Weak order polytopes

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Abstract

The weak order polytope $\mathbf{P}^n_{WO}$ is defined as the convex hull in $\mathbb{R}^{n(n-1)}$ of the characteristic vectors of all reflexive weak orders on $\{1, 2, \ldots, n\}$. The paper focuses on facet-defining inequalities, vertex adjacency and symmetries. We relate $\mathbf{P}^n_{WO}$ to the theories of probabilistic choice and preference aggregation, prove a basic lifting lemma that carries facet defining inequalities for $\mathbf{P}^n_{WO}$ into $\mathbf{P}^{n+1}_{WO}$, identify complete sets of facet-defining inequalities for $n \leq 4$, give a complete account of vertex adjacency, and determine all symmetries of $\mathbf{P}^n_{WO}$.

Key words: Weak order. Polytope. Preference. Choice probability.

1 Introduction

The present study explicates structural aspects of the weak order polytope $\mathbf{P}^n_{WO}$ for $n \geq 2$. The vertices of $\mathbf{P}^n_{WO}$ represent members of the family of reflexive weak orders $\precsim$ on $\mathbf{n} = \{1, 2, \ldots, n\}$. Our motivation is threefold. The first aspect is the pre-eminence of weak orders in theories of preference, comparative probability, and social choice as represented during the past century by de Finetti [7], von Neumann and Morgenstern [32], Arrow [2], Savage [30], Anscombe and Aumann [1], Fishburn [12,13] and Wakker [33]. The second is probabilistic preference and choice theory pioneered by Thurstone [31] and popularized by Marschak [24] with later contributions by Dridi [9], McFadden and Richter [27], Cohen and Falmagne [6], Gilboa [16], McLennan [28], Fishburn [14], Leung and Lee [23] and Koppen [22], among others. The third aspect

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is the development of the theory of polytopes during the past two decades for families of binary relations as an aid to understanding and analyzing problems in mathematical programming and combinatorial optimization. The most relevant contributions to polytope theory for the present study include the analyses of linear ordering polytopes by Young [34], Grötschel, Jünger, and Reinelt [17,18], and Bolotashvili, Kovalev and Girlich [3], and investigations of weak order polytopes by Gurgel [19], Gurgel and Wakabayashi [20] and Doignon and Fiorini [8].

We begin with a discussion of binary relations and associated polytopes, follow this by an outline of ensuing sections, and conclude the introduction with comments on probabilistic preferences and social choice. Throughout, $\mathbf{n} = \{1, \ldots, n\}$, $n \geq 2$, each coordinate of $\mathbb{R}^{n(n-1)}$ is identified with a distinct member of $\{(i,j) : i,j \in \mathbf{n},\ i \neq j\}$, and x_{ij} or $x_{(i,j)}$ denotes the value of $x \in \mathbb{R}^{n(n-1)}$ on that coordinate. Given a binary relation $\precsim$ on $\mathbf{n}$, we define its induced relations $\prec$ and $\sim$ on $\mathbf{n}$ by

$$
\begin{aligned}
&i \prec j \quad \text{if} \quad i \precsim j \text{ and not } (j \precsim i) \\
&i \sim j \quad \text{if} \quad i \precsim j \text{ and } j \precsim i.
\end{aligned}
$$

A binary relation on $\mathbf{n}$ is a *preorder* if it is reflexive and transitive, a *weak order* if it is a complete preorder, and a *linear order* if it is an antisymmetric weak order. An *ordered partition* of $\mathbf{n}$ is a tuple $(A_1, \ldots, A_m)$ of nonempty and mutually disjoint subsets of $\mathbf{n}$ whose union equals $\mathbf{n}$. Ordered partitions on $\mathbf{n}$ bijectively correspond to weak orders on $\mathbf{n}$. The weak order $\precsim$ corresponding to ordered partition $(A_1, \ldots, A_m)$ has $i \precsim j$ if and only if $i \in A_\alpha$ and $j \in A_\beta$ for some α, β in $\mathbf{m} = \{1, 2, \ldots, m\}$ such that $\alpha \leq \beta$. We will sometimes identify a weak order with its corresponding ordered partition. When B is an ordering relation on $\mathbf{n}$, we sometimes express "$i\,B\,j$" by "$i \precsim j$ in B", "$i\,B\,j$ and not $(j\,B\,i)$" by "$i \prec j$ in B", and so forth. The *dual* of B is $B^d = \{(j,i) : (i,j) \in B\}$.

In decision theory, $i \precsim j$ could mean that i is not preferred to (is not more probable than, is not socially more acceptable than) j. With $\succ$ the inverse of $\prec$, $i \succ j$ indicates that i is strictly preferred to j; and $i \sim j$ that i and j are equally valued.

Let $\mathcal{B}$ denote a family of binary relations on $\mathbf{n}$. We assume that all $B \in \mathcal{B}$ are reflexive, or all are irreflexive, so that the relationship of every i to itself is unambiguously uniform. The *polytope* of $\mathcal{B}$ is the convex hull $\mathbf{P}^n_{\mathcal{B}}$ of the set

$\{x(B) : B \in \mathcal{B}\}$ where, for all ordered pairs (i, j) of distinct points in $\mathbf{n}$,

$$x(B)_{ij} = \begin{cases} 1 \text{ if } i\,B\,j \\ 0 \text{ otherwise.} \end{cases}$$

The vertex set of $\mathbf{P}^n_{\mathcal{B}}$ is $\{x(B) : B \in \mathcal{B}\}$. When $\mathcal{B}$ is the family of all weak orders on $\mathbf{n}$, $\mathbf{P}^n_{\mathcal{B}}$ is the *weak order polytope* $\mathbf{P}^n_{WO}$; when $\mathcal{B}$ is the family of linear orders on $\mathbf{n}$, $\mathbf{P}^n_{\mathcal{B}}$ is the *linear ordering polytope* $\mathbf{P}^n_{LO}$.

The *dimension* $\dim(\mathbf{P})$ of polytope $\mathbf{P}$ is the dimension of its affine hull. The polytope in $\mathbb{R}^N$ is said to be *full-dimensional* if its dimension equals N. A linear inequality $\langle a, x\rangle \leq b$ is *valid* for $\mathbf{P}$ if every point in $\mathbf{P}$ satisfies it. Here and henceforth, $\langle \cdot, \cdot \rangle$ denotes inner product. A *face* of $\mathbf{P}$ is the set of points of $\mathbf{P}$ that satisfy a valid inequality for $\mathbf{P}$ with equality. For instance, the linear ordering polytope $\mathbf{P}^n_{LO}$ is a face of the weak order polytope $\mathbf{P}^n_{WO}$. The faces of dimension 0, 1 and $\dim(\mathbf{P}) - 1$ are referred to as *vertices, edges*, and *facets*, respectively.

A fundamental theorem of polytopes (e.g., Ziegler [35], p. 29) states that every polytope $\mathbf{P}$ has an equivalent characterization by a finite system of linear inequalities. When the members of such a system are indexed by $u = 1, \ldots, U$, we have

$$\mathbf{P} = \bigcap_{u=1}^{U} \{x \in \mathbb{R}^N : \langle a_u, x\rangle \leq b_u\}.$$

If $\mathbf{P}$ is full-dimensional, the inequalities that remain in a minimum set that characterizes $\mathbf{P}$ will be *facet-defining inequalities*, or *FDI's* for short.

Because $\dim(\mathbf{P}^n_{WO}) = n(n-1)$ (see Gurgel [19]), there is no equality for weak order polytopes and only FDI's are sought for. The FDI's for weak orders that follow directly from $x(\precsim)_{ij} \in \{0, 1\}$, completeness and transitivity are: for all $x \in \mathbf{P}^n_{WO}$ and all distinct $i, j, k \in \mathbf{n}$,

$$x_{ij} \leq 1 \tag{1}$$
$$-x_{ij} - x_{ji} \leq -1 \tag{2}$$
$$x_{ij} + x_{jk} - x_{ik} \leq 1, \tag{3}$$

with $x_{ij} \geq 0$ by the first two inequalities. The first two inequalities completely characterize $\mathbf{P}^2_{WO}$. We prove later that all three completely characterize $\mathbf{P}^3_{WO}$ but are insufficient for $n = 4$.

The next section of the paper establishes the lifting lemma which shows that every FDI for $\mathbf{P}_{WO}^{n}$ extends in a straightforward way to an FDI for $\mathbf{P}_{WO}^{n+1}$. Section 3 briefly discusses FDI's (1)–(3) and presents a complete set for $n = 4$, obtained with the aid of the **porta** program, developed by Christof [4]. Section 4 offers a combinatorial description of the vertex-adjacency relation of $\mathbf{P}_{WO}^{n}$ and relates this to transitive orientations of a generalized graph with two types of edges and a betweenness relation. Section 5 concludes the paper with a characterization of combinatorial symmetries of $\mathbf{P}_{WO}^{n}$.

We conclude our introduction with connections to decision theory, where it has been customary to consider the probability p_{ij} for $i \neq j$ that i is strictly preferred to j ($i \succ j$) or that i will be chosen over j in a choice between the two. Although p_{ij} usually applies to an individual's preferences or choices between decision alternatives, it is relevant also to social choice where it could denote the proportion of individuals in a group who favor i over j. The p_{ij} are known as *binary choice probabilities*, and $\{p_{ij} : i, j \in \mathbf{n}, i \neq j\}$ is a *binary probability system.*

It is often assumed that $p_{ij}+p_{ji} = 1$, which could indicate that an individual is never indifferent between two alternatives, or that a definite choice is always required between i and j, or that every individual in a group expresses a preference for i over j or j over i in a comparison between them. However, it may be more natural to assume only $p_{ij} + p_{ji} \leq 1$ to allow for expressed indifference, refusal to make a definite choice between i and j, or abstentions in binary votes.

Three restrictions on binary choice probabilities for distinct indices are

$$\begin{aligned} p_{ij} &\geq 0 \\ p_{ij} + p_{ji} &\leq 1 \\ p_{ik} &\leq p_{ij} + p_{jk}. \end{aligned}$$

The third restriction is the *triangle inequality* for a binary probability system. Unlike the first two restrictions, it is vulnerable to refutation by individual choice or voting data and should be viewed as a testable assumption. This could also be said of $p_{ij} + p_{ji} = 1$ when it is presumed instead of $p_{ij} + p_{ji} \leq 1$.

The natural connection between binary choice probabilities and our interpretation of x_{ij} is specified by

$$p_{ij} = 1 - x_{ji}. \tag{4}$$

Given (4), the displayed restrictions on p in the preceding paragraph are identical to (1)–(3), and $p_{ij} + p_{ji} = 1$ is tantamount to the equations $x_{ij} + x_{ji} = 1$ for the linear ordering polytope.

A central question addressed by Marschak [24] and others for binary probability systems is: what must be true of a binary probability system so that there exists a probability distribution μ on the set $\mathcal{L}$ of linear orders on $\mathbf{n}$ for which

$$p_{ij} = \sum\{\mu(L) : L \in \mathcal{L} \text{ and } i \succ j \text{ in } L\}$$

for all distinct i and j ? The question is motivated by the choice model in which a member of $\mathcal{L}$ is randomly chosen according to some probability distribution and the relationship between i and j in the chosen L is then revealed. The answer is that such a μ exists if and only if the x that corresponds to p by (4) is in $\mathbf{P}^n_{LO}$. Equivalently, because $(1,\ldots,1)-x$ is in $\mathbf{P}^n_{LO}$ if and only if $x \in \mathbf{P}^n_{LO}$, there exists such a μ if and only if $p \in \mathbf{P}^n_{LO}$. Hence all FDIs for $\mathbf{P}^n_{LO}$ apply directly to the binary probability system.

The situation for $p_{ij}+p_{ji} \leq 1$ is similar except for the final duality observation. In particular, with $\mathcal{W}$ the set of weak orders on $\mathbf{n}$, there exists a probability distribution μ on $\mathcal{W}$ for which

$$p_{ij} = \sum\{\mu(W) : W \in \mathcal{W} \text{ and } i \succ j \text{ in } W\}$$

for all distinct i and j if and only if $(1,\ldots,1)-p$ is in $\mathbf{P}^n_{WO}$. Hence all FDIs for $\mathbf{P}^n_{WO}$ translate into necessary and sufficient conditions for μ under (4).

2 Lifting lemma

In this section we prove that trivial lifting of facet-defining inequalities applies to weak order polytopes. Gurgel [19] states and prove a weak version of the trivial lifting lemma for the weak order polytopes. Her result is less general than Theorem 1 because it needs an extra hypothesis on the facet being lifted. In particular, the latter must contain the characteristic vector of a linear order.

Theorem 1 (lifting lemma) *Let $\langle a, x\rangle \leq b$ be a facet-defining inequality for $\mathbf{P}^n_{WO}$, and let $\bar{a}$ in $\mathbb{R}^{(n+1)n}$ be defined by*

$$\bar{a}_{ij} = \begin{cases} a_{ij} \text{ if } i,j \in \mathbf{n}, i \neq j \\ 0 \quad \text{if } (i = n+1, j \in \mathbf{n}) \text{ or } (i \in \mathbf{n}, j = n+1). \end{cases}$$

Then the inequality $\langle \bar{a}, x\rangle \leq b$ defines a facet of $\mathbf{P}^{n+1}_{WO}$.

PROOF. We refer to inequality $\langle a, x\rangle \leq b$ as the *original inequality*, and to inequality $\langle \bar{a}, x\rangle \leq b$ as the *lifted inequality*. The lifted inequality is easily seen to be valid for $\mathbf{P}_{WO}^{n+1}$. Now suppose that the face $\bar{F}$ it defines is contained in a hyperplane $\bar{H}$ distinct from the hyperplane associated to the lifted inequality. Let $\langle \bar{c}, x\rangle = d$ be an equation for $\bar{H}$, and let y, z be two points in $\mathbb{R}^n$ defined by $y_i = \bar{c}_{(i,n+1)}$ and $z_j = \bar{c}_{(n+1,j)}$. We will show that $y = z = 0$. This contradicts the fact that the original inequality defines a facet of $\mathbf{P}_{WO}^{n}$, so the lifted inequality is facet defining for $\mathbf{P}_{WO}^{n+1}$.

Let $W = (A_1, \ldots, A_t)$ be a weak order on $\mathbf{n}$ whose characteristic vector $w = x(W)$ is a vertex of the facet defined by the original inequality $\langle a, x\rangle \leq b$, and let $s \in \mathbf{t} = \{1, 2, \ldots, t\}$. The three extensions

$$\begin{aligned}
&(A_1, \ldots, \{n+1\}, A_s, \ldots, A_t),\\
&(A_1, \ldots, A_s \cup \{n+1\}, \ldots, A_t), \text{ and}\\
&(A_1, \ldots, A_s, \{n+1\}, \ldots, A_t)
\end{aligned}$$

of W on $\mathbf{n} \cup \{n+1\}$ determine three vertices of $\bar{F}$. Because these vertices all belong to $\bar{H}$, we have

$$\sum_{i \in A_s} y_i = \sum_{j \in A_s} z_j = 0. \tag{5}$$

For each k in $\mathbf{n}$, we introduce an affine map β_k from $\mathbb{R}^{n(n-1)}$ to $\mathbb{R}^n$ as follows

$$\beta_k(x)_i = \begin{cases} x_{ik} + x_{ki} - 1 & \text{if } i \neq k,\\ 1 & \text{otherwise.} \end{cases}$$

The image of w by β_k is a 0/1-vector whose i-th coordinate is 1 iff $i \sim k$ in W. Using one of these affine maps for some k in A_s, equation (5) can be rewritten $\langle \beta_k(w), y\rangle = \langle \beta_k(w), z\rangle = 0$.

Now let $w_1, \ldots, w_m$ be $m = n(n-1)$ affinely independent vertices of the facet defined by the original inequality. Thus $w_1, \ldots, w_m$ affinely span the hyperplane $H = \{x \in \mathbb{R}^{n(n-1)} : \langle a, x\rangle = b\}$. We have

$$\langle \beta_k(w_l), y\rangle = \langle \beta_k(w_l), z\rangle = 0 \quad \forall k \in \mathbf{n}, \forall l \in \mathbf{m}. \tag{6}$$

As we show below, this system of equations has only the trivial solution $y = z = 0$.

Choose k such that there exist elements i, j distinct from k with $a_{ij} \neq 0$ (this is possible when $n > 2$, we leave it to the reader to work out the case of $n = 2$ separately). It can be easily verified that $\beta_k(H)$ is the hyperplane Π in $\mathbb{R}^n$ defined by equation $x_k = 1$ (it is clear that we have $\beta_k(H) \subseteq \Pi$; to prove the other inclusion we check that the affine subspace $\beta^{-1}(x)$ intersects H for every $x \in \Pi$). This implies that we can find n linearly independent points among $\beta_k(w_1), \ldots, \beta_k(w_m)$, so system (6) has only the trivial solution $y = z = 0$. □

3 Facet-defining inequalities

The present section gives basic FDI's for weak order polytopes. We first discuss the axiomatic inequalities, and then give a complete linear description for $n = 4$. Further FDI's for weak order polytopes can be found in Gurgel [19] and Doignon and Fiorini [8].

The axiomatic inequalities were introduced in Section 1. They are respectively: the *trivial inequalities* (1), the *completeness inequalities* (2) and the *transitivity inequalities* (3). As can be easily verified with a convex hull code (see below), the FDI's for $\mathbf{P}^3_{WO}$ are exactly the axiomatic inequalities. After several successive uses of Theorem 1 and one application of a relabeling symmetry (see Section 5), we conclude that axiomatic inequalities define facets of $\mathbf{P}^n_{WO}$ for all $n \geq 3$.

We now give a complete linear description of the weak order polytopes for $n = 4$. The inequalities are represented diagrammatically (see Figure 1) with isolated vertices omitted. The seven diagrams define seven classes of inequalities, denoted by C_1 through C_7. After a distinct number in $\mathbf{n}$ has been chosen for each node, each diagram defines an inequality $\langle a, x\rangle \leq b$ as follows. Coefficient a_{ij} is equal to 0, 1 or -1 respectively when there is no arc, a plain arc or a dotted arc from node i to node j in the diagram. The right member b can be found in the little circle attached to the picture. The FDI's were found using the **porta** program developed by Christof [4]. Tables 1 and 2 contain supplementary information about $\mathbf{P}^n_{WO}$ for $n \leq 4$.

n	dimension	#vertices	#facets	#facets up to symmetry
2	2	3	3	1
3	6	13	15	3
4	12	75	106	7

Table 1
Aspects of weak order polytopes

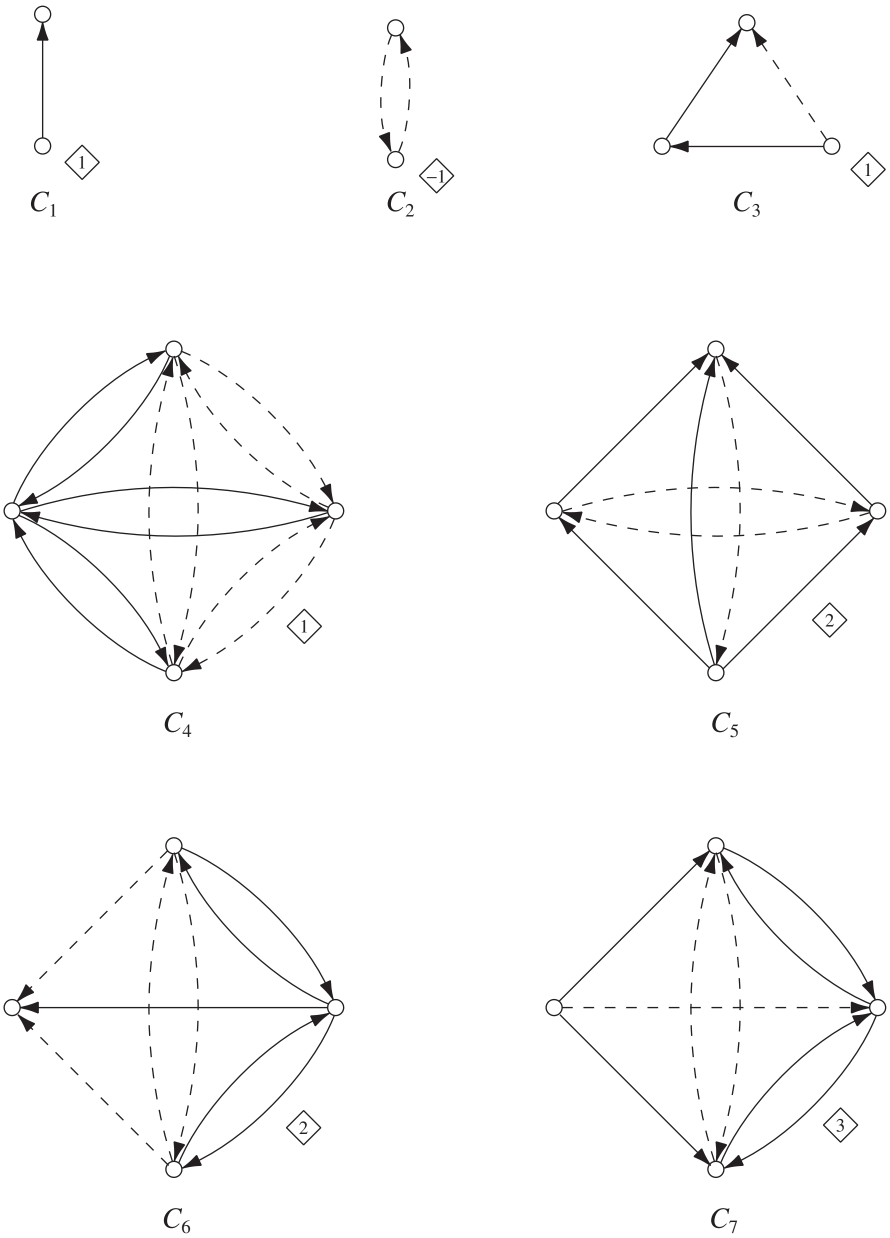

Fig. 1. Facet-defining inequalities for $n \leq 4$

4 Adjacency on weak order polytopes

In this section we give a combinatorial description of the adjacency relation on weak order polytopes, together with a simple and computationally efficient test to decide whether two vertices are adjacent. In a recent paper, Gurgel and Wakabayashi [20] provide necessary and sufficient conditions for adjacency of two vertices of $\mathbf{P}^n_{WO}$ in two particular cases. The results of this section cover all possible situations and extend a characterization due to Young [34] of the adjacency relation on linear ordering polytopes.

facet class	#facets in class	#vertices in facet
C_1	12	44
C_2	6	62
C_3	24	44
C_4	4	24
C_5	12	29
C_6	24	13
C_7	24	18
total	106	

Table 2
Numbers of facets and vertices in facets for $n = 4$

Our characterization of adjacency on weak order polytopes makes use of certain graphs with additional structure that we call 'bicolored graphs'. A *bicolored graph* is a quadruple $G = (V, B, R, \tau)$, where V is any set, B and R are disjoint irreflexive and symmetric relations on V, and $\tau : \mathcal{C} \to V$ is a map from $\mathcal{C} = \{\{i, j, k\} : i, j, k \in V \text{ and } iRjRkRi\}$ to V satisfying $\tau(C) \in C$ for all C in $\mathcal{C}$. Sets V, B and R are respectively referred to as the *vertex set*, the *blue edge set* and the *red edge set* of G. The *edge set* of G is $E = B \cup R$. The elements of V, B, R and E are called *vertices, blue directed edges, red directed edges* and *directed edges* of G, respectively. Each directed edge of G is an ordered pair (i, j) with i and j distinct in V; its *opposite* directed edge — which also belongs to the edge set of G — is (j, i). Thus an *edge* of G corresponds to an unordered pair of opposite directed edges of G. Map τ is the *articulation vertex map* and its domain $\mathcal{C}$ is the set of *red 3-cliques* of G.

Suppose that W_1 and W_2 are two weak orders on a set X. The *conflict graph* $G = G(W_1, W_2)$ of W_1 and W_2 is the bicolored graph defined as follows. The vertex set of G is X. A pair (i, j) is a blue directed edge of G exactly when i and j are equivalent neither in W_1 nor in W_2, and are ordered oppositely in W_1 and W_2. A pair (i, j) is a red directed edge of G exactly when i and j are equivalent in one of W_1 and W_2, but not both. Then a red 3-clique $\{i, j, k\}$ of G is such that i, j and k are equivalent in W_1 or W_2 and linearly ordered in the other; we then set $\tau(\{i, j, k\})$ equal to j if j is between i and k in this linear order. An example with $|X| = 9$ can be found in Figure 2 (plain and dotted lines are used for blue and red edges, respectively, the articulation vertex map τ can easily be deduced from the picture; for instance we have $\tau(\{2, 6, 8\}) = 2$).

Let $G = (V, B, R, \tau)$ be a bicolored graph. A *transitive orientation* of G is a partial order P on V such that $P \cup P^d = B \cup R \cup \{(i, i) : i \in V\}$ and such that $i \prec j \prec k$ in P and iRk imply that $C = \{i, j, k\}$ is a red 3-clique of

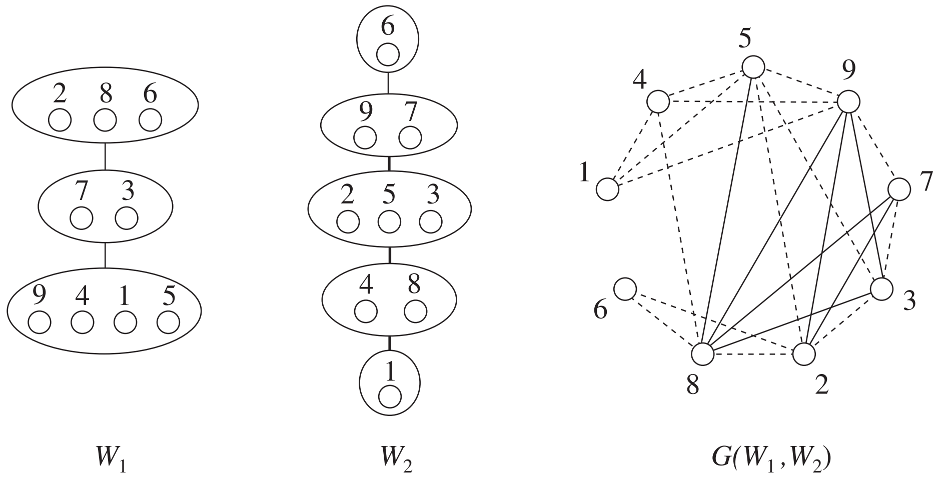

Fig. 2. Two weak orders and their conflict graph

G with $\tau(C) = j$. Obviously, if P is a transitive orientation of G, then P^d is also a transitive orientation of G. Bicolored graph G is said to be *uniquely transitively orientable* if G has no edge or if G has exactly two dual transitive orientations.

Transitive orientations of bicolored graphs generalize transitive orientations of (ordinary) graphs, which have been widely studied: see, e.g., Gallai [15], Kelly [21] or Möhring [29]. Trivially, when $G = (V, B, R, \tau)$ is a bicolored graph with no red directed edge (i.e., $R = \tau = \emptyset$), the transitive orientations of G exactly correspond to the transitive orientations of the graph (V, B). The results of the classic theory of transitive orientations of graphs (also known as the theory of comparability graphs) include characterizations of the graphs that admit a transitive orientation, and of the graphs that are uniquely transitively orientable. It turns out that these results can be adapted to the bicolored case as discussed in Fiorini [10].

Let W_1 and W_2 be two weak orders on X. The *common equivalence relation* of W_1 and W_2 is defined by $\approx = (W_1 \cap W_1^d) \cap (W_2 \cap W_2^d)$. We say that W_1 and W_2 have a *trivial* common equivalence if $\approx = \{(i,i) : i \in X\}$. Let $X/{\approx}$ denote the set of equivalence classes of $\approx$. Then W_1 naturally determines a weak order $W_1/{\approx}$ on $X/{\approx}$ with $A \succsim B$ in $W_1/{\approx}$ if $a \succsim b$ in W_1 for all $a \in A$ and for all $b \in B$, where $A, B \in X/{\approx}$. Similarly, W_2 determines weak order $W_2/{\approx}$ on $X/{\approx}$. We say that $W_\alpha/{\approx}$ is W_α *reduced up to common equivalence* for $\alpha = 1, 2$. When W_1 and W_2 have a trivial common equivalence, their conflict graph $G = G(W_1, W_2)$ always admits two dual transitive orientations, namely $P = W_1 \cap W_2^d$ and $P^d = W_1^d \cap W_2$. The following theorem characterizes adjacency on weak order polytopes. Its proof is given later in the section.

Theorem 2 *Two distinct vertices $x(W_1)$ and $x(W_2)$ of the weak order polytope $\mathbf{P}^n_{WO}$ are adjacent if and only if the conflict graph of W_1' and W_2' is uniquely transitively orientable, where W_1' and W_2' respectively denote W_1 and*

W_2 *reduced up to common equivalence.*

A polytope $\mathbf{P}$ is referred to as a *combinatorial polytope* if for any unordered pair $\{p_1, p_2\}$ of vertices of $\mathbf{P}$, either p_1 and p_2 are adjacent or there exists another unordered pair $\{q_1, q_2\}$ of vertices of $\mathbf{P}$ that have the same midpoint as $\{p_1, p_2\}$, i.e., $\frac{1}{2}p_1 + \frac{1}{2}p_2 = \frac{1}{2}q_1 + \frac{1}{2}q_2$. Clearly, any polytope affinely equivalent to a combinatorial polytope is also a combinatorial polytope. Adjacency on combinatorial 0/1-polytopes can be rephrased in purely combinatorial terms, hence the name. An *equality constrained 0/1-polytope* is a polytope whose vertex set is the set of all 0/1-vectors that satisfy a given system of linear equations. As was noted by Matsui and Tamura [25], any equality constrained 0/1-polytope is a combinatorial polytope.

Proposition 3 *For all $n \geq 2$, the weak order polytope $\mathbf{P}^n_{WO}$ is affinely equivalent to an equality constrained 0/1-polytope.*

PROOF. Let $d = n(n-1)/2 + n(n-1) + n(n-1)(n-2)$. We define an affine map π from $\mathbb{R}^{n(n-1)}$ to $\mathbb{R}^d$ as follows. Each vector of $\mathbb{R}^d$ is identified with a triple (y, z, t) where $y \in \mathbb{R}^{n(n-1)/2}$, $z \in \mathbb{R}^{n(n-1)}$ and $t \in \mathbb{R}^{n(n-1)(n-2)}$. Using natural notations for coordinates, the affine map π is defined by

$$\begin{aligned} \pi(x) = (y, z, t) \text{ with } y_{\{i,j\}} &= x_{ij} + x_{ji} - 1 && \text{for } 1 \leq i < j \leq n, \\ z_{ij} &= x_{ij} && \text{for } i, j \text{ distinct in } \mathbf{n}, \\ t_{ijk} &= x_{ij} + x_{jk} - x_{ik} && \text{for } i, j, k \text{ distinct in } \mathbf{n}. \end{aligned}$$

By construction, π is one-to-one so $\mathbf{P}^n_{WO}$ is affinely equivalent to its image. The vertex set of $\pi(\mathbf{P}^n_{WO})$ is

$$\begin{aligned} \{(y, z, t) \in \{0,1\}^d : \forall i, j, k \in \mathbf{n}, i, j, k \text{ distinct} : y_{\{i,j\}} &= z_{ij} + z_{jk} - 1, \\ t_{ijk} &= z_{ij} + z_{jk} - z_{ik}\}\,. \end{aligned}$$

In conclusion, $\pi(\mathbf{P}^n_{WO})$ is an equality constrained 0/1-polytope affinely equivalent to $\mathbf{P}^n_{WO}$. □

A consequence of Proposition 3 is that $\mathbf{P}^n_{WO}$ is a combinatorial polytope. Hence the next lemma follows from Proposition 3.

Lemma 4 *Two distinct vertices $x(W_1)$ and $x(W_2)$ of weak order polytope $\mathbf{P}^n_{WO}$ are* not *adjacent in $\mathbf{P}^n_{WO}$ if and only if there exist two weak orders Z_1 and Z_2 distinct from W_1 and W_2 such that $Z_1 \cup Z_2 = W_1 \cup W_2$ and $Z_1 \cap Z_2 = W_1 \cap W_2$.*

Let X be any set and let W_1 and W_2 be two distinct weak orders on X. The *sandwich set* of W_1 and W_2, denoted by $\mathcal{S}(W_1, W_2)$, is defined as the set of weak orders W on X containing $W_1 \cap W_2$, contained in $W_1 \cup W_2$ and such that $W \Delta W_1 \Delta W_2$ is a weak order on X. Here, Δ denotes the *symmetric difference* of sets, defined by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Lemma 4 can be restated as follows: vertices $x(W_1)$ and $x(W_2)$ are adjacent if and only if $\mathcal{S}(W_1, W_2) = \{W_1, W_2\}$.

When studying adjacency on the weak order polytope, we can restrict ourselves to weak orders W_1 and W_2 on $\mathbf{n}$ which are reduced up to common equivalence: if the common equivalence relation $\approx$ is not trivial, then $W \mapsto W/{\approx}$ defines a bijection between $\mathcal{S}(W_1, W_2)$ and $\mathcal{S}(W_1/{\approx}, W_2/{\approx})$. Consequently, vertices $x(W_1)$ and $x(W_2)$ are adjacent if and only if $\mathcal{S}(W_1/{\approx}, W_2/{\approx}) = \{W_1/{\approx}, W_2/{\approx}\}$.

Proposition 5 *If W_1 and W_2 are weak orders on $\mathbf{n}$ $(n \geq 2)$ with trivial common equivalence, then there exists a bijection between the sandwich set of W_1 and W_2, and the set of all transitive orientations of the conflict graph of W_1 and W_2.*

PROOF. Let $\mathcal{S} = \mathcal{S}(W_1, W_2)$ be the sandwich set of W_1 and W_2, let $G = G(W_1, W_2) = (V, B, R, \tau)$ be the conflict graph of W_1 and W_2, let E be the edge set of G, and let $\mathcal{O}$ be the set of all transitive orientations of G. Consider the map $\varphi : W \mapsto W \cap (W \Delta W_1 \Delta W_2)^d$ with domain $\mathcal{S}$. Our first claim is that φ is a map from $\mathcal{S}$ to $\mathcal{O}$.

Let W be an element of $\mathcal{S}$, and let $P = \varphi(W)$. By definition of $\mathcal{S}$, the relation $W^* = W \Delta W_1 \Delta W_2$ is a weak order. Note that we have $W \cap W^* = W_1 \cap W_2$ and $W \cup W^* = W_1 \cup W_2$, and that W and W^* have a trivial common equivalence. Because it is the intersection of two weak orders, P is reflexive and transitive. Moreover, P is antisymmetric because $i \precsim j$ and $j \precsim i$ in P imply $i \precsim j$ in W, $j \precsim i$ in W^*, $j \precsim i$ in W and $i \precsim j$ in W^*, so $i \sim j$ in W and $j \sim i$ in W^*, hence $i = j$ because W and W^* have a trivial common equivalence. By preceding remarks, we have

$$\begin{aligned} P \cup P^d &= (W \cap (W^*)^d) \cup (W^d \cap W^*) \\ &= (W \cup W^d) \cap (W \cup W^*) \cap (W^* \cup W)^d \cap ((W^*)^d \cup W^*) \\ &= (W_1 \cup W_2) \cap (W_1 \cup W_2)^d \\ &= E \cup \{(i,i) : i \in \mathbf{n}\} \,. \end{aligned}$$

Now suppose that $i \prec j \prec k$ in P and iRk for some elements i, j and k. By definition of P, we have $i \precsim j \precsim k$ in W and $k \precsim j \precsim i$ in W^*. Because the union (resp. the intersection) of W and W^* is equal to the union (resp. the intersection) of W_1 and W_2, the conflict graph of W and W^* is G. Since

(i,k) is a red directed edge of G, we have either $i \sim k$ in W and $i \not\sim k$ in W^*, or $i \not\sim k$ in W and $i \sim k$ in W^*. Suppose, without loss of generality (the roles of W and W^* are symmetric), that $i \sim k$ in W. Then $i \sim j \sim k$ in W, hence $k \prec j \prec i$ in W^*, for W and W^* have a trivial common equivalence. Therefore, $\{i,j,k\}$ is a red 3-clique in G and $\tau\{i,j,k\} = j$. In conclusion, P is a transitive orientation of G and φ is a map from $\mathcal{S}$ to $\mathcal{O}$. Our second claim is that map φ is bijective.

Consider the map $\psi : P \mapsto P \cup (W_1 \cap W_2)$ with domain $\mathcal{O}$. We prove that map φ is bijective in three steps. First, we show that ψ is a map from $\mathcal{O}$ to $\mathcal{S}$; then, that $\psi \circ \varphi$ is the identity map on $\mathcal{S}$; finally, that ψ is one-to-one. Let P be a transitive orientation of G and $W = \psi(P)$ be the image of P under map ψ. Clearly, W contains $W_1 \cap W_2$ and is contained in $W_1 \cup W_2$. In particular, W is reflexive. Moreover, W is complete because if i,j are two distinct elements and if pairs (i,j) and (j,i) are not in $W_1 \cap W_2$, then they are both blue directed edges of G so either (i,j) or (j,i) belongs to P, and to W because it contains P. Now suppose that W is not transitive. Then there exist distinct elements i,j,k with $(i,j) \in W$, $(j,k) \in W$ and $(i,k) \notin W$. Without loss of generality (by transitivity of P and $W_1 \cap W_2$ and because the other case is similar), we assume that (i,j) is in P but not in $W_1 \cap W_2$ and that (j,k) is in $W_1 \cap W_2$ but not in P. Because P is contained in $W_1 \cup W_2$, we have $i \precsim j \precsim k$ in W_1 or $i \precsim j \precsim k$ in W_2, so (i,k) is in $W_1 \cup W_2$. By hypothesis, (i,k) is neither in $W_1 \cap W_2$ not in P, so (i,k) is in $W_1 \Delta W_2$ and (k,i) is in P. By transitivity of P, we have $(k,j) \in P$. In fact, (k,j) is a red directed edge of G. Because P is a transitive orientation of G with $k \prec i \prec j$ in P and kRj, vertices i,j and k form a red 3-clique in G and $\tau(\{i,j,k\}) = i$. It follows that (k,i) and (i,j) are in $W_1 \cap W_2$. But the linear order induced by $W_1 \cap W_2$ on $\{i,j,k\}$ is $j \prec k \prec i$, so $\tau(\{i,j,k\}) = k$, a contradiction. Consequently, W is transitive, so W is a weak order. A second application of the same arguments to P^d instead of P shows that $\psi(P^d)$ is a weak order. It is not hard to see that $\psi(P^d) = W \Delta W_1 \Delta W_2$ (consider the union and the intersection of $W = \psi(P)$ and of $\psi(P^d)$). Therefore W is a member of $\mathcal{S}$. In conclusion, ψ is a map from $\mathcal{O}$ to $\mathcal{S}$.

Now let W be an element of $\mathcal{S}$, and let $W^* = W \Delta W_1 \Delta W_2$, as before. We have

$$\begin{aligned}\psi(\varphi(W)) &= \psi(W \cap (W^*)^d) = (W \cap (W^*)^d) \cup (W_1 \cap W_2) \\ &= (W \cap (W^*)^d) \cup (W \cap W^*) = W \cap (W^* \cup (W^*)^d) = W\end{aligned}$$

so $\psi \circ \varphi$ is the identity map on $\mathcal{S}$. Finally, suppose that P and Q are two different transitive orientations of G. Therefore, there exist two elements i and j such that $i \prec j$ in P and $j \prec i$ in Q. Because (i,j) is a directed edge of G, one of (i,j) and (j,i) is in $W_1 \Delta W_2$, so $\psi(P)$ and $\psi(Q)$ are different. Consequently, ψ is one-to-one. Finally, we conclude that φ is a bijection from

$\mathcal{S}$ to $\mathcal{O}$. $\square$

Proof of Theorem 2. Lemma 4 implies that vertices $x(W_1)$ and $x(W_2)$ are adjacent on the weak order polytope $\mathbf{P}^n_{WO}$ if and only if the sandwich set of W_1 and W_2 has exactly two elements, that is, if and only if the sandwich set of W_1' and W_2' has exactly two elements. By Proposition 5, this holds if and only if the conflict graph of W_1' and W_2' is uniquely transitively orientable. $\square$

Finally, we give an efficient algorithm for testing if two distinct vertices of a weak order polytope are adjacent (see Algorithm 1). Given two distinct weak orders W_1 and W_2 on $\mathbf{n}$, the algorithm first computes $W_1' = W_1/\approx$ and $W_2' = W_2/\approx$ where $\approx$ denotes, as usual, the common equivalence relation of W_1 and W_2. Then it constructs a graph $\mathcal{G}$ as follows. The graph $\mathcal{G}$ has one vertex $\{i, j\}$ for each directed edge (i, j) of the conflict graph G of W_1' and W_2'. Two vertices of $\mathcal{G}$ are adjacent iff they correspond to two incident edges of G such that the orientation of one edge forces the orientation of the other (in any transitive orientation). More precisely, $\{i, j\}$ is adjacent to $\{i, k\}$ in $\mathcal{G}$ iff $j \neq k$ and (j, k) is not a blue directed edge of G (in which case (j, k) is a red directed edge of G or not a directed edge of G). By construction, G is uniquely transitively orientable whenever $\mathcal{G}$ is connected. As shown in Fiorini [10], the converse is also true. Now the correctness of Algorithm 1 follows from Theorem 2.

If a depth-first search algorithm is used to test for connectedness of $\mathcal{G}$, then the overall running time of the algorithm is $O(n^3)$. We mention that adjacency on the linear ordering polytopes can be tested in $O(n^2)$ time, e.g., by results of McConnell and Spinrad [26]. We leave open the question of whether their results can be adapted to the bicolored case.

Algorithm 1.
Input: two distinct weak orders W_1 and W_2 on $\mathbf{n}$.
Output: a Boolean variable *adj* which is TRUE iff $x(W_1)$ and $x(W_2)$ are adjacent on $\mathbf{P}^n_{WO}$.

$\approx \leftarrow (W_1 \cap W_1^d) \cap (W_2 \cap W_2^d)$
$W_1' \leftarrow W_1/\approx$
$W_2' \leftarrow W_2/\approx$
$\mathcal{V} \leftarrow \{\{i, j\} : (i, j) \in W_1' \Delta W_2'\}$

$$\begin{aligned}\mathcal{E} \leftarrow \{\{\{i, j\}, \{i, k\}\} : \ & \{i, j\}, \{i, k\} \in \mathcal{V}, \\ & (j \precsim k \text{ in } W_1' \text{ or } k \precsim j \text{ in } W_2'), \text{ and} \\ & (k \precsim j \text{ in } W_1' \text{ or } j \precsim k \text{ in } W_2')\}\end{aligned}$$

if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is connected then
  $adj \leftarrow$ TRUE
else
  $adj \leftarrow$ FALSE
end if

5 Symmetries of weak order polytopes

This section establishes the group of combinatorial automorphisms of $\mathbf{P}^n_{WO}$ for each $n \geq 3$. In this section, we always assume that $n \geq 3$. A *combinatorial automorphism* of a polytope $\mathbf{P}$ is an inclusion-preserving permutation of the collection of faces of $\mathbf{P}$. The set of all automorphisms of $\mathbf{P}$, equipped with composition product, is a group denoted by $\mathrm{Aut}(\mathbf{P})$. Let α be an automorphism of $\mathbf{P}$, and let F be a face of $\mathbf{P}$. Then F is said to be *fixed* by α if $\alpha(F) = F$. Trivially, the dimension of F equals the dimension of $\alpha(F)$. When F is a vertex of $\mathbf{P}$, i.e., $F = \{v\}$ for some v in $\mathbf{P}$, we write $\alpha(v)$ to denote the vertex w of F such that $\alpha(\{v\}) = \{w\}$.

Combinatorial automorphisms (in short: *automorphisms*) are symmetries of a very general kind. They are useful for analyzing polytopes. For example, they are efficiently exploited in state-of-the-art facet enumeration algorithms (Christof and Reinelt [5]). Sometimes, a systematic analysis of the automorphism group of a polytope can reveal automorphisms not easily found at first sight, as is the case for linear ordering polytopes: see Bolotashvili, Kovalev and Girlich [3] and Fiorini [11].

Obvious automorphisms for weak order polytopes include 'duality' and 'relabeling' automorphisms. Given any permutation σ of the set $\mathbf{n}$, we define a linear operator R_σ on $\mathbb{R}^{n(n-1)}$ by $R_\sigma(x)_{\sigma(i)\sigma(j)} = x_{ij}$. This linear operator maps the vertex set of the weak order polytope $\mathbf{P}^n_{WO}$ to itself, hence it induces an automorphism of $\mathbf{P}^n_{WO}$ that is referred to as a *relabeling* automorphism. Similarly, the linear operator D defined by $D(x)_{ij} = x_{ji}$ induces an automorphism of $\mathbf{P}^n_{WO}$ that is called the *duality* automorphism.

We will show that, for each $n \geq 3$, the automorphism group of $\mathbf{P}^n_{WO}$ is spanned by relabeling and duality automorphisms. Therefore, the group $\mathrm{Aut}(\mathbf{P}^n_{WO})$ is isomorphic to the direct product $\mathbb{Z}_2 \times \mathrm{Sym}(n)$, where $\mathbb{Z}_2$ is the two-element group and $\mathrm{Sym}(n)$ is the symmetric group on n elements. So, contrary to linear ordering polytopes, weak order polytopes do not have nonobvious automorphisms.

Lemma 6 *Let G be a bicolored graph with $m \geq 3$ vertices. Then G has at* *most $m!$ transitive orientations. Moreover, if G has exactly $m!$ transitive orientations, then it is a complete graph whose edges are all blue.*

PROOF. Let G' denote the graph obtained from G by coloring all red edges of G blue. Also let t and t' be the numbers of transitive orientations of G and G', respectively. We trivially have $t \le t'$. It is well known that the number of transitive orientations of an ordinary graph on m vertices is bounded above by $m!$ (e.g., Möhring [29], Theorem 1.20), with equality if and only if the graph is complete. In particular, we have $t' \le m!$, so $t \le m!$. Now suppose that $t = m!$, hence $t = t'$. It follows that G' is complete, so G also is complete. Moreover, if G has at least one directed red edge, we have $t < t'$, a contradiction. $\square$

Lemma 7 *Let $w_1 = x(W_1)$ and $w_2 = x(W_2)$ be two distinct vertices of $\mathbf{P}^n_{WO}$, let W_1' and W_2' denote W_1 and W_2 reduced up to common equivalence, respectively, let F be the (inclusion) minimal face of $\mathbf{P}^n_{WO}$ containing w_1 and w_2, and let G be the conflict graph of W_1' and W_2'. Then there is a bijection between the vertices of F and the transitive orientations of G.*

PROOF. Let V denote the vertex set of F. By Proposition 5, it suffices to prove that the equality $V = \{x(W) : W \in \mathcal{S}(W_1, W_2)\}$ holds. By definition of $\mathcal{S}(W_1, W_2)$, to each point $w = x(W)$ with $W \in \mathcal{S}(W_1, W_2)$ there corresponds another point $w^* = x(W^*)$ with $W^* \in \mathcal{S}(W_1, W_2)$ such that the midpoint of w and w^* equals the midpoint of w_1 and w_2. Consequently, V contains $\{x(W) : W \in \mathcal{S}(W_1, W_2)\}$. To prove the other inclusion, we use Proposition 3 and the fact that, in an equality constrained 0/1-polytope, the minimal face containing two vertices is centrally symmetric with respect to the midpoint of these vertices. $\square$

Lemma 8 *For $n \ge 3$, every automorphism of $\mathbf{P}^n_{WO}$ fixes $\mathbf{P}^n_{LO}$.*

PROOF. Let α be an automorphism of $\mathbf{P}^n_{WO}$ and v be a vertex of $\mathbf{P}^n_{LO}$. We have to show that $\alpha(v)$ is again a vertex of $\mathbf{P}^n_{LO}$. The minimal face containing v and its image $D(v)$ by the duality automorphism D is $\mathbf{P}^n_{LO}$. So the image of this face under α is a face of $\mathbf{P}^n_{WO}$ which has exactly $n!$ vertices and is the minimal face containing the images of v and $D(v)$ under α. Let W_1 and W_2 be the weak orders on $\mathbf{n}$ such that $x(W_1) = \alpha(v)$ and $x(W_2) = \alpha(D(v))$, and let W_1' and W_2' denote W_1 and W_2 reduced up to common equivalence, respectively. By Lemma 7, the vertices of face $\alpha(\mathbf{P}^n_{LO})$ are in bijection with the transitive orientations of the conflict graph of W_1' and W_2'. Therefore Lemma 6 implies that the latter bicolored graph is a complete graph on n vertices whose edges are all blue. In particular, W_1 is a linear order so that $\alpha(v)$ is a vertex of $\mathbf{P}^n_{LO}$. $\square$

It is well known that the trivial inequality (1) and the *3-dicycle inequality* $x_{ij} + x_{jk} + x_{ki} \leq 2$ define facets of the linear ordering polytope $\mathbf{P}^n_{LO}$, for all i, j, k distinct in $\mathbf{n}$: see Grötschel, Jünger and Reinelt [18]. We refer to the corresponding facets as *trivial* and *3-dicycle* facets, respectively.

Lemma 9 *Let T be a trivial facet of $\mathbf{P}^n_{LO}$. As a face of $\mathbf{P}^n_{WO}$, T is contained in exactly $\binom{n}{2} + 1$ facets of $\mathbf{P}^n_{WO}$ which are: the $\binom{n}{2}$ completeness facets plus a unique trivial facet.*

PROOF. Because T is a trivial facet of $\mathbf{P}^n_{LO}$, there exist distinct elements u, v in $\mathbf{n}$ such that the vertices of T exactly correspond to linear orders L with $u \prec v$ in L. Clearly, T is contained in at least $\binom{n}{2} + 1$ distinct facets of $\mathbf{P}^n_{WO}$ which are the facets defined by the inequalities $x_{ij} + x_{ji} \geq 1$ (completeness facets) for all $i, j \in \mathbf{n}$ with $i < j$, plus the trivial facet of $\mathbf{P}^n_{WO}$ defined by the inequality $x_{uv} \leq 1$ (whose vertices correspond to weak orders W such that $u \precsim v$ in W). Now suppose that F is a facet of $\mathbf{P}^n_{WO}$ containing T and let $\langle a, x\rangle \geq b$ be any inequality defining F. We claim that $a_{ij} = a_{ji} \geq 0$, for all $i, j \in \mathbf{n}$ with $i < j$ and $ij \notin \{uv, vu\}$, and that $a_{vu} \geq a_{uv}$, $a_{vu} \geq 0$.

Define S as the set of all pairs ij with i, j in $\mathbf{n}$, $i < j$ and $ij \notin \{uv, vu\}$. Let $ij \in S$. There exists a linear order L such that $u \prec v$ in L, $i \prec j$ in L, and $L' = L \setminus \{ij\} \cup \{ji\}$ is a linear order (the latter condition amounts to the requirement that there is no k with $i \prec k \prec j$ in L). Moreover, $u \prec v$ in L', and $W = L \cup \{ji\}$ is a weak order. We have

$$\langle a, x(L)\rangle = b = \langle a, x(L')\rangle \Rightarrow \langle a, x(L) - x(L')\rangle = 0 \Rightarrow a_{ij} = a_{ji}$$

and

$$a_{ji} + b = a_{ji} + \langle a, x(L)\rangle = \langle a, x(W)\rangle \geq b \Rightarrow a_{ji} \geq 0.$$

Let L'' be a linear order such that $u \prec v$ in L'', and $L''' = L'' \setminus \{uv\} \cup \{vu\}$ is a linear order, and let $W' = L'' \cup \{vu\}$. Then W' is a weak order and

$$a_{vu} - a_{uv} + \langle a, x(L'')\rangle = \langle a, x(L''')\rangle \geq b \Rightarrow a_{vu} \geq a_{uv}$$

and

$$a_{vu} + \langle a, x(L'')\rangle = \langle a, x(W')\rangle \geq b \Rightarrow a_{vu} \geq 0.$$

Now consider the positive combination of the valid inequalities $x_{ij} + x_{ji} \geq 1$ (ij in S), $x_{uv} + x_{vu} \geq 1$ and $-x_{uv} \geq -1$, with coefficients respectively equal to a_{ij} ($ij \in S$), a_{vu} and $a_{vu} - a_{uv}$. By construction, this combination is equal to the inequality $\langle a, x\rangle \geq \sum_{ij \in S} a_{ij} + a_{uv}$. It defines a face of $\mathbf{P}^n_{WO}$ which contains the face T and is parallel to the facet F. Consequently, this face is equal to F and we have

$$\sum_{ij \in S} a_{ij} + a_{uv} = b.$$

Because F is a facet, it cannot be defined by an inequality that is a positive combination of two or more valid inequalities, so the coefficients a_{ij} ($ij \in S$), a_{vu} and $(a_{vu} - a_{uv})$ are all equal to zero, but one. In conclusion, F is a completeness facet or the trivial facet defined by the inequality $x_{uv} \leq 1$. □

Theorem 10 *For $n \geq 3$, the automorphism group of $\mathbf{P}^n_{WO}$ is spanned by duality and relabeling automorphisms and is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n)$.*

PROOF. Consider the map ψ from $\mathrm{Aut}(\mathbf{P}^n_{WO})$ to $\mathrm{Aut}(\mathbf{P}^n_{LO})$ mapping each automorphism α of $\mathbf{P}^n_{WO}$ to its restriction to the set of faces of $\mathbf{P}^n_{LO}$. Lemma 8 ensures that ψ is well defined. It is a group homomorphism. Let α be an automorphism of $\mathbf{P}^n_{WO}$ whose restriction to the set of faces of $\mathbf{P}^n_{LO}$ is the identity permutation. In particular, α fixes all faces of $\mathbf{P}^n_{WO}$ that are trivial facets of $\mathbf{P}^n_{LO}$. By Lemma 9, if T is a trivial facet of $\mathbf{P}^n_{LO}$ then α must permute among themselves the $\binom{n}{2} + 1$ facets of $\mathbf{P}^n_{WO}$ ($n \geq 3$) containing T. But in $\mathbf{P}^n_{WO}$, completeness facets contain strictly more vertices than trivial facets, hence α cannot map a trivial facet on a completeness facet. So α also fixes the unique trivial facet of $\mathbf{P}^n_{WO}$ containing T. Because this is true for any choice of T, α fixes all trivial facets of $\mathbf{P}^n_{WO}$. Consequently, all vertices, hence all faces of $\mathbf{P}^n_{WO}$ are fixed by α. In conclusion, ψ is one-to-one.

As a face of $\mathbf{P}^n_{WO}$, any 3-dicycle facet of $\mathbf{P}^n_{LO}$ (see Grötschel, Jünger and Reinelt [18]) is contained in at least $\binom{n}{2} + 3$ facets of $\mathbf{P}^n_{WO}$, namely all completeness facets, plus three transitivity facets. By Lemma 9, an automorphism of $\mathbf{P}^n_{WO}$ cannot map a trivial facet of $\mathbf{P}^n_{LO}$ on a 3-dicycle facet of $\mathbf{P}^n_{LO}$. Exploiting this property, the structure of the automorphism group of the linear ordering polytope $\mathbf{P}^n_{LO}$, and the fact that ψ is a one-to-one group homomorphism, we conclude that the automorphism group of $\mathbf{P}^n_{WO}$ is spanned by the duality and relabeling automorphisms and is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n)$. □

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