet x$ can take only two values on the set of vertices of $P^{m\times n}_{\mathrm{Bio}}$ (one at any vertex of the considered facet, the other at any vertex of the image of this facet by the complementation automorphism). But evaluating this form at the origin O, the basis vectors e_{ij} (for $i \in X$, $j \in Y$), and the point J, which all are vertices of $P^{m\times n}_{\mathrm{Bio}}$, gives the values 0, c_{ij} and $\sum_{ij\in X\times Y} c_{ij}$. Consequently, the form is nothing else than a multiple of $x \mapsto x_{ij}$ for some $(i,j) \in X \times Y$, and the inequality $c \bullet x \leq \delta$ is equivalent to a trivial inequality. ∎

Lemma 3 *The trivial facets of the biorder polytope $P^{m\times n}_{\mathrm{Bio}}$ form an orbit for the action of* $\mathrm{Aut}(P^{m\times n}_{\mathrm{Bio}})$ *on the collection of all facets of* $P^{m\times n}_{\mathrm{Bio}}$.

Proof. In view of Lemma 2, the orbit of a trivial facet contains only trivial facets. On the other hand, the relabelling automorphisms together with the complementation automorphism show that all trivial facets fall in the same orbit. ∎

Proof. (of Theorem 1.) We set $G = \mathrm{Aut}(P^{m\times n}_{\mathrm{Bio}})$. The first case, with $m = 1$, being obvious, let us assume from now on $2 \leq m \leq n$. From Lemmas 2 and 3, we know that all facets with a maximum number of vertices form an orbit. Now let us look for vertices contained each in the minimum number of facets. Any vertex belongs to exactly $m \cdot n$ trivial facets. The origin O is a vertex contained in no other facet; so, the vertices O and J each minimize the number of facets containing them. Every other minimizing vertex v cannot belong to any axiomatic facet; it must then encode a biorder having all columns equal one to the other, or all rows equal one to the other. In both cases, we get a vertex v such that any vertex of the 0/1-hypercube which differs from v in exactly one coordinate is again a vertex of $P^{X\times Y}_{\mathrm{Bio}}$. It follows that any facet of $P^{X\times Y}_{\mathrm{Bio}}$ containing v is a trivial facet. Denote by $\mathcal{C}$ the set of vertices which encode biorders having all columns equal one to the other, and by $\mathcal{R}$ the set of vertices which encode biorders having all rows equal one to the other. Thus $\mathcal{C} \cup \mathcal{R}$ is a union of orbits for the action of $G = \mathrm{Aut}(P^{X\times Y}_{\mathrm{Bio}})$ on the set of vertices.

For any vertex v of $P_{\mathrm{Bio}}^{m\times n}$, let $\mathcal{T}(v)$ be the collection of trivial facets containing v. If w is also a vertex, set

$$d(v,w) \quad = \quad \tfrac{1}{2}\,|\mathcal{T}(v)\Delta\mathcal{T}(w)|.$$

As trivial facets form an orbit (Lemma 3), both mapping $\mathcal{T}$ and d are invariant under the actions of G. Notice that $d(v,w)$ is nothing else than the number of cells at which the biorders encoded by v and w differ, that is the Hamming distance between the two biorders. The minimum distance $d(v,w)$ of a vertex v in $\mathcal{C}\cup\mathcal{R}$ to another vertex w in $\mathcal{C}\cup\mathcal{R}$ equals (remember $m\le n$)

$$\begin{cases} m & \text{if } v\in\mathcal{R},\\ n & \text{if } v\in\mathcal{C}\setminus\{O,J\}. \end{cases} \tag{5}$$

When $m<n$, we conclude that both $\mathcal{R}$ and $\mathcal{C}\setminus\{O,J\}$ are unions of orbits. Notice that, in all cases, $\mathcal{R}$ with the distance d has the structure of a hypercube of dimension n, and similarly for $\mathcal{C}$ but with dimension m. Moreover, each automorphism fixing all vertices in $\mathcal{C}\cup\mathcal{R}$ must fix all vertices, because the following mapping defined on the set of vertices of $P_{\mathrm{Bio}}^{X\times Y}$ is injective:

$$v\mapsto\{F\cap(\mathcal{C}\cup\mathcal{R}) : F \text{ a trivial facet, with } v\in F\}.$$

If $m=2<n$, the relabelling automorphisms obtained from the identical permutation on X and all permutations of Y, together with the automorphisms σ_j, $j\in Y$, induce on $\mathcal{R}$ the group of the hypercube C_n. An automorphism fixing all vertices of $\mathcal{R}$ either fixes or exchanges the two vertices of $\mathcal{C}\setminus\mathcal{R}$. Hence, $|G|\le n!\cdot 2^n\cdot 2$. In view of the known automorphisms, we have equality here, and the group in the statement results for $2=m<n$. When $2=m=n$, the 14 vertices of $P_{\mathrm{Bio}}^{2\times 2}$ are the vertices of a 4-dimensional hypercube minus two opposite vertices, hence the statement is also correct in this case.

Now, if $2<m<n$, we remark that the minimum distance $d(v,w)$ for $v\in\{O,J\}$ and $w\in\mathcal{C}\setminus\{O,J\}$ with $v\ne w$ equals n, while if we take $v\in\mathcal{R}\setminus\{O,J\}$ it is more than n. Thus $\{O,J\}$ forms an orbit, as well as $\mathcal{R}\setminus\{O,J\}$. As $\mathcal{C}$ with distance d has the structure of a hypercube of dimension m with O and J diametrically opposite vertices, and moreover $\mathcal{C}\cap\mathcal{R}=\{O,J\}$, we derive $|G|\le 2\cdot m!\cdot n!$. With the known automorphisms, this establishes the statement in case $2<m<n$.

There remains to treat the case $2<m=n$. By counting the other elements of $\mathcal{C}\cup\mathcal{R}$ at distance m from a certain element in $\mathcal{C}\cup\mathcal{R}$, one deduces that $\{O,J\}$ is an orbit, and thus $(\mathcal{C}\cup\mathcal{R})\setminus\{O,J\}$ is a union of orbits. Notice that, after identification of X and Y, the transposition

automorphism exchanges $\mathcal{C}$ and $\mathcal{R}$ and fixes both O and J. The values of d on $(\mathcal{C} \cup \mathcal{R}) \setminus \{O, J\}$ lead to the conclusion in case $2 < m = n$. ∎

4. The minimum linear description in case $m \leq 2$

A complete description of the biorder polytope $P_{\text{Bio}}^{m \times n}$ in cases $m \leq 2$ or $n \leq 2$ will be provided. When $m = 1$ or $n = 1$, $P_{\text{Bio}}^{m \times n}$ is a hypercube. The following theorem covers the remaining cases.

Theorem 4 *The biorder polytope $P_{\text{Bio}}^{2 \times n}$, with $2 \leq n$, is completely described by the trivial and axiomatic inequalities.*

Proof. Let P^n be the polytope defined in $\mathbb{R}^{X \times Y}$ by all trivial and axiomatic inequalities. Clearly, we have $P_{\text{Bio}}^{2 \times n} \subseteq P^n$. Let us prove that any vertex v of P^n belongs to $P_{\text{Bio}}^{2 \times n}$; the equality $P_{\text{Bio}}^{2 \times n} = P^n$ follows, and so the thesis also.

A first step is to show that all coordinates of v are 0/1. For the sake of simplicity, we let $X = \{1, 2\}$. Fix j in Y. Because v is in P^n, we have $0 \leq v_{1j} \leq 1$ and $0 \leq v_{2j} \leq 1$. If $0 < v_{1j} < 1$ and $0 < v_{2j} < 1$, there exists a positive real number ε with $0 < v_{1j} - \varepsilon < v_{1j} + \varepsilon < 1$ and $0 < v_{2j} - \varepsilon < v_{2j} + \varepsilon < 1$. Then we have

$$v \quad = \quad \frac{1}{2}\left(v + \varepsilon(e_{1j} + e_{2j})\right) + \frac{1}{2}\left(v - \varepsilon(e_{1j} + e_{2j})\right). \tag{6}$$

Moreover $v \pm \varepsilon(e_{1j} + e_{2j})$ belongs to P^n, because trivial inequalities are satisfied by the choice of ε, and axiomatic inequalities are unchanged. But then Equation (6) is impossible for a vertex, so $v_{1j} \in \{0, 1\}$ or $v_{2j} \in \{0, 1\}$. Using the automorphisms σ_j from previous section, we may thus assume $v_{1j} \in \{0, 1\}$, for all j in Y.

Suppose now that some other coordinate of v is not in $\{0, 1\}$. Notice that the axiomatic inequality $x_{1j} + x_{2l} - x_{1\ell} - x_{2j} \leq 1$ can be satisfied with equality only if $v_{2j}, v_{2\ell} \in \{0, 1\}$ or

$$v_{1j} = 1, \quad v_{1\ell} = 0, \quad \text{ and } 0 < v_{2j} = v_{2\ell} < 1.$$

For some positive ε, define the two points p and q by

$$\begin{aligned}
&p_{1j} = q_{1j} = v_{1j} \;\; \text{for all } j \in Y,\\
&p_{2j} = q_{2j} = v_{2j} \;\; \text{for all } j \in Y \text{ such that } v_{2j} \in \{0,1\},\\
&\left.\begin{aligned} p_{2j} &= v_{2j} + \varepsilon \\ q_{2j} &= v_{2j} - \varepsilon \end{aligned}\right\} \;\; \text{for all } j \in Y \text{ such } v_{1j} = 1 \text{ and } 0 < v_{2j} < 1,\\
&\left.\begin{aligned} p_{2j} &= v_{2j} - \varepsilon \\ q_{2j} &= v_{2j} + \varepsilon \end{aligned}\right\} \;\; \text{for all } j \in Y \text{ such that } v_{1j} = 0 \text{ and } 0 < v_{2j} < 1.
\end{aligned}$$

Then ε can be chosen in such a way that $p, q \in P^n$. As v is the midpoint of p and q, this contradicts the fact that v is a vertex of P^n. We conclude that all coordinates of v are 0/1. It is then easily checked that v is the characteristic matrix of a biorder, hence belongs to $P_{\text{Bio}}^{2\times n}$. This completes the proof. ∎

5. A General Scheme of Facets

For all pairs (m, n) with $m \geq 3$ and $n \geq 3$, the biorder polytope $P_{\text{Bio}}^{m\times n}$ has other facets than the trivial and axiomatic ones. As an example, the following inequality defines a facet of $P_{\text{Bio}}^{X\times Y}$ when $\{1, 2, \ldots, h\} \subseteq X \cap Y$ and $h \geq 1$ (for $h = 1$, it is a trivial inequality, and for $h = 2$, an axiomatic inequality):

$$\sum_{i=1}^{h} x_{ii} \;-\; \sum_{\substack{i,j=1\\ i\neq j}}^{h} (x_{ij} + x_{ji}) \quad\leq\quad 1. \tag{7}$$

This inequality is called the *fence inequality* because it is similar to a facet-defining inequality known for the linear ordering polytope P_{LO}^{q} under the same name [18]. Similarly as it was shown for P_{LO}^{q}, the fence inequality can be generalized to a very large family of facet-defining inequalities which interestingly relates to 'stability-critical graphs'. When investigating the case of $P_{\text{Bio}}^{m\times n}$, we uncover an extension of the classical theory which delivers new facets even for P_{LO}^{q}. The results for $P_{\text{Bio}}^{m\times n}$ are exposed here, while their consequences for P_{LO}^{q} are collected in Doignon and Fiorini [8].

Before emphasizing a particular scheme of facet-defining inequalities, we record five other isolated examples obtained from the software `porta` [3]. To display them in Equations (8)-(10), we make the matrix c explicit in the inequality $c \bullet x \leq \delta$ (see the Introduction for the meaning of $\bullet$). Notice that we recall first a fence inequality, and that we write $+$ for $+1$, $-$ for -1:

$$\begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix} \bullet x \leq 1; \qquad \begin{pmatrix} - & + & + \\ + & - & + \\ + & + & - \end{pmatrix} \bullet x \leq 4; \tag{8}$$

$$\begin{pmatrix} - & - & - & 2 \\ - & + & + & - \\ + & - & + & - \end{pmatrix} \bullet x \leq 3; \qquad \begin{pmatrix} + & + & + & -2 \\ + & - & - & + \\ - & + & - & + \end{pmatrix} \bullet x \leq 4; \tag{9}$$

$$\begin{pmatrix} + & + & - & - & - \\ - & 0 & + & - & + \\ 0 & - & + & + & - \end{pmatrix} \bullet x \leq 3; \qquad \begin{pmatrix} - & - & + & + & + \\ + & 0 & - & + & - \\ 0 & + & - & - & + \end{pmatrix} \bullet x \leq 4. \quad (10)$$

The above two-by-two grouping reflects the action of the complementation automorphism. For $m = 3$ and $n \in \{3, 4, 5\}$, we know, according to `porta`, all facet-defining inequalities of $P_{\mathrm{Bio}}^{m\times n}$. Their numbers for all types, as well as the total counts, are provided in Table I (Christophe [4]).

Table I. Numbers of facets of $P_{\mathrm{Bio}}^{3\times n}$ for $3 \leq n \leq 5$.

(m,n)	(2) trivial	(3) axiom.	(8) (co)fence	(9)	(10)	Total nb.
$(3,3)$	9+9	18	6+6	-	-	48
$(3,4)$	12+12	36	24+24	72+72	-	252
$(3,5)$	15+15	60	60+60	360+360	360+360	1650

The next proposition (which will be used intensively later) implies that inequalities as in Equations (7)-(10) remain facet defining for all larger values of m and n. In polyhedral combinatorics, it is called a "trivial lifting lemma" (see e.g. Reinelt [18]).

Proposition 5 *Let $c{\bullet}x \leq \delta$ be an inequality on $\mathbb{R}^{X\times Y}$, and let $X \subseteq \bar{X}$, $Y \subseteq \bar{Y}$, with $|\bar{X}| = \bar{m}$, $|\bar{Y}| = \bar{n}$. The following assertions then hold true.*

1. *The inequality $c \bullet x \leq \delta$ is valid for $P_{\mathrm{Bio}}^{X\times Y}$ iff it is valid for $P_{\mathrm{Bio}}^{\bar{X}\times\bar{Y}}$.*
2. *If the latter holds and $c \bullet x \leq \delta$ defines a face F of $P_{\mathrm{Bio}}^{X\times Y}$ and a face $\bar{F}$ of $P_{\mathrm{Bio}}^{\bar{X}\times\bar{Y}}$, then $\dim \bar{F} = \bar{m}\cdot\bar{n} - m\cdot n + \dim F$.*
3. *The inequality $c \bullet x \leq \delta$ is facet defining for $P_{\mathrm{Bio}}^{X\times Y}$ iff it is facet defining for $P_{\mathrm{Bio}}^{\bar{X}\times\bar{Y}}$.*

Proof. Assertion 1 derives from the following two easy facts: the restriction to $X \times Y$ of any biorder from $\bar{X}$ to $\bar{Y}$ is again a biorder, and any biorder from X to Y is also a biorder from $\bar{X}$ to $\bar{Y}$.

As Assertion 2 follows easily from the case $m = \bar{m}$ and $n = \bar{n} - 1$, we may assume $X = \bar{X}$ and $\bar{Y} \setminus Y = \{\bar{y}\}$. With $t = \dim F$, select in $\mathbb{R}^{X\times Y}$ vertices $v_0, v_1, \ldots, v_t$ of F which form an affine basis of F. By adding coordinates $x_{i\bar{y}}$ for $i \in X$ having zero values, we obtain vertices $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_t$ of $\bar{F}$ which are affinely independent. Denoting by B the biorder corresponding to v_0, select linear orderings L of X and M of Y in such a way that the resulting 0/1-array encoding B is step-like. For any $h \in X$, the relation $B \cup \{(i, \bar{y}) : i \in X \text{ and } iLh\}$ is a biorder from X to $\bar{Y}$. Write w_h for the corresponding vertex of $P_{\mathrm{Bio}}^{X \times \bar{Y}}$ and notice $w_h \in \bar{F}$. Furthermore, $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_t$ together with all these vertices w_h, for $h \in X$, are affinely independent points. Hence $\dim \bar{F} \geq \bar{m} \cdot \bar{n} - m \cdot n + \dim F$. The opposite inequality comes from considering the natural projection $\mathbb{R}^{X \times \bar{Y}} \to \mathbb{R}^{X \times Y}$.

Assertion 3 is the subcase of Assertion 2 with $\dim F = m \cdot n - 1$. ∎

In the rest of this paper, we work in the 'square' case, that is $m = n$. Remark that, in general, any facet-defining inequality $c \bullet x \leq \delta$ can be written with $c_{ij}, \delta \in \mathbb{Z}$ for all $(i, j) \in X \times Y$ (multiplication by a well-chosen factor always makes this true). We will investigate inequalities $c \bullet x \leq \delta$ satisfying the following assumption:

Working Assumption. There exists some bijection f from X to Y such that

1. $c_{ij}, \delta \in \mathbb{Z}$, for $(i, j) \in X \times Y$;
2. $c_{ij} \in \{-1, 0\}$ for all pairs (i, j) in $(X \times Y) \setminus f$;
3. for $i, j \in X$, we have $c_{if(j)} = c_{jf(i)}$.

Requirement 3 means that, after identification of Y to X according to f, the transposition automorphism preserves the inequality $c \bullet x \leq \delta$. This Working Assumption allows us to strongly generalize techniques which Koppen [13] designed for the linear ordering polytope. In fact, we use the bijection f in order to identify Y with X, and then the coefficients c_{ij} to define a weighted graph on X (Koppen considered only non-weighted graphs, see Proposition 13 below).

Our terminology for graphs generally follows Diestel [6]. A *weighted graph* is a pair (G, μ) where G is a graph and μ is a function from the node set $V(G)$ of G to $\mathbb{Z}$; thus a weight is specified for each node of G. The *worth* (or net weight) of a set S contained in $V(G)$ is the difference between the weight $\mu(S)$ of S and the number $||G[S]||$ of edges in the subgraph $G[S]$ of G induced on S. In particular, the worth of the empty set is zero. If S is of maximum worth, it is *tight*.

Let (G, μ) be a weighted graph whose node set $V(G)$ is contained in X (and whose edge set is denoted as $E(G)$). The *graphical inequality* of (G, μ) is the inequality on $\mathbb{R}^{X \times X}$ which reads

$$\sum_{v \in V(G)} \mu(v) x_{vv} \; - \sum_{\{v,w\} \in E(G)} (x_{vw} + x_{wv}) \quad \leq \quad \alpha(G, \mu), \qquad (11)$$

where $\alpha(G,\mu)$ is defined by

$$\alpha(G,\mu) \quad = \quad \max_{S\subseteq V(G)} \Big(\mu(S) - ||G[S]||\Big). \tag{12}$$

When G is a complete graph and $\mu = \mathbb{1}$ (the constant mapping with value 1), the graphical inequality (11) becomes the fence inequality (7).

Proposition 6 *Let (G,μ) be a weighted graph with $V(G) \subseteq X$. The graphical inequality of (G,μ) is valid for $P_{\text{Bio}}^{X\times X}$. A vertex x^B of $P_{\text{Bio}}^{X\times X}$ belongs to the face F defined by this inequality if and only if the biorder B on X contains some linear ordering L of some tight set of (G,μ) in such a way that $(i,j) \in B \setminus L$ implies $i \neq j$ and $\{i,j\} \notin E(G)$.*

Proof. Let B be a biorder on X, and let $S = \{v \in V(G) : (v,v) \in B\}$. If v and w are in S, then (v,w) or (w,v) belongs to B because B is a biorder. Therefore, the left-hand side of Inequality (11) evaluated at the characteristic matrix x^B is at most $\alpha(G,\mu)$. This proves the first assertion. Now if S is any tight set of (G,μ) and L is any linear order on L, then x^L satisfies Inequality (11) with equality. The same conclusion remains true if L is replaced with any biorder B as in the statement.

To prove the converse, let B be a biorder such that $x^B \in F$, and set again $S = \{v \in V(G) : (v,v) \in B\}$. The restriction of B to $S \times S$ is a reflexive biorder B'. There exists some linear ordering L of S with $L \subseteq B'$ (this is better proved by working with the complement relations: any interval order on S can be extended to an irreflexive linear ordering of S). If some $(i,j) \in B \setminus L$ is such that $\{i,j\} \in E(G)$, then x^L gives to the left-hand side of (11) a strictly larger value than x^B does, which is impossible. Consequently, x^B and x^L give the same value $\alpha(G,\mu)$, and S needs to be a tight set. ∎

We will call a biorder B as in Proposition 6 an *elite biorder* of (the graphical inequality of) the weighted graph (G,μ). Note that any linear ordering of any tight set of (G,μ) is necessarily an elite biorder.

To determine the dimension of the face of $P_{\text{Bio}}^{X\times X}$ defined by the graphical inequality (11), the following tool will prove useful. Given a weighted graph (G,μ) with $V(G) \subseteq X$, consider one real variable y_v for each node v in $V(G)$, and similarly one real variable $y_{\{v,w\}}$for each edge $\{v,w\}$ in $E(G)$.

The *system* of (G,μ) has one equation per tight set T of (G,μ), namely

$$\sum_{v\in T} y_v \ + \sum_{\substack{\{v,w\}\in E(G),\\ v,w\in T}} y_{\{v,w\}} \ = \ \alpha(G,\mu). \tag{13}$$

In view of the definition of a tight set, the system of (G,μ) always admits the *obvious solution* given by $y_v = \mu(v)$ and $y_{\{v,w\}} = -1$.

A node of (G,μ) is called *degenerate* if its degree and weight both equal zero.

Proposition 7 *Let (G,μ) be a weighted graph with $V(G) \subseteq X$, such that G has no degenerate node, and moreover any node (resp. edge) is included in some tight set. Denoting by F the face defined by the graphical inequality of (G,μ) and by $\mathcal{Y}$ the affine space of solutions to the system of (G,μ), we have*

$$\dim F \quad = \quad m^2 - 1 - \dim \mathcal{Y}. \tag{14}$$

Proof. In view of Proposition 5(ii), we may assume $V(G) = X$. Notice $\alpha(G,\mu) \neq 0$ follows from our assumption on (G,μ). Then by geometric considerations, the dimension of F equals $(m^2 - 1)$ minus the dimension of the solution space $\mathcal{Z}$ of the system in the unknowns z_{ij} for $i, j \in X$ having one equation per elite biorder B of (G,μ):

$$\sum_{(i,j)\in B} z_{ij} = \alpha(G,\mu). \tag{15}$$

This is because the m^2 variables of the latter system represent the coefficients of a linear equation $z \bullet x = \alpha(G,\mu)$ on $\mathbb{R}^{X\times X}$ satisfied by all vertices on the face defined by the graphical inequality of (G,μ). It suffices now to show that the affine spaces $\mathcal{Y}$ and $\mathcal{Z}$ have the same dimension. The following two claims about any z from $\mathcal{Z}$ will be used for this goal.

Claim 1. $z_{ij} = 0$ when $i \neq j$ and $\{i,j\} \notin E(G)$. By assumption, there exists some tight set T with $\{i,j\} \subseteq T$ or $\{i,j\} \cap T = \{i\}$. In the first case, let B_1 be any linear order on T containing a pair (i,j) with j immediately after i in T, and let $B_2 = B_1 \cup \{(j,i)\}$. Then B_1 and B_2 are elite biorders. Write the equations as in Equation (15) for B_1 and B_2, and take their difference; this shows $z_{ij} = 0$. In the second case, that is $\{i,j\} \cap T = \{i\}$, take any linear ordering B_3 of T with i as its minimum element, and let $B_4 = B_3 \cup \{(i,j)\}$. Again taking the difference of the equations corresponding to two elite biorders, namely B_3 and B_4 this time, we infer $z_{ij} = 0$.

Claim 2. $z_{ji} = z_{ij}$ when $\{i,j\} \in E(G)$. Take some tight set T containing $\{i,j\}$ and then any linear ordering of T in which j comes immediately after i. This linear ordering is an elite biorder B_5, as well as $B_6 = (B_5 \setminus \{(i,j)\}) \cup \{(j,i)\}$. The difference of the two equations for respectively B_5 and B_6 give the claim.

Now define on $\mathcal{Z}$ a mapping $g : (z_{ij}) \mapsto (y_v, y_{\{v,w\}})$ by

$$\begin{aligned} y_v &= z_{vv} && \text{for } v \in V(G), \\ y_{\{v,w\}} &= (z_{vw} + z_{wv})/2 && \text{for } \{v,w\} \in E(G). \end{aligned} \tag{16}$$

Then $g(\mathcal{Z}) \subseteq \mathcal{Y}$. Indeed, from $(z_{ij}) \in \mathcal{Z}$, we derive $(y_v, y_{\{v,w\}}) = g((z_{ij})) \in \mathcal{Y}$. For any tight set T, consider some linear order B on T; in view of the two claims above, we have

$$\sum_{(i,j)\in B} z_{ij} = \sum_{v\in T} y_v + \sum_{\substack{\{v,w\}\in E(G),\\ v,w\in T}} y_{\{v,w\}}, \tag{17}$$

so both quantities take value $\alpha(G,\mu)$.

Define on $\mathcal{Y}$ a mapping $h : (y_v, y_{\{v,w\}}) \mapsto (z_{ij})$ by

$$z_{ii} = y_i \qquad \text{for } i \in X, \tag{18}$$

$$z_{ij} = \begin{cases} y_{\{i,j\}} & \text{if } \{i,j\} \in E(G), \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Then $h(\mathcal{Y}) \subseteq \mathcal{Z}$, as we now show. Let $(y_v, y_{\{v,w\}}) \in \mathcal{Y}$, and set $(z_{ij}) = h(y_v, y_{\{v,w\}})$. Given any elite biorder B, apply Proposition 6 to it, with T the resulting tight set. Then for these B and T, the left-hand members of Equations (13) and (15) take the same value (because of Proposition 6). This shows $h(\mathcal{Y}) \subseteq \mathcal{Z}$.

Finally, notice that obviously $g \circ h$ is the identity on $\mathcal{Y}$. As both $\mathcal{Y}$ and $\mathcal{Z}$ are affine spaces with finite dimensions and both g and h are affine maps, the equality of these dimensions derives. ■

In the above Proposition 7 the assumption that any edge is contained in some tight set cannot be dispensed with, as testified by $(K_{3,3}, \mathbb{1})$, the bipartite graph on $3+3$ vertices with all weights 1 (we leave the easy verification to the reader).

Corollary 8 *For any weighted graph (G,μ) with $V(G) \subseteq X$ and $\alpha(G,\mu) \neq 0$, the graphical inequality of (G,μ) defines a facet of $P_{\mathrm{Bio}}^{X\times X}$ if and only if the system of (G,μ) has no other solution than the obvious one.*

Quite naturally in this paper, a weighted graph as in Corollary 8 *defines a facet*, or *is facet defining*.

Proof. Assume first that (G,μ) defines some facet F. Any node $\{v\}$ must belong to some tight set of (G,μ), otherwise Proposition 6 implies that (v,v) does not belong to any elite biorder B, and so all vertices of F would also satisfy $x_{vv} = 0$, which is clearly impossible. A similar argument shows that any edge is included in some tight set. Now Proposition 7 directly implies that the system of (G, μ) has only one solution.

To prove the converse, assume the system of (G, μ) has only one solution. Then clearly any unknown must appear in some equation, so any node or edge is in some tight set. Again, Proposition 7 leads to the desired conclusion, namely that (G, μ) is facet defining. ∎

Although Corollary 8 may first look rather technical, it is quite convenient to establish facets of $P_{\text{Bio}}^{X \times X}$ as illustrated in the following example. Simpler conditions, either necessary or sufficient, for a graph to be facet defining will be established in the next section.

Example 9 Take a weighing $\mu : V \to \{1, 2\}$, with at least four elements in V receiving weight 2 and at least one receiving weight 1. Select in V some element v_0 of weight 2, and consider the graph on V in which all vertices are two-by-two adjacent, except that v_0 is nonadjacent to all vertices with weight 1. Then (G, μ) defines a facet, as we now check by applying Corollary 8. First, notice $\alpha(G, \mu) = 3$ is easily verified. Moreover, any sets $\{u, v\}$ and $\{u, v, w\}$ of vertices satisfying $\mu(u) = \mu(v) = \mu(w) = 2$ are tight. This gives in the system of (G, μ) the equations

$$y_u + y_v + y_{\{u,v\}} = 3, \tag{20}$$

$$y_u + y_v + y_w + y_{\{u,v\}} + y_{\{v,w\}} + y_{\{w,u\}} = 3. \tag{21}$$

By subtracting, we get

$$y_w + y_{\{w,v\}} + y_{\{w,u\}} = 0. \tag{22}$$

Keeping v and w fixed but letting u vary, we conclude that $y_{\{w,u\}}$ is constant. As v and w are arbitrary, we derive that for all distinct vertices w, u of weight 2, the variable $y_{\{w,u\}}$ has some constant value, say $-\lambda$. Going back to Equations (22) and (20), we get $y_w = 2 \cdot \lambda$ and then $\lambda = 1$.

Now for any vertices u, t with $\mu(u) = 2$, $\mu(t) = 1$, we see that $\{v_0, t\}$ $\{v_0, t, u\}$ are tight, hence

$$y_{v_0} + y_t = 3, \tag{23}$$

$$y_{v_0} + y_t + y_u + y_{\{v_0,u\}} + y_{\{t,u\}} = 3. \tag{24}$$

As we yet know $y_{v_0} = y_u = 2$ and $y_{\{v_0,u\}} = -1$, we derive $y_t = 1$ and $y_{\{t,u\}} = -1$. Finally, using in a similar way the tight sets $\{v_0, t\}$ $\{v_0, t, t'\}$ where $\mu(t') = 1$, we get $y_{\{t,t'\}} = -1$.

6. Necessary and Sufficient Conditions

We now turn to more tractable conditions for the graphical inequality to define a facet, starting with necessary conditions. Removing any degenerate node from a weighted graph leaves unchanged the graphical inequality. Consequently, it is often convenient to assume that G has no degenerate node. If G has only one node, then Inequality (11) is either vacuous or a trivial inequality. If G has two nondegenerate nodes, then it follows from Theorem 4 and Proposition 5 that (11) is facet defining if and only if G is complete and $\mu = \mathbb{1}$, in which case (11) is an axiomatic inequality.

If there exist weighted graphs (H_1, λ_1) and (H_2, λ_2) both having at least one nondegenerate node and satisfying the three following conditions:

(i) $G = H_1 \cup H_2$ and $E(H_1) \cap E(H_2) = \varnothing$,

(ii) $\mu = \bar{\lambda}_1 + \bar{\lambda}_2$,

(iii) $\alpha(G, \mu) = \alpha(H_1, \lambda_1) + \alpha(H_2, \lambda_2)$,

where $\bar{\lambda}_i(v) = \lambda_i(v)$ if $v \in V(H_i)$ and $\bar{\lambda}_i(v) = 0$ if $v \in V(G) \setminus V(H_i)$ (for $i \in \{1, 2\}$), then we say that (G, μ) is *decomposed into* (H_1, λ_1) and (H_2, λ_2), and that (G, μ) is *decomposable.* In this case, Inequality (11) is the sum of two nonvacuous valid inequalities, namely, the graphical inequality of (H_1, λ_1) and the graphical inequality of (H_2, λ_2), hence is not facet defining. This proves the following lemma.

Lemma 10 *Let (G, μ) be a weighted graph with $V(G) \subseteq X$. If (G, μ) is decomposable, then (G, μ) is not facet defining.*

Without going into details, we notice that the converse of Lemma 10 does not hold. A counterexample is obtained by putting the constant weight 2 on a graph which has at least 14 vertices and which is the complement of a perfect matching. We now apply this lemma to obtain more concrete conditions which must necessarily be satisfied by facet-defining weighted graphs. In the next proposition, $\mu - e$ denotes the weight function defined by $(\mu - e)(v) = \mu(v) - 1$ if v is an end of edge e and $(\mu - e)(v) = \mu(v)$ otherwise.

Proposition 11 *Let (G, μ) be a weighted graph with $V(G) \subseteq X$, having at least three nodes and no degenerate node. If (G, μ) defines a facet of $P_{\mathrm{Bio}}^{X \times X}$, then the following conditions necessarily hold:*

(C1) G is connected;

(C2) for all $e \in E(G)$, we have $\alpha(G-e,\mu) = \alpha(G,\mu)+1$;

(C3) for all $e \in E(G)$, we have $\alpha(G-e,\mu-e) = \alpha(G,\mu)$;

(C4) for all $v \in V(G)$, we have $1 \le \mu(v) \le \deg(v)-1$.

Proof. By contradiction, suppose that G is not connected. Then $G = H_1 \cup H_2$ for some graphs H_1 and H_2 with disjoint node sets. Let λ_1 and λ_2 denote the restrictions of μ to $V(H_1)$ and $V(H_2)$ respectively. It is easily verified that $\alpha(G,\mu) = \alpha(H_1,\lambda_1)+\alpha(H_2,\lambda_2)$. By Lemma 10, Inequality (11) is not facet defining, a contradiction. Consequently, Condition (C1) holds.

Let $e = \{a,b\}$ be an edge of G. Because the worth of every set of nodes in $(G-e,\mu)$ is at least its worth in (G,μ) and at most 1 plus its worth in (G,μ), we have $\alpha(G,\mu) \le \alpha(G-e,\mu) \le \alpha(G,\mu)+1$. If equality holds in the first inequality, then (G,μ) can be decomposed into the weighted graph formed by edge e with zero weights on the nodes and $(G-e,\mu)$. By Lemma 10, Condition (C2) holds. Because the worth of every set of nodes in $(G-e,\mu-e)$ is at least its worth in (G,μ) minus 1 and at most its worth in (G,μ), we have $\alpha(G,\mu)-1 \le \alpha(G-e,\mu-e) \le \alpha(G,\mu)$. If equality holds in the first inequality, then (G,μ) can be decomposed into the weighted graph formed by edge e with unit weights on the nodes and $(G-e,\mu-e)$. By Lemma 10, Condition (C3) holds.

Let v be a node of G with $\mu(v) \le 0$. By (C1), G is connected, so there is a node w of G adjacent to v. By (C2), there exists a tight set S containing both v and w. But S cannot be tight because the worth of $S \setminus \{v\}$ is strictly greater than the worth of S, a contradiction. Finally, let v be a node of G with $\mu(v) \ge \deg(v)$, and let w be any neighbor of v. By (C3), there is a tight set S such that neither v nor w belongs to S. The worth of $S \cup \{v\}$ is strictly greater than the worth of S, a contradiction. ■

Observe that Conditions (C2) and (C3) of Proposition 11 are respectively equivalent to:

(C2') for all $e \in E(G)$, there is a tight set containing both ends of e;

(C3') for all $e \in E(G)$, there is a tight set containing neither end of e.

The necessary conditions given in Proposition 11 are not sufficient. For instance the weighted graph depicted in Figure 1 satisfies Conditions (C1)–(C4) but is not facet defining because it can be decomposed into two triangles with unit weights on the nodes (when the weight of a node is not shown in such a figure, it equals 1; this convention will be used also later).

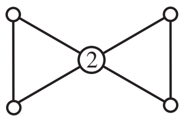

Figure 1. A decomposable weighted graph

We now provide a manageable, sufficient condition for the weighted graph (G, μ) to be facet defining.

Proposition 12 *Let (G, μ) be a connected weighted graph satisfying $V(G) \subseteq X$, $1 \leq \mu(v) \leq deg(v) - 1$ for every node v, and also the following condition: For any nodes $v, w_1, w_2, \ldots, w_k$ with $k = \mu(v)$ and $\{v, w_1\}, \{v, w_2\}, \ldots, \{v, w_k\} \in E(G)$, there exists a tight set containing v, w_1, w_2, $\ldots$, w_k. Then the graphical inequality of (G, μ) is facet defining for $P_{\text{Bio}}^{X \times X}$.*

Proof. Our goal is to apply Corollary 8. Keeping notation as in the statement, we see that $T \setminus \{v\}$ is also a tight set. The difference of the two equations (13) for T and $T \setminus \{v\}$ gives

$$y_v + y_{\{v,w_1\}} + y_{\{v,w_2\}} + \ldots + y_{\{v,w_k\}} \; = \; 0. \tag{25}$$

As $k \leq \deg(v) - 1$, we can write a similar equation for any other choice of k vertices adjacent to v. Taking the difference of two such equations, we derive $y_{\{v,w\}} = y_{\{v,w'\}}$ for any vertices w, w' adjacent to v. Because the graph G is connected, all variables $y_{\{v,w\}}$ for $\{v, w\} \in E(G)$ must take the same value, say $-\lambda$. Back to Equation (25), we now derive $y_v = \mu(v) \cdot \lambda$. Finally, Equation (13) gives $\lambda = 1$. The thesis follows from Corollary 8. ■

The sufficient condition provided in Proposition 12 is not necessary for facet-definiteness: a counter-example is given by Example 9.

7. Applications

Now putting to use the various conditions established so far, we provide two variants of results known for the linear ordering polytope, plus additional findings.

The following proposition is the translation to biorder polytopes of a result on linear ordering polytopes obtained by Koppen [13]. We need some further notions about an unweighted graph G. The *stability number* $\alpha(G)$ of G is the size of the largest stable set in $V(G)$; notice $\alpha(G) = \alpha(G, \mathbb{1})$. The graph G is *stability critical* if $\alpha(G) > \alpha(G - e)$ for every edge e of G (this is exactly (C2) in the case $\mu = \mathbb{1}$). For information and additional references about stability-critical graphs, the reader is refered to Section 7 of Koppen [13]. Notice that our facet-defining graphs generalize stability-critical graphs because of the following proposition. (We mention in passing that another generalization, also linked to facets of a polytope, appear in Lipták and Lovász [15].)

Proposition 13 *Let G be a graph with $V(G) \subseteq X$. Then $(G, \mathbb{1})$ is facet defining if and only if G is connected and stability critical.*

Proof. Assume $|G| \geq 3$. The forward direction follows from Proposition 11 (Condition (C2) is equivalent to G being stability critical). The backward direction follows from Proposition 12, because any edge of $(G, \mathbb{1})$ is contained in some tight set when G is stability critical. ■

Here are further propositions which can be used to produce additional facet-defining graphs.

Proposition 14 *Let (G, μ) be a weighted graph with $V(G) \subseteq X$ and $\mu \geq 1$. Assume some particular node v_0 satisfies $\mu(v_0) = \alpha(G, \mu)$. Then (G, μ) is facet defining if and only if the graph $G - v_0$ is stability critical and connected.*

Proof. The assumption $\mu(v_0) = \alpha(G, \mu)$ has several consequences. First, for $v \in V(G) \setminus \{v_0\}$, we have $\mu(v) = 1$ and $\{v, v_0\} \in E(G)$ (otherwise $\{v_0, v\}$ would have a worth greater than $\alpha(G, \mu)$). Second, the tight sets of (G, μ) containing v_0 are exactly the sets $\{v_0\} \cup S$ with S the empty set or a stable set in $G - v_0$, and those not containing v_0 (assuming there is at least one such) are exactly the tight sets in $(G - v_0, \mathbb{1})$. Thus, in the system of (G, μ), the equations corresponding to tight sets containing v_0 carry exactly the following information: $y_{v_0} = \alpha(G, \mu) = \mu(v)$, and $y_{\{v_0, v\}} = -y_v$ for $\{v_0, v\} \in E(G)$. The other equations form exactly the system of the weighted graph $(G - v_0, \mathbb{1})$. The statement then follows from the application of Corollary 8 to both weighted graphs (G, μ) and $(G - v_0, \mathbb{1})$. ■

The backward direction of the next proposition has an analog for linear ordering polytopes, proved by Leung and Lee [14] and independently by Suck [20].

Proposition 15 *Let (G, μ) be a complete weighted graph with at least three nodes and such that $V(G) \subseteq X$. Then (G, μ) is facet defining if and only if there exists some natural number m such that $1 \leq m \leq |G| - 2$ and $\mu(v) = m$ for all v in $V(G)$.*

Proof. To prove the forward implication, let $v_{\max}$ (resp. $v_{\min}$) be a node at which μ takes its maximum (resp. minimum) value. By Proposition 11 (C3), there exists a tight set S with $v_{\max} \notin S$. Then for any w in S, we must have $\mu(w) = \mu(v_{\max})$: otherwise, exchanging w with $v_{\max}$ would increase the worth of the set (because two sets of vertices having the same cardinality contain the same number of edges). Similarly, there exists a tight set T containing $v_{\min}$, and T must contain any node w such that $\mu(w) = \mu(v_{\max})$. Thus $S \subsetneq T$.

Now $\mu(v_{\max}) \leq |S|$ (otherwise adding $v_{\max}$ to S would result in a set with strictly higher worth than the tight set S). Also, $\mu(v_{\min}) \geq |T| - 1$ (otherwise deleting $v_{\min}$ from T would give a set with strictly larger worth than T). Hence $\mu(v_{\max}) \leq \mu(v_{\min})$, and μ is constant. The rest of the thesis follows from Proposition 11.

To prove the backward direction, assume μ has constant value M. The worth of any k-set in $V(G)$ is $k \cdot M - \binom{k}{2}$. Hence, a set of nodes is tight iff it is of cardinality M or $M + 1$. So Proposition 12 directly applies and the graphical inequality of (G, μ) is facet defining. ■

Propositions 13 and 15 show the usefulness of Proposition 12 for producing facets of the biorder polytope. We now turn to another construction, which is illustrated in Figure 2.

Proposition 16 *Let (G, μ) be a facet-defining weighted graph of which $\{v, w\}$ is an edge, with $|G| \geq 3$ and G having no degenerate node. Define a new weighted graph $(\tilde{G}, \tilde{\mu})$ by keeping all nodes of G with their weights, making v and w nonadjacent, and inserting a complete graph K_p on p vertices with weight 1, where $p \geq 2$, and additional edges from the nodes of K_p to either v or w, in such a way that any node of K_p is connected to exactly one of v and w, and at least one such node is connected to v and another one to w. Then the resulting weighted graph $(\tilde{G}, \tilde{\mu})$ is facet defining if and only if (G, μ) is facet defining. In any case, we have*

$$\alpha(\tilde{G}, \tilde{\mu}) \quad = \quad 1 + \alpha(G, \mu). \tag{26}$$

Proof. Any subset S of $V(\tilde{G})$ has worth in $(\tilde{G}, \tilde{\mu})$ at most one plus the worth of $S \cap V(G)$ in (G, μ); moreover, equality can be attained

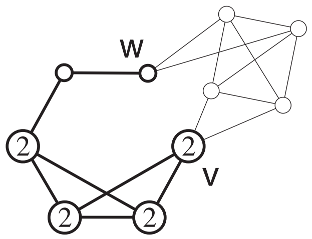

Figure 2. Illustration of Proposition 16: the weighted graph $\tilde{G}$, after K_4 was added to G.

here (for instance by taking any tight set of (G, μ) containing both v and w, which exists by Proposition 11). This establishes Equation (26).

The following four assertions hold true for all vertices a and b of $V(K_p)$ and tight set T of (G, μ).

1. If T avoids v and w, then both sets $T \cup \{a\}$ and $T \cup \{a, b\}$ are tight in $(\tilde{G}, \tilde{\mu})$.

2. If T contains v but not w and if neither a nor b are adjacent to v, both sets $T \cup \{a\}$ and $T \cup \{a, b\}$ are tight in $(\tilde{G}, \tilde{\mu})$.

3. A similar result holds when v and w are interchanged.

4. If T contains v and w, then both sets T and $T \cup \{a\}$ are tight in $(\tilde{G}, \tilde{\mu})$.

Assume now that the original graph (G, μ) is facet defining. By Corollary 8, its system has only the obvious solution. Using the same corollary, we proceed by contradiction to prove that $(\tilde{G}, \tilde{\mu})$ is facet defining. So let $\tilde{y}$ be a solution to the system of $(\tilde{G}, \tilde{\mu})$ which is distinct from the obvious solution. By considering tight sets listed above, it is easy to derive $\tilde{y}_a = \tilde{y}_b = -\tilde{y}_{\{a,b\}}$ for a, $b \in V(K_p)$, and when $\{v, b\}$, $\{w, a\} \in E(\tilde{G})$ moreover $\tilde{y}_a = -\tilde{y}_{\{v,b\}} = -\tilde{y}_{\{w,a\}}$. We next consider two cases.

If $\tilde{y}_a = 1$ for $a \in V(K_p)$, we set for $u \in V(G)$

$$y_u = \tilde{y}_u \tag{27}$$

and

$$y_{\{v,w\}} = -1, \qquad y_{\{t,u\}} = \tilde{y}_{\{t,u\}} \tag{28}$$

for $\{t, u\} \in E(G) \setminus \{\{v, w\}\}$. Then y is a solution to the system of (G, μ) which differs from the obvious one, in contradiction to the fact that (G, μ) is facet defining.

If $\tilde{y}_a \neq 1$, we may assume $\tilde{y}_a = 0$ (because the obvious solution has 1 for its a-coordinate). We then set for $u \in V(G)$

$$y_u = \tilde{y}_u \frac{\alpha(G,\mu)}{\alpha(G,\mu)+1} \tag{29}$$

and for $\{t,u\} \in E(G)$

$$y_{\{t,u\}} = \begin{cases} \tilde{y}_{\{t,u\}} \frac{\alpha(G,\mu)}{\alpha(G,\mu)+1}, & \text{if } \{t,u\} \neq \{v,w\}, \\ 0 & \text{if } \{t,u\} = \{v,w\}. \end{cases} \tag{30}$$

Then y is again a solution to the system of (G,μ) distinct from the obvious one, a contradiction again.

This completes the proof of sufficiency. The necessary part is done using similar arguments. ∎

More facet-defining graphs will now be derived. We first show that the worth of a set S of nodes in $(G, \deg - \mu)$ equals the worth of its complement in (G,μ) minus the worth of $V = V(G)$ in (G,μ). This is due to the following calculations, where $E(S, V \setminus S)$ denotes the set of edges of G connecting S to $V \setminus S$:

$$\begin{aligned}
(\deg - \mu)(S) - ||G[S]|| &= \sum_{v \in S} \deg(v) - \mu(S) - ||G[S]|| \\
&= 2||G[S]|| + |E(S, V \setminus S)| - \mu(S) - ||G[S]|| \\
&= ||G|| - ||G[V \setminus S]|| - \mu(S) \\
&= \Big(\mu(V \setminus S) - ||G[V \setminus S]||\Big) - \Big(\mu(V) - ||G||\Big).
\end{aligned}$$

Consequently, we have

$$\alpha(G, \deg - \mu) = \alpha(G,\mu) - \Big(\mu(V) - ||G||\Big). \tag{31}$$

Moreover, $S \mapsto V \setminus S$ defines a bijection between the tight sets of (G,μ) and the tight sets of $(G, \deg - \mu)$. This simple observation opens the way to the next proposition. We mention that the similar statement for the linear ordering polytope is apparently new, and reported in Doignon and Fiorini [8].

Proposition 17 *For any weighted graph (G,μ) with at least three nodes and no degenerate node, the two following assertions are equivalent:*

(i) (G,μ) is facet defining;

(ii) $(G, \deg - \mu)$ *is facet defining.*

Proof. Without loss of generality, we consider $P_{\text{Bio}}^{X \times X}$ with $X = V(G)$ (cf. Proposition 5). It is sufficient to show that (i) implies (ii), so suppose (G, μ) is facet defining. By Corollary 8, the system of (G, μ) has only the obvious solution. We now derive the same assertion for the system of $(G, \deg - \mu)$.

A bijective, affine permutation of the affine space A consisting of all points with real coordinates c_v and $c_{\{v,w\}}$, for $v \in V(G)$ and $\{v, w\} \in E(G)$, is defined by

$$c'_v = 1 - c_v, \tag{32}$$
$$c'_{\{v,w\}} = 1 + c_{\{v,w\}} - c_v - c_w. \tag{33}$$

We now view Equation (13), for the tight set T of (G, μ), as its vector of coefficients, that we call the T-point in A. As easily checked, f transforms this T-point into another point of A which is the $(V \setminus T)$-point with $V \setminus T$ a tight set of $(G, \deg - \mu)$. Notice that the affine space generated by all T-points of (G, μ) does not contain the origin (since the obvious solution to Equation (13) provides a non-homogeneous linear equation satisfied by all T-points); a similar result holds for the graph $(G, \deg - \mu)$. Thus the system of (G, μ) and the system of $(G, \deg - \mu)$ have solution spaces with the same dimension because their matrices have the same rank. ■

Proposition 17 generates a fairly large collection of facet-defining inequalities: for instance, it may be applied to any stability-critical graph G with $\mu = \mathbb{1}$. Finally, we mention that applying the complementation automorphism to any facet-defining graphical inequality delivers a facet-defining inequality for $P_{\text{Bio}}^{X \times X}$ (which generally does not satisfy our Working Assumption). Nevertheless, we suspect that the biorder polytope has still many more facets, in a sense which needs to be explained by further investigation. An obvious generalization of the investigation we have reported stems from the replacement, in our Working Assumption, of Condition 2 with

2* $c_{ij} \in \{\ldots, -2, -1, 0\}$ for all pairs (i, j) in $(X \times Y) \setminus f$.

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