MECA H 303:
Kinematics and dynamics of machines

Partim: Dynamics and vibrations

6. Rotor dynamics

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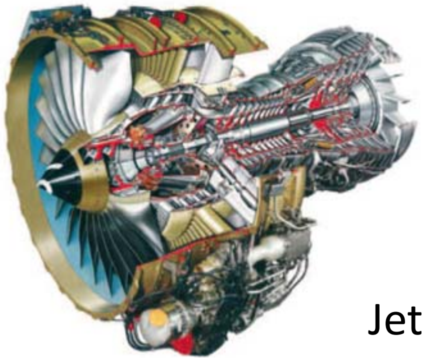
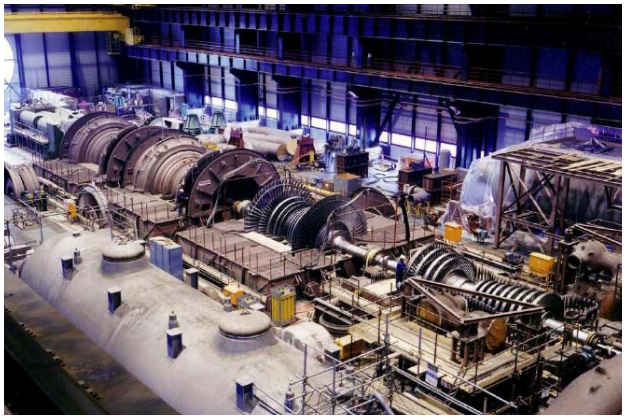
Introduction

Figure 1 - Photograph of Refrigeration Compressor Rotor



Jet engines

Machine tools



Electricity power plant

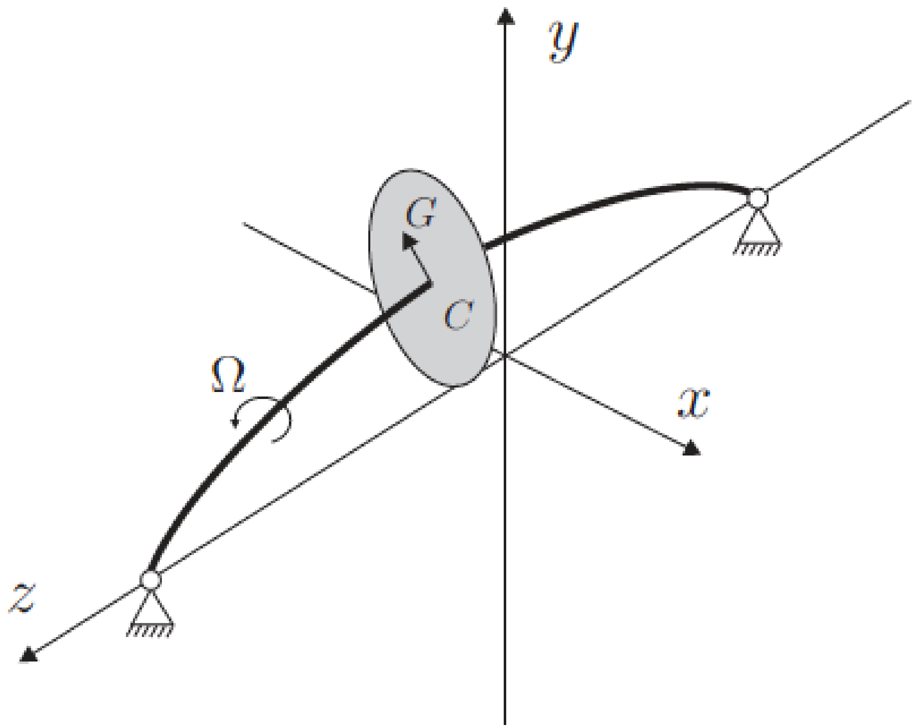
Outline

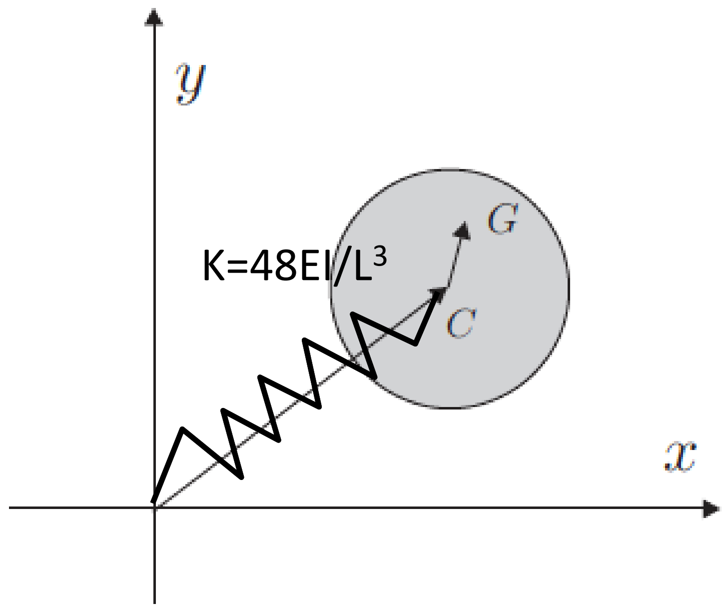
- Jefcott model without damping
  - Free whirl
  - Unbalance response
- Jeffcott model with damping
  - Free response
  - Unbalance response
- Jeffcott model including gyroscopic effects

Reference: G. Genta, Dynamics of rotating systems (ch.2 & 3), Springer (2005)

Jeffcott model without damping

(1919)





$$\begin{cases} x_G = x_C + \varepsilon \cos(\Omega t) \\ y_G = y_C + \varepsilon \sin(\Omega t) \end{cases}$$

$$\begin{cases} m\ddot{x}_C + kx_C = m\varepsilon\Omega^2 \cos \Omega t \\ m\ddot{y}_C + ky_C = m\varepsilon\Omega^2 \sin \Omega t \end{cases}$$

1. Free whirling (general solution)

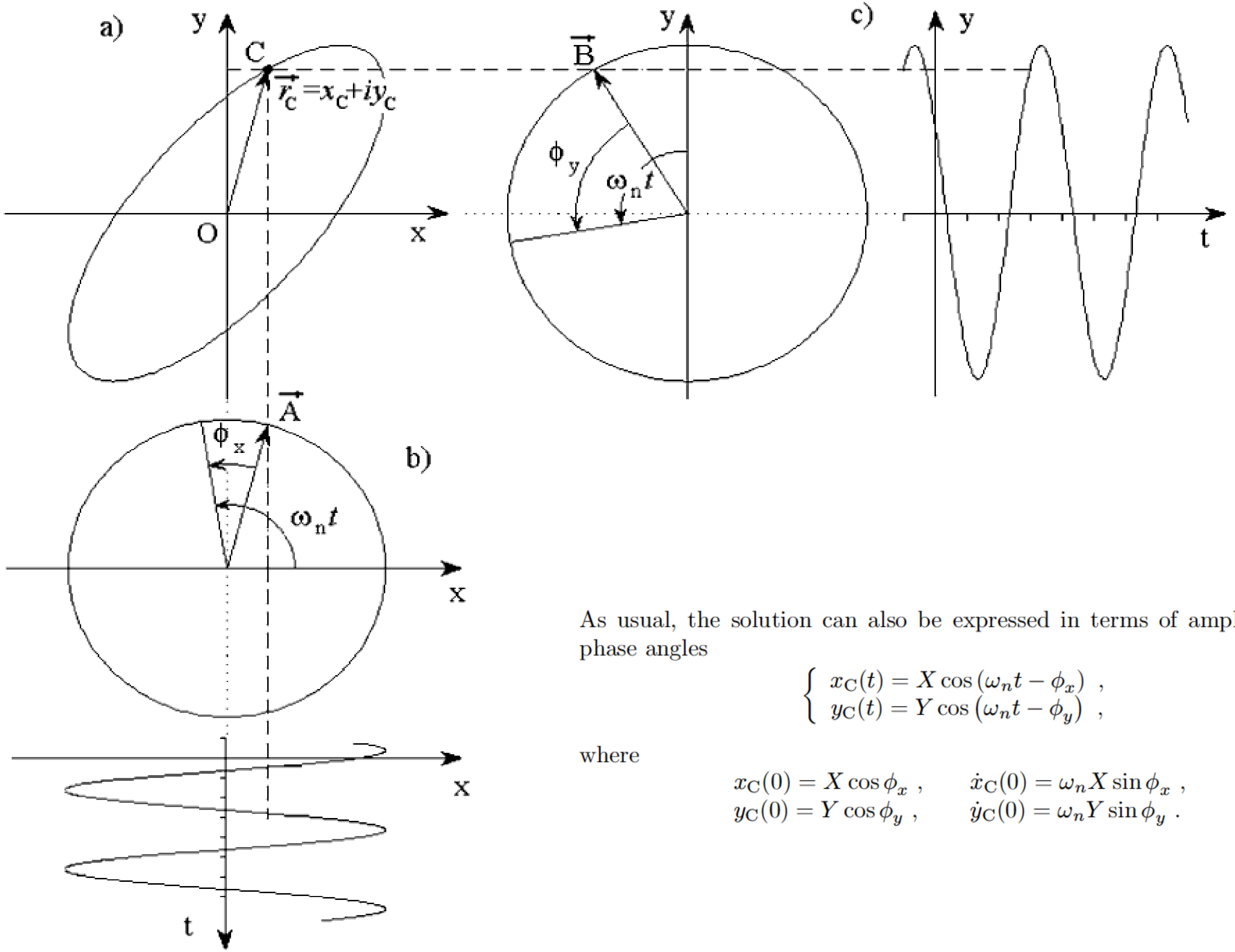
$$\begin{cases} m\ddot{x}_C + kx_C = 0 \\ m\ddot{y}_C + ky_C = 0 \end{cases}$$

$$\begin{cases} x_C = X_0 e^{st} \\ y_C = Y_0 e^{st} \end{cases}$$

$$\begin{cases} (ms^2 + k)X_0 e^{st} = 0 \\ (ms^2 + k)Y_0 e^{st} = 0 \end{cases}$$

$$\begin{cases} x_c(t) = X_1 e^{i\omega_0 t} + X_2 e^{-i\omega_0 t} \\ y_c(t) = Y_1 e^{i\omega_0 t} + Y_2 e^{-i\omega_0 t} \end{cases}$$

$$\begin{cases} x_C(t) = x_C(0)\cos(\omega_0 t) + \frac{1}{\omega_0}\dot{x}_C(0)\sin(\omega_0 t) \\ y_C(t) = y_C(0)\cos(\omega_0 t) + \frac{1}{\omega_0}\dot{y}_C(0)\sin(\omega_0 t) \end{cases}$$



As usual, the solution can also be expressed in terms of amplitude and phase angles

$$\begin{cases} x_C(t) = X\cos\left(\omega_n t - \phi_x\right) \ , \\ y_C(t) = Y\cos\left(\omega_n t - \phi_y\right) \ , \end{cases} \tag{2.18}$$

where

$$\begin{aligned} x_C(0) &= X\cos\phi_x \ , \qquad & \dot{x}_C(0) &= \omega_n X \sin\phi_x \ , \\ y_C(0) &= Y\cos\phi_y \ , \qquad & \dot{y}_C(0) &= \omega_n Y \sin\phi_y \ . \end{aligned} \tag{2.19}$$

2. Unbalance response

$$\begin{cases} m\ddot{x}_C + kx_C = m\varepsilon\Omega^2\cos\Omega t \\ m\ddot{y}_C + ky_C = m\varepsilon\Omega^2\sin\Omega t \end{cases}$$

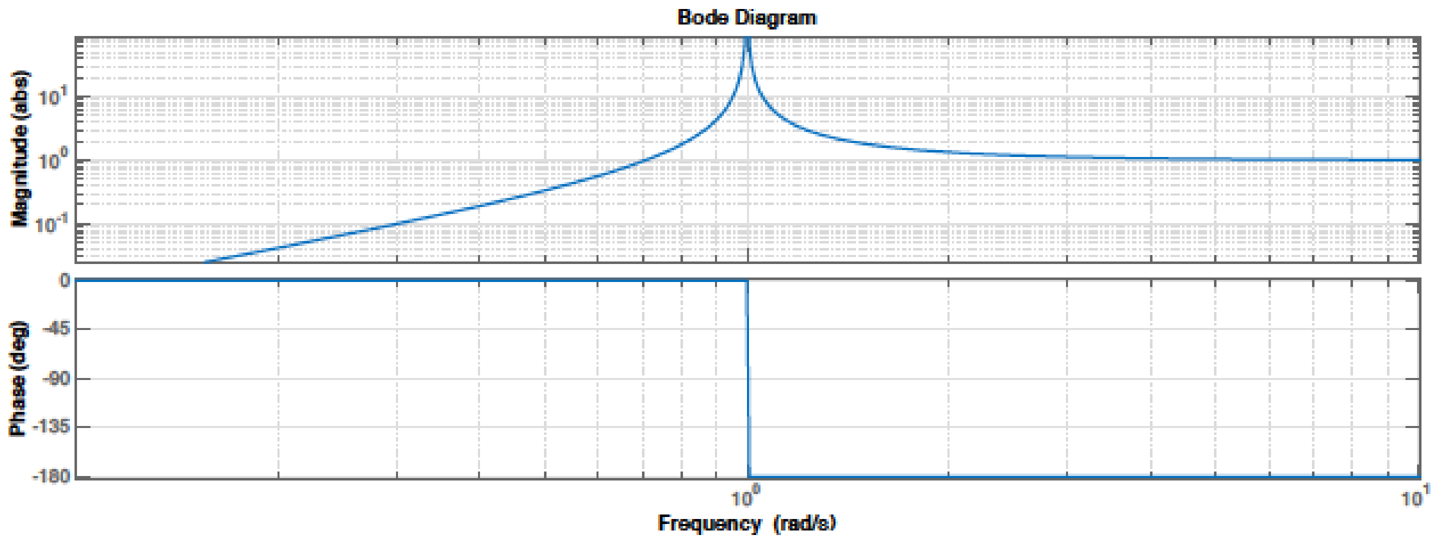
The solution is:

$$\mathbf{r}(t) = R_0(\cos\Omega t\mathbf{1}_x + \sin\Omega t\mathbf{1}_y)$$

$$\rightarrow \quad (k - \Omega^2 m)R_0 = \varepsilon m\Omega^2$$

$$\frac{R_0}{\varepsilon} = \frac{\Omega^2/\omega_0^2}{1 - \Omega^2/\omega_0^2}$$

Speed is “critical” when it matches a resonance of the rotor



The critical speed corresponds to the resonance frequency of the shaft

Estimation of the critical speed

- When both the mass and the static stiffness are known, $\Omega_c = \sqrt{k/m}$
- Measure the free response to a hammer test.
- Measure the static deflection. If the mass of the rotor is known the measurement of the static deflection y_0 allows to evaluate the stiffness equal to $k = mg/y_0$
- Rotate the shaft at increasing speed, and directly measure the critical speed (which is of course dangerous)

Complex coordinates

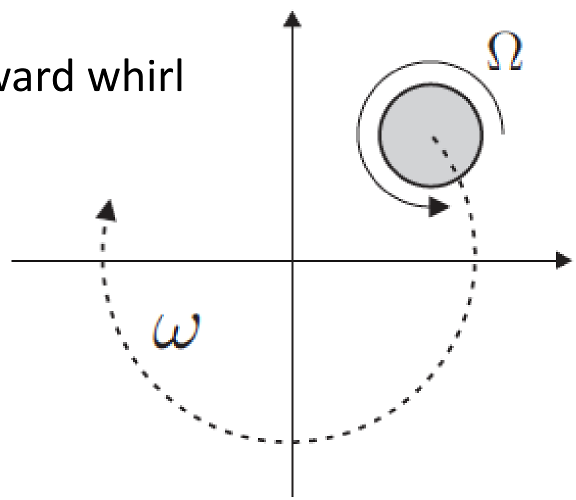
$$r_C(t) = x_C(t) + iy_C(t)$$

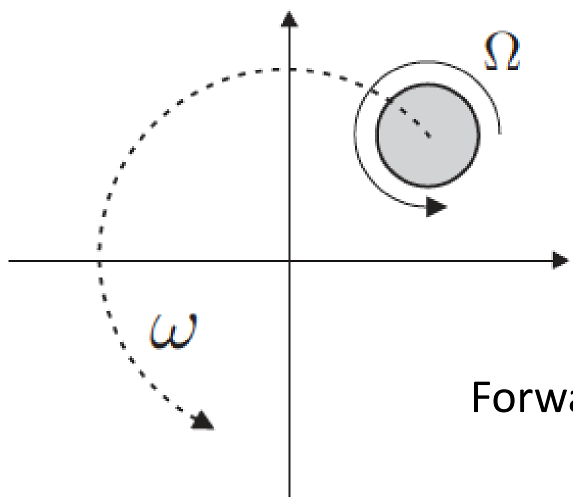
$$m\ddot{r}_C + kr_C = m\varepsilon\Omega^2 e^{i\Omega t}$$

1. Free whirling : $r_C(t) = r_0 e^{st}$ ➔ $(ms^2 + k)X_0 e^{st} = 0$

$$r_C(t) = R_1 e^{i\omega t} + R_2 e^{-i\omega t} = r_r(t) + r_b(t)$$

Backward whirl





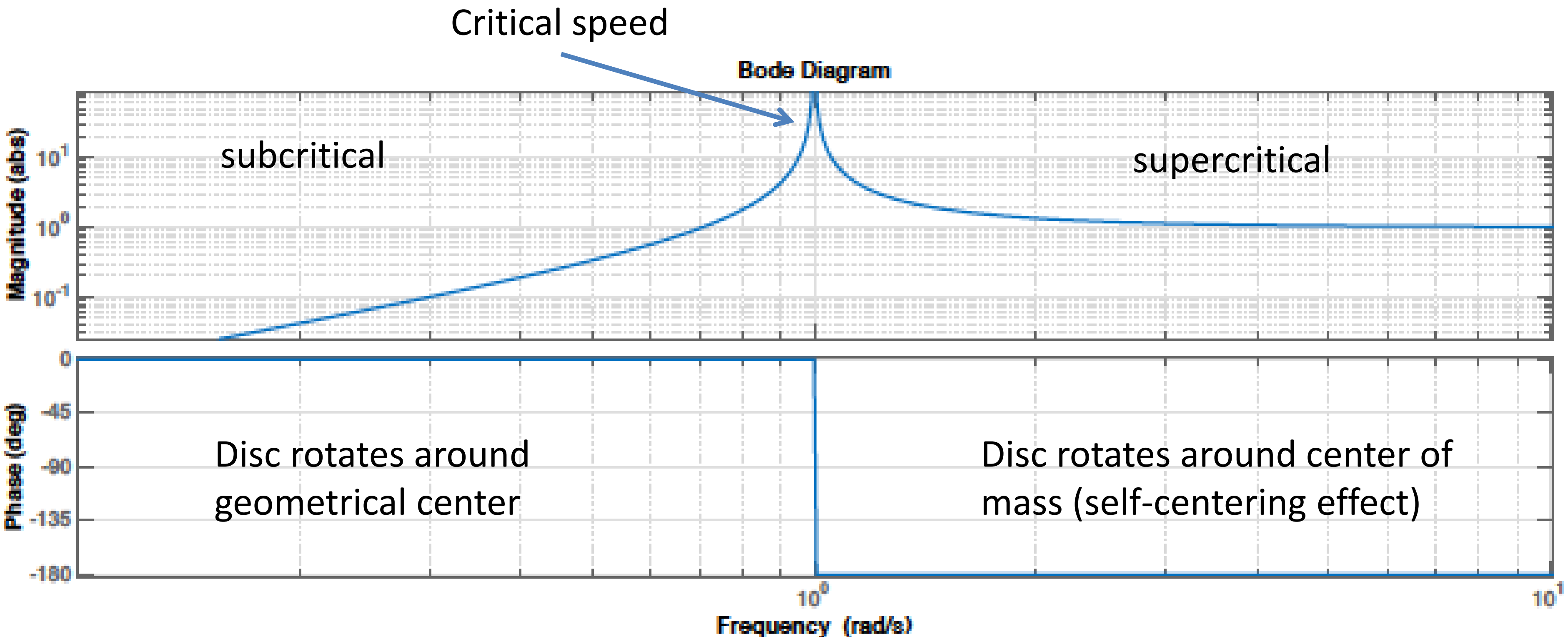
Forward whirl

Complex coordinates

2. Unbalance response $r_C(t) = r_0 e^{i\Omega t}$

$$\rightarrow \quad (-m\Omega^2 + k) r_{C_0} = m\varepsilon\Omega^2$$

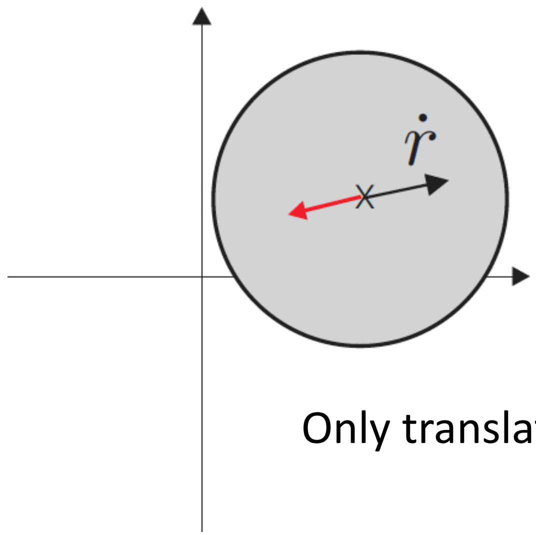
$$\frac{r_{C_0}}{\varepsilon} = \frac{\Omega^2}{\Omega_c^2 - \Omega^2}$$



Jeffcott rotor with viscous damping

Non-rotating damping

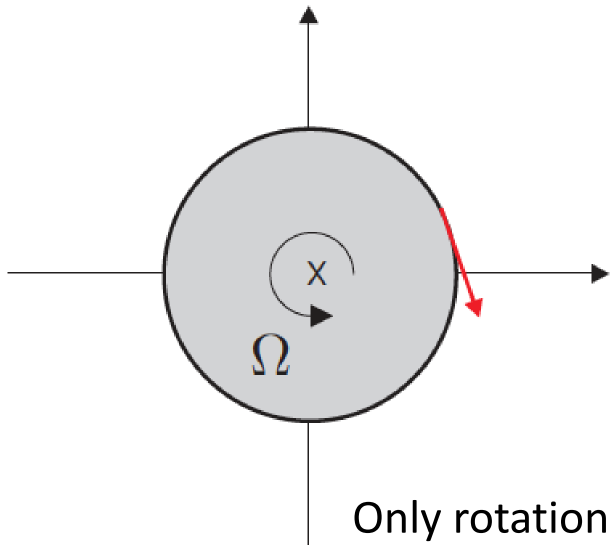
$$\mathbf{F}_n = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = -c_n \begin{Bmatrix} \dot{x}_C \\ \dot{y}_C \end{Bmatrix}$$



Only translation

Rotating damping

$$\mathbf{F}_r = \begin{Bmatrix} F_u \\ F_v \end{Bmatrix} = -c_r \begin{Bmatrix} \dot{u}_C \\ \dot{v}_C \end{Bmatrix}$$



Only rotation

Jeffcott rotor with viscous damping

Express rotation damping in fixed frame :

$$\begin{Bmatrix} u_C \\ v_C \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} x_C \\ y_C \end{Bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix}$$

$$\begin{Bmatrix} \dot{u}_C \\ \dot{v}_C \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} \dot{x}_C \\ \dot{y}_C \end{Bmatrix} + \dot{\mathbf{R}} \begin{Bmatrix} x_C \\ y_C \end{Bmatrix} \qquad \dot{\mathbf{R}} = \Omega \begin{bmatrix} -\sin(\Omega t) & \cos(\Omega t) \\ -\cos(\Omega t) & -\sin(\Omega t) \end{bmatrix}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_C \\ \ddot{y}_C \end{Bmatrix} + \begin{bmatrix} c_n + c_r & 0 \\ 0 & c_n + c_r \end{bmatrix} \begin{Bmatrix} \dot{x}_C \\ \dot{y}_C \end{Bmatrix} + \left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} + \Omega \begin{bmatrix} 0 & c_r \\ -c_r & 0 \end{bmatrix} \right) \begin{Bmatrix} x_C \\ y_C \end{Bmatrix} = m\varepsilon\Omega^2 \begin{Bmatrix} \cos(\Omega t) \\ \sin(\Omega t) \end{Bmatrix}$$

Jeffcott rotor with viscous damping

$$r_C = x_C + iy_C$$

$$m\ddot{r}_C + (c_r + c_n)\dot{x}_C + (k - ic_r\Omega)r_C = m\varepsilon\Omega^2 e^{i\Omega t}$$

Complex stiffness
= negative damping

Rotating damping performs two tasks:

1. dissipating energy
2. transferring energy from the rotation of the system to its vibration

1. Free whirling:

$$r_C = r_0 e^{st} \quad \Rightarrow \quad ms^2 + (c_r + c_n)s + k - i\Omega c_r = 0$$

$$s = \sigma + i\omega = -\frac{c_r + c_n}{2m} \pm \sqrt{\frac{(c_r + c_n)^2 - 4m(k - i\Omega c_r)}{4m^2}}$$

$$\boxed{r = R_1 e^{(\sigma_1 + i\omega_1)t} + R_2 e^{(\sigma_2 + i\omega_2)t}}$$

R_1 and R_2 are determined with initial conditions

$$\begin{cases} \sigma_{1,2} = -\frac{c_r + c_n}{2m} \pm \frac{1}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + (\frac{\Omega c_r}{m})^2} - \Gamma} \\ \omega_{1,2} = \pm \frac{\mathrm{sgn}(\Omega)}{2m} \pm \frac{1}{\sqrt{2}}\sqrt{\sqrt{\Gamma^2 + (\frac{\Omega c_r}{m})^2} + \Gamma} \end{cases} \qquad \Gamma = \frac{k}{m} - \frac{(c_r + c_n)^2}{4m^2}$$

<u>Homework:</u> Show that the system is stable ($\sigma_1 < 0$) if $\Omega < \sqrt{\frac{k}{m}}(1 + \frac{c_n}{c_r})$

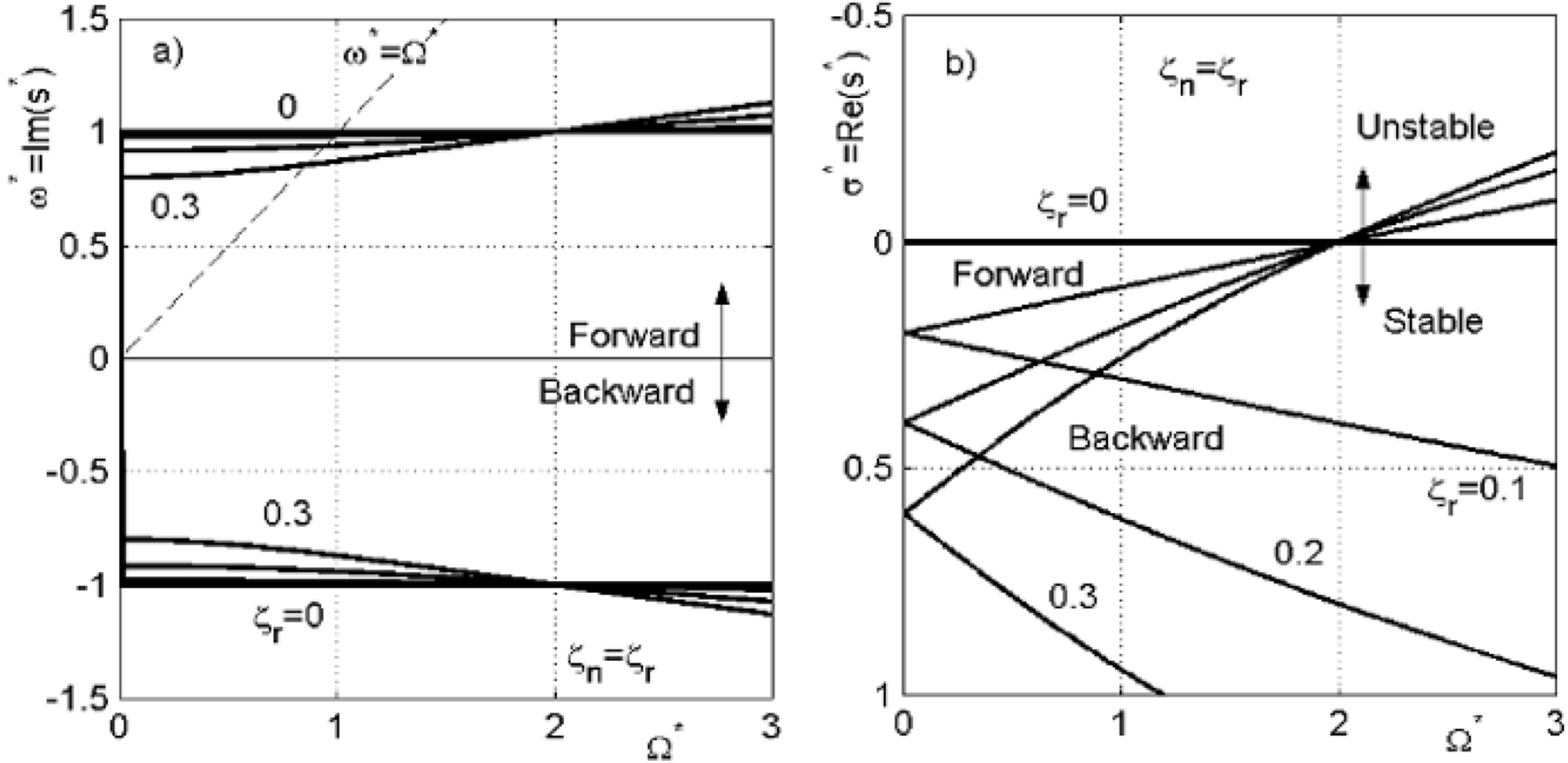
Roots can be expressed in nondimensional form:

$$\begin{cases} \sigma^* = -(\zeta_r + \zeta_n) \pm \sqrt{\sqrt{\Gamma^{*2} + \Omega^{*2}\zeta_r^2} - \Gamma^*} \ , \\ \omega^* = \pm \mathrm{sgn}(\Omega^*)\sqrt{\sqrt{\Gamma^{*2} + \Omega^{*2}\zeta_r^2} + \Gamma^*} \ , \end{cases}$$

$$\omega^* = \omega/\sqrt{k/m} \ , \qquad \sigma^* = \sigma/\sqrt{k/m} \ ,$$
$$\Omega^* = \Omega/\sqrt{k/m} \ , \qquad \Gamma^* = [1 - (\zeta_n + \zeta_r)^2]/2 \ ,$$
$$\zeta_r = c_r/2\sqrt{km} \ , \qquad \zeta_n = c_n/2\sqrt{km} \ .$$

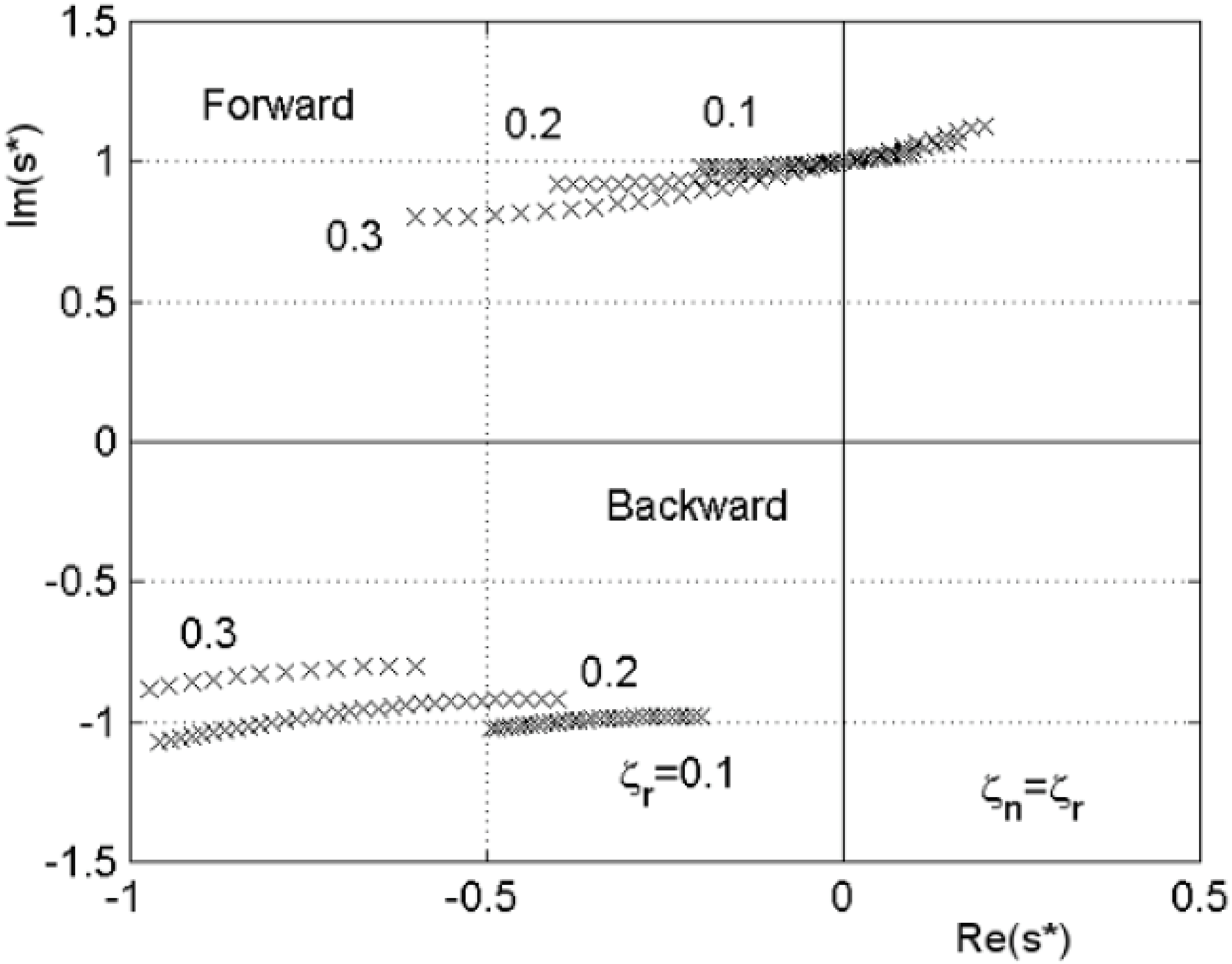
The nondimensional values of σ^* and ω^* are then functions of the spin speed Ω^* and of only two parameters ζ_n and ζ_r.

Campbell diagram　　　　　　Decay rate plot



For increasing rotation speed, the forward whirl becomes unstable !

Root locus: Motion of the roots in the complex plane for increasing value of the rotating frequency

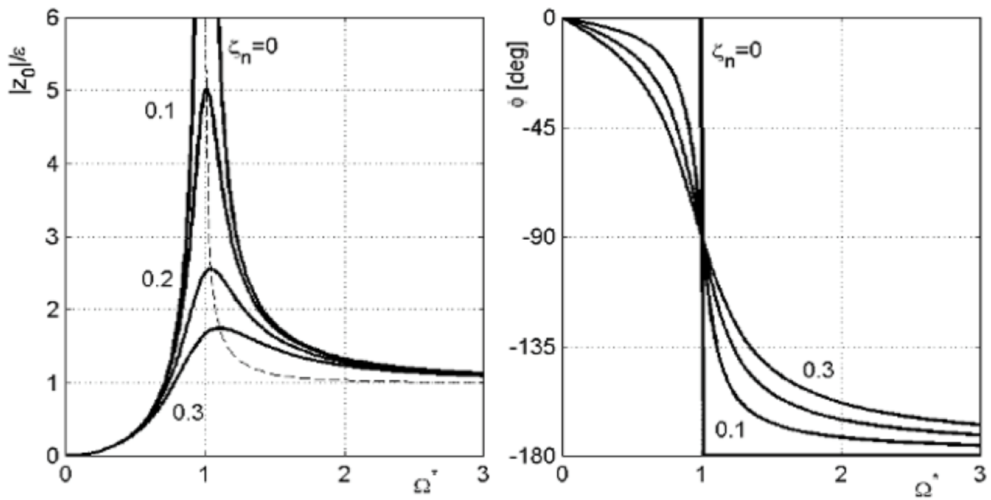


2. Unbalanced response $r_C = r_{C_0} e^{i\Omega t}$

is solution of $$m\ddot{r}_C + (c_r + c_n)\dot{r}_C + (k - ic_r\Omega)r_C = m\epsilon\Omega^2 e^{i\Omega t}$$

if $$r_{C_0}(-m\Omega^2 + i\Omega c_n + k) = m\epsilon\Omega^2$$

Rotating damping does not enter in the equation



$$|r_{C_0}| = \epsilon \frac{{\Omega^*}^2}{\sqrt{(1 - {\Omega^*}^2)^2 + 4\zeta_n^2 {\Omega^*}^2}}$$

$$\Phi = \arctan\left(\frac{-2\Omega^*\zeta_n}{1 - {\Omega^*}^2}\right).$$

Summary

- Rotation speed is critical when it corresponds to a resonance
- Without damping: the resonances do not depend on the rotation frequency
- With rotating damping: resonances depend on rotation and one pole becomes unstable for high rotation speed