Rössler systems

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Rössler systems were introduced in the 1970s as prototype equations with the minimum ingredients for continuous-time chaos.

Since the Poincaré-Bendixson theorem precludes the existence of other than steady, periodic, or quasiperiodic attractors in autonomous systems defined in one- or two-dimensional manifolds such as the line, the circle, the plane, the sphere, or the torus (Hartman, 1964), the minimal dimension for chaos is three. On this basis, Otto Rössler came up with a series of prototype systems of ordinary differential equations in three-dimensional phase spaces (Rössler 1976a,c, 1977a, 1979a). He also proposed four-dimensional systems for hyperchaos, that is chaos with more than one positive Lyapunov exponent (Rössler 1979a,b).

Rössler was inspired by the geometry of flows in dimension three and, in particular, by the reinjection principle, which is based on the feature of relaxation-type systems to often present a Z-shaped slow manifold in their phase space. On this manifold, the motion is slow until an edge is reached whereupon the trajectory jumps to the other branch of the manifold, allowing not only for periodic relaxation oscillations in dimension two (see Fig. 1a), but also for higher types of relaxation behavior (see Fig. 1b) as noted by Rössler (1979a). In dimension three, the reinjection can induce chaotic behavior if the motion is spiraling out on one branch of the slow manifold (see Fig. 1b). In this way, Rössler invented a series of systems, the most famous of which is probably (Rössler 1979a)

\[
\begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= bx - cz + xz
\end{align*}
\]

This system is minimal for continuous chaos for at least three reasons: Its phase space has the minimal dimension three, its nonlinearity is minimal because there is a single quadratic term, and it generates a chaotic attractor with a single lobe, in contrast to the Lorenz attractor which has two lobes. In Eq. (1), \((x, y, z)\) are the three variables that evolve in the continuous time \(t\) and \((a, b, c)\) are three parameters. The linear terms of the two first equations create oscillations in the variables \(x\) and \(y\). These oscillations can be amplified if \(a > 0\), which results into a spiraling-out motion. The motion in \(x\) and \(y\) is then coupled to the \(z\) variable ruled by the third equation, which contains the nonlinear term and which induces the reinjection back to the beginning of the spiraling-out motion.

System (1) possesses two steady states: one at the origin \(x = y = z = 0\), around which the motion spirals out, and another one at some distance of the origin due to the quadratic nonlinearity. This system presents stationary, periodic, quasiperiodic, and chaotic attractors depending on the value of the parameters \((a, b, c)\). These attractors are interconnected by bifurcations, in particular, a Hopf bifurcation from the stationary to periodic attractors and a period-doubling cascade from periodic to chaotic attractors. The resulting chaotic attractor has a single lobe and is referred to as spiral-type chaos, which mainly manifests itself in irregular amplitudes for the oscillations (see Fig. 2a).

A transition occurs to a screw-type chaos in which the oscillations are irregular not only in their amplitudes but also in the reinjection times (see Fig. 2b). The screw-type chaos is closely related to the presence of a Shil’nikov homoclinic orbit (see Fig. 2c). This homoclinic orbit is attached to the origin \(x = y = z = 0\), which is a saddle-focus with a one-dimensional stable manifold for the reinjection and a two-dimensional unstable manifold where the motion is spiraling out. The Shil’nikov criterion for chaos is that the reinjection is faster than the spiraling-out motion (Shil’nikov 1965) and it is satisfied in the attractor of Fig. 2b. As a consequence, the homoclinic system contains periodic and nonperiodic orbits belonging to multiple horseshoes which can be described in terms of symbolic dynamics. Away from homoclinicity, the system undergoes complex bifurcation cascades generating successive periodic and chaotic attractors.

Chaotic behavior and Shil’nikov homoclinic orbit in the Rössler system (1) can also be understood as originating from an oscillatory-stationary double instability taking place around the origin \(x = y = z = 0\) and the parameter
FIG. 1: Illustration of the reinjection principle between the two branches of a Z-shaped slow manifold allowing (a) periodic
relaxation oscillations in dimension two and (b) higher types of relaxation behavior in dimension three.

FIG. 2: Phase portraits of the Rössler system (1) in the phase space of the variables $(x, y, z)$: (a) spiral-type chaos for $a = 0.32,$
$b = 0.3$, and $c = 4.5$; (b) screw-type chaos for $a = 0.38$, $b = 0.3$, and $c = 4.820$ in which case there also exists the Shil’nikov-type
homoclinic orbit (c). The ticks are separated by unity. (Adapted from Gaspard & Nicolis 1983.)

values $b = 1$, $-\sqrt{2} < a = c < +\sqrt{2}$.

In his work on continuous chaos, Rössler has been motivated by the search for chemical chaos, that is, chaotic
With Willamowski, Rössler proposed the following chemical reaction scheme:

$$
\begin{align*}
A_1 + X & \xrightleftharpoons[1]{k_1} 2X, \\
X + Y & \xrightleftharpoons[2]{k_2} 2Y, \\
A_5 + Y & \xrightleftharpoons[3]{k_3} A_2, \\
X + Z & \xrightleftharpoons[4]{k_4} A_3, \\
A_4 + Z & \xrightleftharpoons[5]{k_5} 2Z,
\end{align*}
$$

which features two autocatalytic steps (reactions 1 and 5) involving the species $X$ and $Z$ coupled to another au-
tocatalytic step (reaction 2) involving another species $Y$ and two further steps (reactions 3 and 4) (Willamowski &
Rössler 1980). The concentrations of the species $A_1, \ldots, A_5$ are held fixed by large chemical reservoirs that maintain the
FIG. 3: Chaotic time evolution of the concentrations of the Willamowski-Rössler chemical reaction scheme (2) for \( k_1 a_1 = 30, k_{-1} = 0.5, k_2 = 1, k_{-2} = 0, k_3 a_5 = 10, k_{-3} = 0, k_4 = 1, k_{-4} = 0, k_5 a_4 = 16.5, \) and \( k_{-5} = 0.5 \): (a) phase portrait in the plane of the concentrations of species \( X \) and \( Y \); (b) concentration of species \( X \) versus time \( t \).

system out of thermodynamic equilibrium. The time evolution of the concentrations \((x, y, z)\) of the three intermediate species \(X, Y,\) and \(Z\) is ruled by a system of three coupled differential equations deriving from the mass action law of chemical kinetics. These equations have quadratic nonlinear terms because of the binary reactive steps and keep the concentrations positive as a consequence of mass action kinetics. The chemical reaction scheme (2) leads to a chaotic attractor very similar to the one of the abstract system (1) (see Fig. 3), and thus provides a mechanistic understanding of chemical chaos in terms of colliding and reacting particles.

In conclusion, Rössler systems are minimal models for continuous-time chaos. The chaotic attractors of Rössler systems are prototypes for a large variety of chaotic behavior, notably, in chemical chaos (Scott 1991).

See also Attractors; Bifurcations; Brusselator; Chaotic dynamics; Chemical kinetics; Hopf bifurcation; Horseshoe and hyperbolicity in dynamical systems; Invariant manifolds and sets; Period doubling; Phase space; Poincaré theorems; Symbolic dynamics.

Further Reading