MECA-H-303
Kinematics and dynamics of machines

Partim: Dynamics and vibrations

Chapter 2: Dynamics

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1. Newtonian dynamics

Translation

$$M\mathbf{a}_G = \mathbf{F} + \mathbf{R}$$

(3 equations)

Joints/constraints appears as reaction forces

Rotation

$$\frac{\mathbf{d}}{\mathbf{dt}}\overline{\mathbf{M}}_{\mathbf{A}} = \mathbf{M}\overline{\mathbf{v}}_{\mathbf{G}} \ \mathbf{x} \ \overline{\mathbf{v}}_{\mathbf{A}} + \overline{\mathbf{m}}_{\mathbf{e,A}}$$

(3 equations)

$$\left.\begin{array}{l} \overline{\mathbf{v}}_{\mathbf{G}} = \mathbf{0} \\ \\ \overline{\mathbf{v}}_{\mathbf{A}} = \mathbf{0} \\ \\ \overline{\mathbf{v}}_{\mathbf{A}} \ // \ \overline{\mathbf{v}}_{\mathbf{G}} \end{array}\right\} \quad \frac{\mathbf{d}}{\mathbf{dt}}\overline{\mathbf{M}}_{\mathbf{A}} = \overline{\mathbf{m}}_{\mathbf{e,\,A}}$$

Angular momentum: $\overline{M}_A = m\,\overline{AG} \times \overline{v}_A + \overline{\overline{I}}_A . \overline{\omega}$

- Solid body rotating <u>around a fixed axis</u>, and A is one point of the axis:

$$\overline{M}_A = \overline{\overline{I}}_A . \overline{\omega}$$

$$\overline{M}_A = -P_{xz}\omega\overline{1}_x - P_{yz}\omega\overline{1}_y + I_z\omega\overline{1}_z$$

In principal axes of the body: $\overline{M}_A = C\omega\overline{1}_z$

- Solid body rotating <u>around a fixed point A</u> :

$$\overline{M}_A = \overline{\overline{I}}_A . \overline{\omega}$$

In principal axes of the body: $\overline{M}_A = Ap\overline{1}_x + Bq\overline{1}_y + Cr\overline{1}_z$

- Solid body rotating <u>around the center of mass</u> :

$$\overline{M}_G = \overline{\overline{I}}_G . \overline{\omega}$$

2. Analytical mechanics

Virtual work principle: $$\delta\tau = \sum_{h=1}^{N} \overline{F}_h \, \delta\overline{r}_h = 0$$

Where $\delta\overline{r_h} = \sum_{i=1}^{n} \frac{\partial\overline{\varphi}_h}{\partial q_i}\delta q_i$ are infinitesimal displacements **compatible with constraints**

$$\overline{r}_h = \overline{\varphi}(q_i)$$

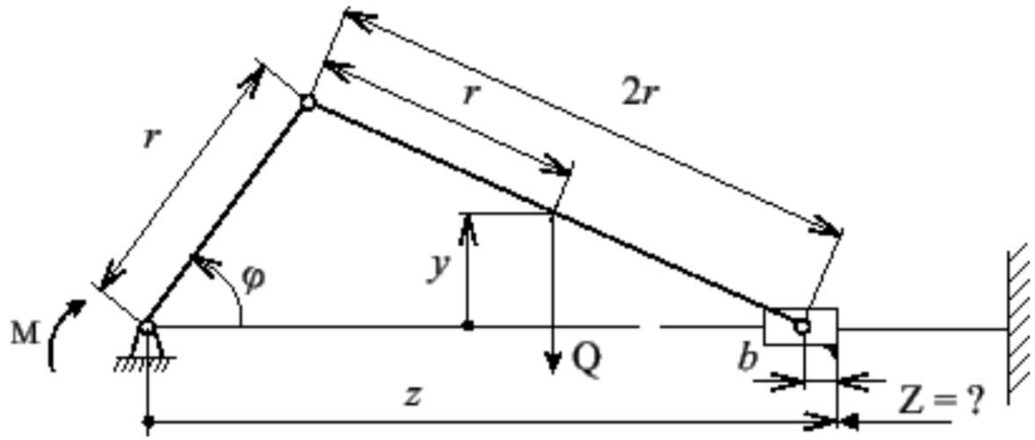
$$\delta\tau = \sum_{i=1}^{n} \mathbf{Q}_i \delta q_i = \mathbf{0}$$ where $$\mathbf{Q_i} = \sum_{h=1}^{N} \overline{\mathbf{F}}_h \cdot \frac{\partial \overline{\boldsymbol{\varphi}}_h}{\partial \mathbf{q}_i}$$

$$Q_i = 0$$ n independent analytical equations

Static equilibrium

Problem: Find *Z* to maintain the slider-crank mechanism in static equilibrium with $\phi = 30^o$.

M = 50 Nm, *Q* = 35 N, *r* = 0.1 m.



Newtonian mechanics 2 equations for translation
1 equation for rotation

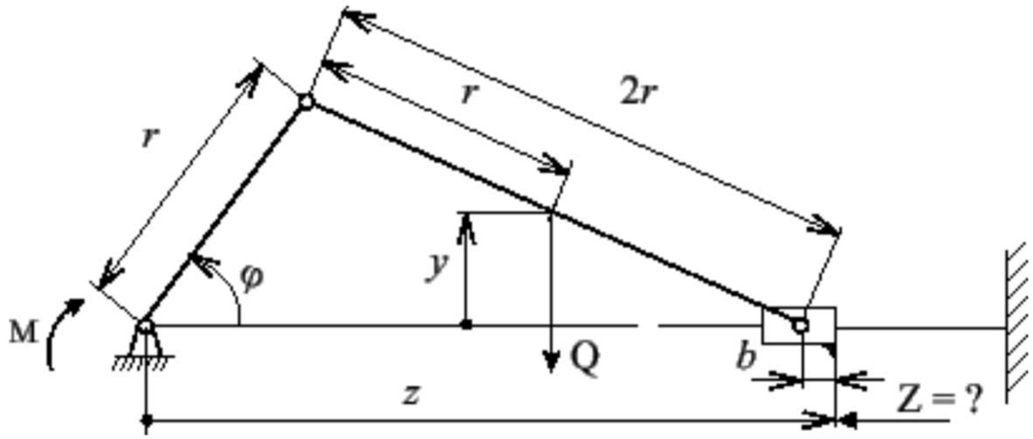
Two bodies ➔ 2 x 3 equations 5 reaction forces + Z ➔ OK

Analytical mechanics

Forces: Q, Z; Moment: M

d.d.l.= 1 ➔ $q_1 = \varphi$

Virtual work:



$$\delta\tau = -M\delta\varphi - Q\delta y - Z\delta z$$

$$= Q_\varphi \delta\varphi$$

$$y = \frac{r}{2}\sin\varphi \quad \Rightarrow \quad \delta y = \frac{r}{2}(\cos\varphi)\delta\varphi$$

$$z = r\cos\varphi + r\sqrt{4-\sin^2\varphi} + b \quad \Rightarrow \quad \delta z = -r\sin\varphi\left(1 + \frac{\cos\varphi}{\sqrt{4-\sin^2\varphi}}\right)\delta\varphi$$

$Q_\varphi = 0$ ➔ $Z = 711.92\ N$

Analytical dynamics

D'Alembert principle: include inertia forces in the equilibrium:

$$m_h \overline{a}_h = \overline{F}_h \Rightarrow m_h \overline{a}_h - \overline{F}_h = 0$$

Virtual work principle: $\delta\tau = \sum_{h=1}^{N} (\overline{F}_h - m_h \overline{a}_h) . \delta\overline{r}_h = 0$ ➔ $\sum_{h=1}^{N} m_h \overline{a}_h \cdot \frac{\partial \overline{\varphi}_h}{\partial q_i} = Q_i$

n equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i$$

where $Q_i = \sum_{h=1}^{N} \overline{F}_h \cdot \frac{\partial \overline{\varphi}_h}{\partial q_i}$

If the generalized forces Q_i can be derived from a potential V such that $Q_i = -\frac{\partial V}{\partial q_i}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$L = T - V$$

Analytical dynamics

With p constraints: $\phi_i(q_1,...,q_n,t)=0 \qquad i=1,...,p$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=Q_j+\sum_{i=1}^{p}\lambda_i\frac{\partial \phi_i}{\partial q_j}$$

If the generalized forces Q_i can be derived from a potential V such that $Q_i=-\frac{\partial V}{\partial q_i}$

n+p equations
n+p variables: $q_1, ..., q_n, \lambda_1, ..., \lambda_p$

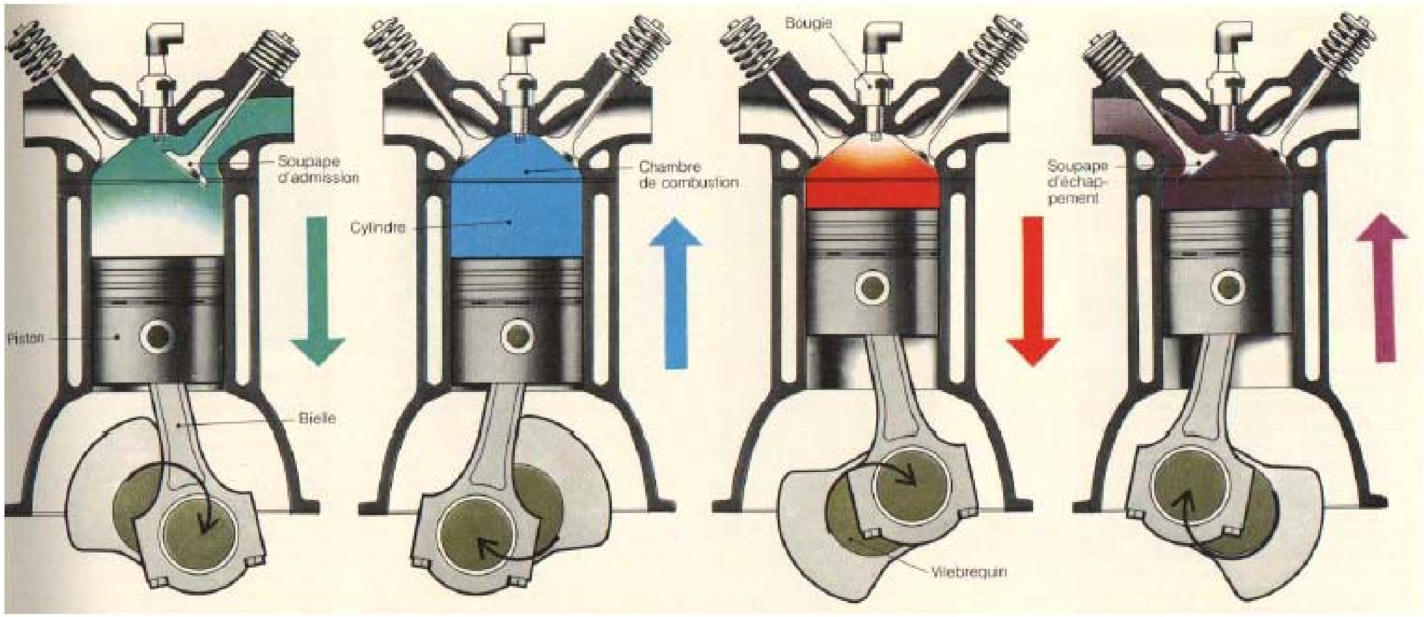
$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j}-\dfrac{\partial L}{\partial q_j}=\sum_{i=1}^{p}\lambda_i\dfrac{\partial \phi_i}{\partial q_j} \\ \phi_i(q_1,...,q_n,t)=0 \end{cases}$$

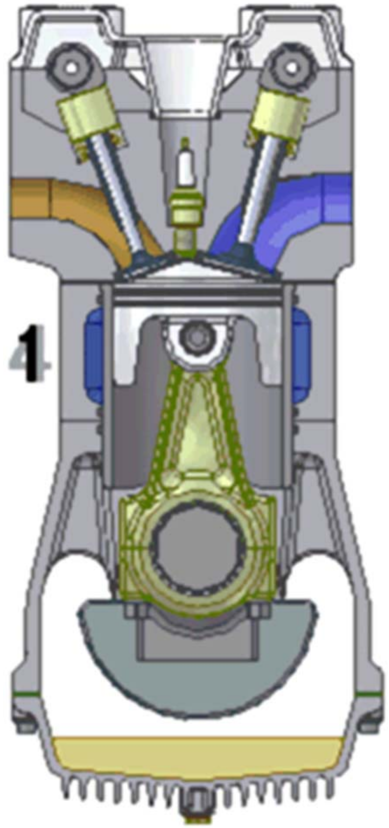
Constraints appear in Lagrange multipliers

Summary

- Newton
  - 6 equations per rigid body
  - Constraints appear as forces
- Lagrange
  - q coordinates
  - N equations of motion
  - p=q-N constraints / Lagrange multipliers

Piston engine

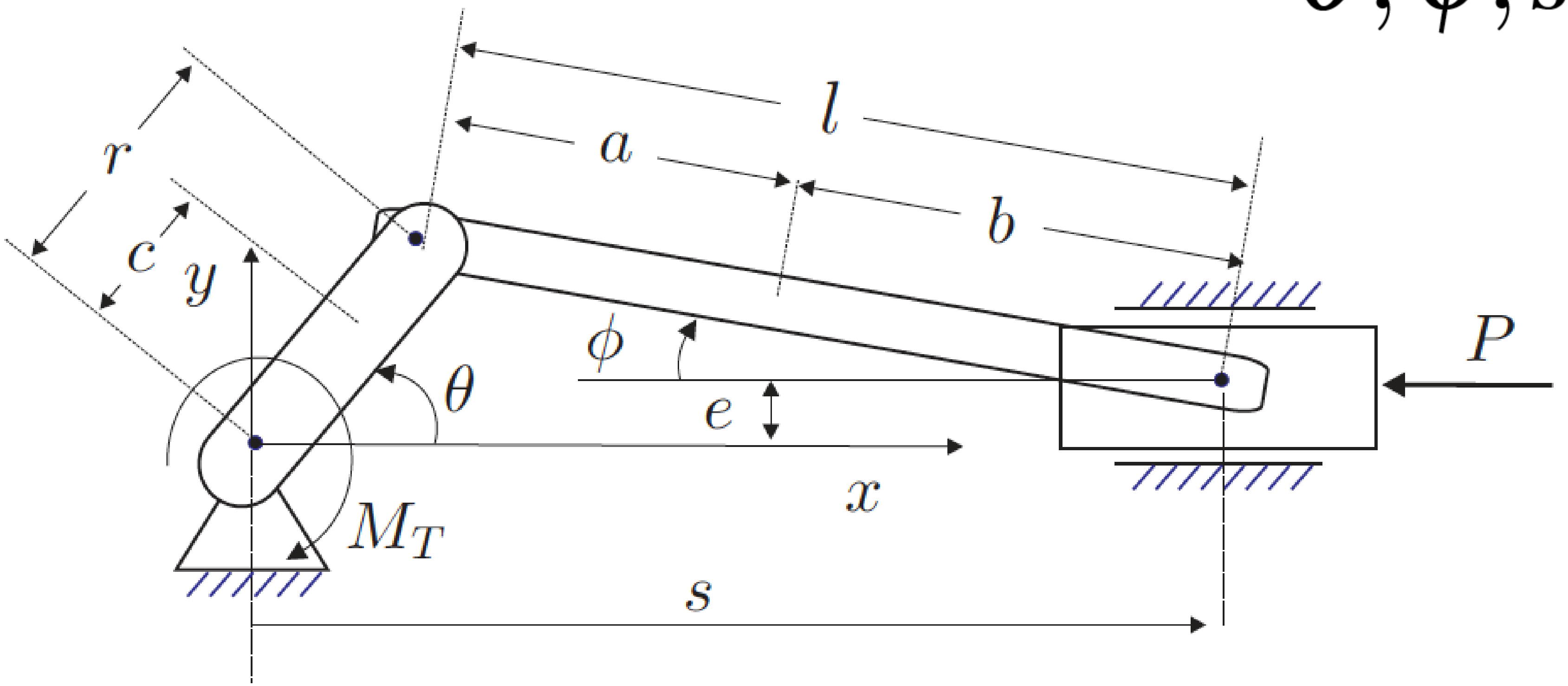




Piston engine

N=1; q=3:

$$\theta, \phi, s$$



p=q-N=2:

$$r\cos\theta + l\cos\phi = s$$

$$r\sin\theta - l\sin\phi = e$$

➔

$$s = r\cos\theta + l\cos\phi = r(\cos\theta + \frac{1}{\lambda}\cos\phi)$$

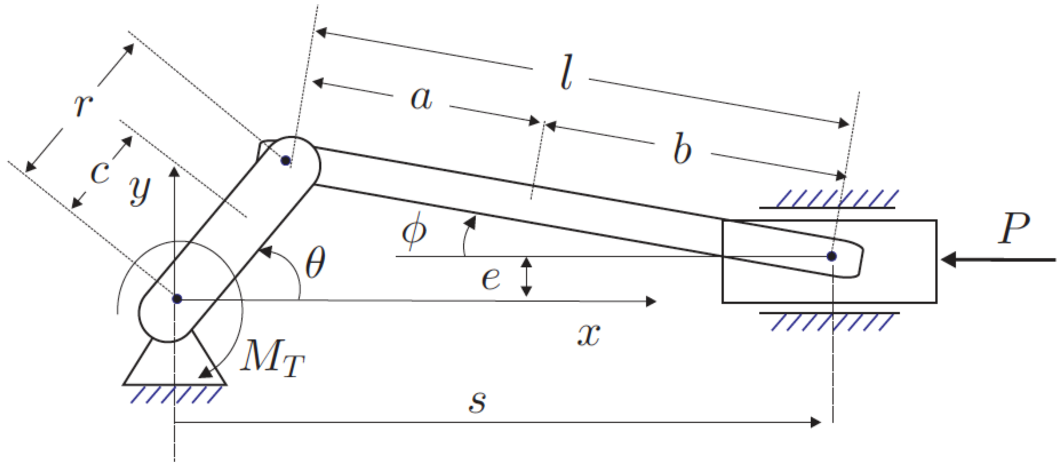
$$\phi = \sin^{-1}(\lambda\sin\theta - \frac{e}{l})$$

Kinetic energy

$$T_1 = \frac{1}{2}(J_1 + m_1 c^2)\dot{\theta}^2 = \frac{1}{2}J_0\dot{\theta}^2$$

$$T_2 = \frac{1}{2}m_2 v_G^2 + \frac{1}{2}J_2\dot{\phi}^2$$

$$T_3 = \frac{1}{2}m_3\dot{s}^2$$



Homework: if $m_A = \frac{m_2 b}{l}$ $\quad m_B = \frac{m_2 a}{l}$

$$T_2 = \frac{1}{2}m_A(r\dot{\theta})^2 + \frac{1}{2}m_B\dot{s}^2 + J_{AB}\dot{\phi}^2$$ where $J_{AB} = J_2 - m_2 ab$

$$T = T_1 + T_2 + T_3 = \frac{1}{2} I(\theta)\dot{\theta}^2$$

$$\begin{cases} I(\theta) = J_0 + m_A r^2 + (m_3 + m_B)k_s^2 + J_{AB} k_\phi^2 \\ k_\phi = \dfrac{\dot{\phi}}{\dot{\theta}} = \dfrac{\lambda \cos\theta}{\cos\phi} = \dfrac{\lambda\cos\theta}{\sqrt{1-(\lambda\sin\theta - \frac{e}{l})^2}} \\ k_s = \dfrac{\dot{s}}{\dot{\theta}} = -r(\sin\theta + \dfrac{k_\phi}{\lambda}\sin\phi) \end{cases}$$

Homework: If $e = 0$ and developments limited after first terms :

$$I(\theta) \simeq J_0 + m_A r^2 + (m_3 + m_B) r^2 \sin^2\theta + J_{AB}\lambda^2 \cos^2\theta = A - B\cos 2\theta$$

$$A = J_0 + m_A r^2 + \frac{1}{2}[(m_3 + m_B) r^2 + J_{AB}\lambda^2]$$

$$B = \frac{1}{2}(m_3 + m_B) r^2 - J_{AB}\lambda^2$$

Calculation of generalized force:

$$\delta W = Q_\theta \delta\theta = -P\delta s - M_T \delta\theta = -(Pk_s + M_T)\delta\theta$$

Dynamic equation: $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

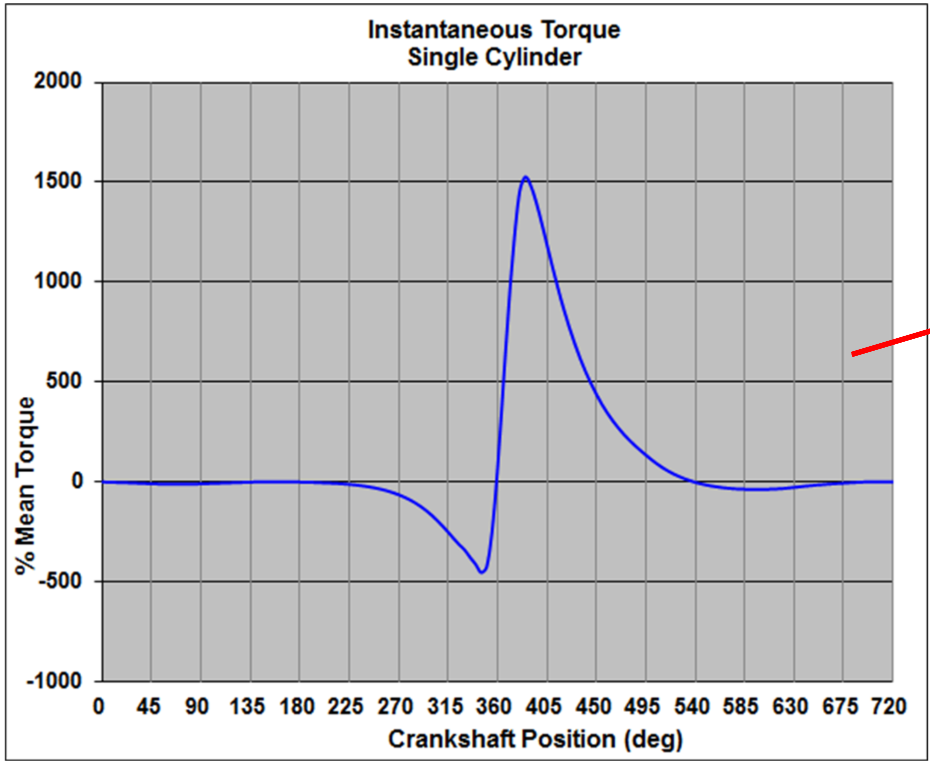
$$I(\theta)\ddot{\theta} + C(\theta)\dot{\theta}^2 = Pk_s + M_T$$ where $$C(\theta) = \frac{1}{2}\frac{dI}{d\theta}$$

Simplification: constant speed $\dot{\theta} = \omega_0$ $P = 0$

$$B \sin 2\theta \, \omega_0^2 = M_T$$

Sinusoïdal variation

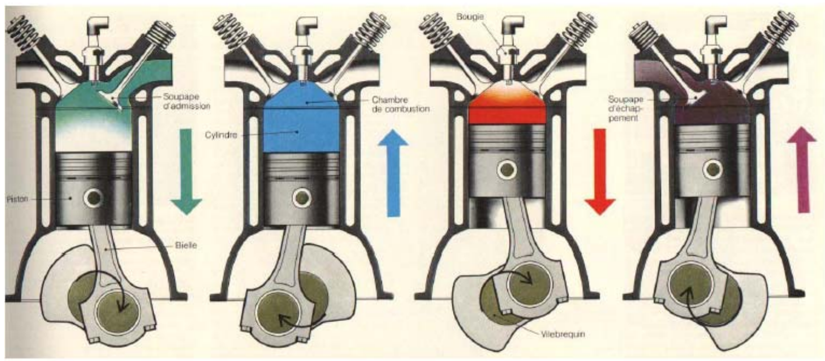
Force *P* applied on piston



Creates noise, vibrations, cracks, fatigue…

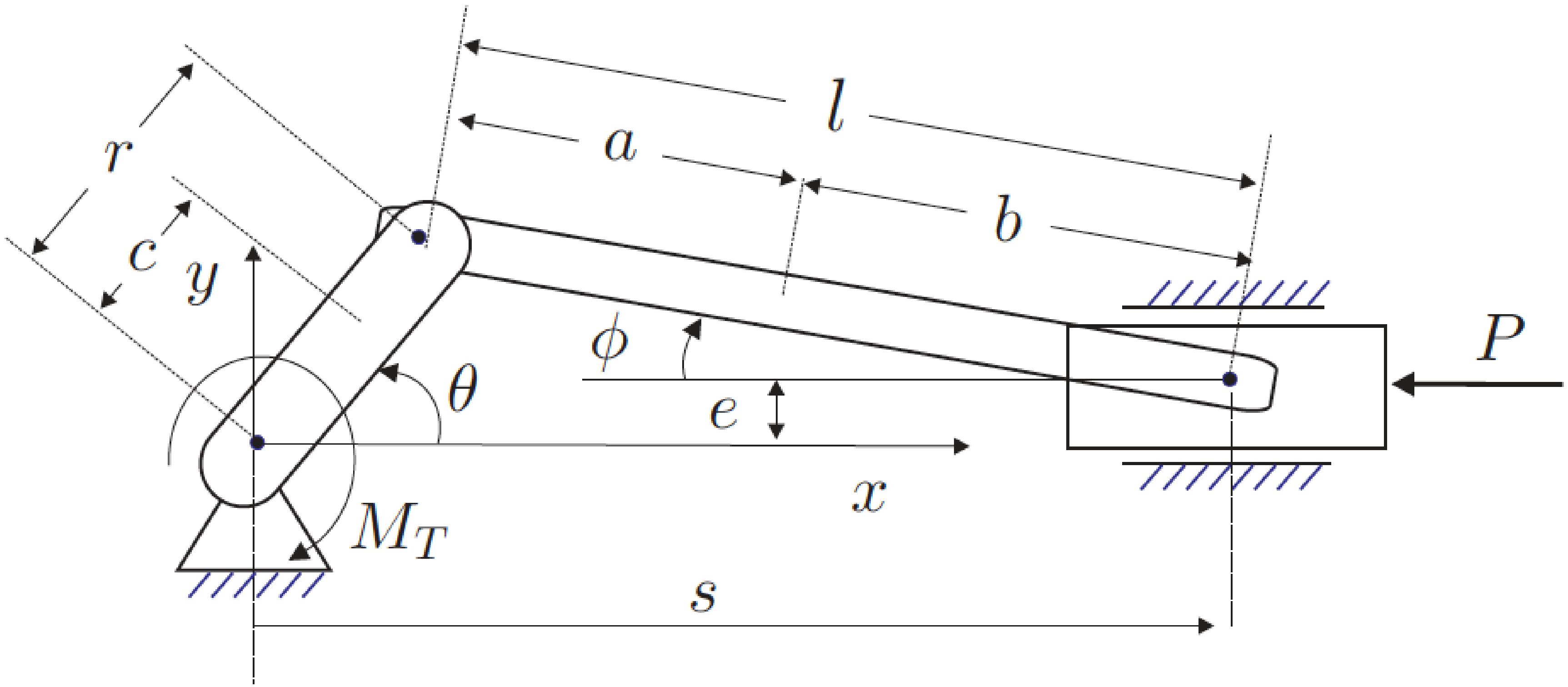
Solutions:

- Vibration absorbers
- Vibration isolators
- Multi-piston motor / Balancing



Calculation of reaction forces

1. Analytical method



Generalized coordinates: $x_1, y_1, \theta, x_2, y_2, \phi, s$

Constraints

$$r\cos\theta + l\cos\phi = s \qquad r\sin\theta\,\delta\theta + l\sin\phi\,\delta\phi + \delta s = 0$$

$$r\sin\theta - l\sin\phi = e \qquad r\cos\theta\,\delta\theta - l\cos\phi\,\delta\phi = 0$$

$$x_1 = c\cos\theta \qquad \delta x_1 + c\sin\theta\,\delta\theta = 0$$

$$y_1 = c\sin\theta \qquad \delta y_1 - c\cos\theta\,\delta\theta = 0$$

$$x_2 = r\cos\theta + a\cos\phi \qquad \delta x_2 + r\sin\theta\,\delta\theta + a\sin\phi\,\delta\phi = 0$$

$$y_2 = b\sin\phi + e \qquad \delta y_2 - b\cos\phi\,\delta\phi = 0$$

Dynamic equations

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}J_1\dot{\theta}^2$$

$$+ \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}J_2\dot{\phi}^2$$

$$+ \frac{1}{2}m_3\dot{s}^2$$

$$m_1\ddot{x}_1 = \lambda_3$$

$$m_1\ddot{y}_1 = \lambda_4$$

$$m_2\ddot{x}_2 = \lambda_5$$

$$m_2\ddot{y}_2 = \lambda_6$$

$$m_3\ddot{s} = \lambda_1 - P$$

$$J_1\ddot{\theta} = -M_r + (\lambda_1 + \lambda_5)r\sin\theta + \lambda_2 r\cos\theta + \lambda_3 e\sin\theta - \lambda_4 e\cos\theta$$

$$J_2\ddot{\phi} = \lambda_1 l\sin\phi - \lambda_2 l\cos\phi + \lambda_5 a\sin\phi - \lambda_6 b\cos\phi$$

2. Newtonian method

$$m_1\ddot{x}_1 - (X_{01} + X_{12}) = 0$$

$$m_1\ddot{y}_1 - (Y_{01} + Y_{12}) = 0$$

$$J_1\ddot{\theta} + M_r - X_{01}e\sin\theta + X_{12}(r-e)\sin\theta + Y_{01}(r-e)\cos\theta = 0$$

$$\lambda_3 = X_{01} + X_{12}$$

$$\lambda_4 = Y_{01} + Y_{12}$$

$$\lambda_1 + \lambda_5 = -X_{12}$$

$$\lambda_2 = Y_{12}$$

Comparison with analytical mechanics