Dynamics of structures

2. Vibrations: single degree of freedom system

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One degree of freedom systems in real life
Example 1:

Reduction of a system to a one dof system

Example 2:
Example 3:

Reduction of a system to a one dof system

Example 4:

Reduction of a system to a one dof system

Offshore platform
Harmonic motion

\[ x = A \sin \omega t \]

\[ \omega = \sqrt{\frac{k}{m}} \]
Harmonic signals

A periodic vibration of which the amplitude can be described by a sinusoidal function:

\[ u(t) = a \cos(\omega t + \phi) \]
\[ u(t) = a \sin(\omega t + \phi) \]

is called an harmonic vibration with:

- amplitude \( a \)
- angular frequency \( \omega = 2\pi f \)
- frequency \( f \)
- period \( T = 1/f \) or \( f = 1/T \)
- phase angle \( \phi \) at \( t=0 \)
- total phase angle \( \omega t + \phi \)

Representation in the complex plane:

\[ u(t) = ae^{i(\omega t + \phi)} \]
\[ = a \cos(\omega t + \phi) + ia \sin(\omega t + \phi) \]

\[ u(t) = Ae^{i\omega t} \]
\[ A = \frac{a \cos \phi + ia \sin \phi}{\sqrt{a^2 + (ia)^2}} \]

Independent of time

Projection of the rotating vector on the real axis is a cosine
Projection of the rotating vector on the imaginary axis is a sine
2. 1DOF system

Harmonic signals

Displacement

\[ u(t) = A e^{i\omega t} \]

Velocity

\[ v(t) = \frac{d}{dt} u(t) = i\omega A e^{i\omega t} = i\omega u(t) \]

Acceleration

\[ a(t) = \frac{d^2}{dt^2} v(t) = -\omega^2 A e^{i\omega t} = -\omega^2 u(t) \]

the phase angle of \( u(t) \) is 90° behind \( v(t) \)

the phase angle of \( v(t) \) is 90° behind \( a(t) \)

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Undamped vibrations under free and forced harmonic excitations

Conservative system: equation of motion

- Spring force: $-kx$
- External force $f$ acting on the mass.

Newton’s law:

$$m\ddot{x} = \sum F_x$$

$$m\ddot{x} + kx = f$$

Diagram:

- Mass $m$ connected to a spring with spring constant $k$.
- Force $f$ applied at the mass.
- Position $x = 0$.
- External force $f$.
2. 1DOF system

Conservative system: equation of motion

What about the effect of gravity?

\[ m\ddot{x} + kx = 0 \]

\[ x = Ae^{rt} \]

Characteristic equation:

\[ mr^2 + k = 0 \]

\[ r = \pm i \sqrt{k/m} \]

\[ x = A\cos \omega_n t + B\sin \omega_n t \]

\[ \omega_n = \sqrt{k/m} \]

In the absence of external excitation force, the motion is oscillatory. The natural angular frequency \( \omega_n \) is defined by the values of \( k \) and \( m \).

The motion is initialized by imposing initial conditions on the displacement and the velocity.
Equation of motion: general solution

\[ x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \]

Alternative representation:

\[ x(t) = a \cos (\omega_n t + \phi) \]

\[ x_0 = a \cos \phi \quad \tan \phi = \frac{\dot{x}_0}{\omega_n x_0} \quad \frac{\dot{x}_0}{\omega_n} = a \sin \phi \]

The motion can be described by a cosine function with a phase. The phase is a function of the initial conditions.
Equation of motion: particular solution for harmonic excitation

\[ f(t) = F e^{i\omega t} \quad x(t) = X e^{i\omega t} \]

\[ m\ddot{x} + kx = f \]

\[ (k - \omega^2 m) X e^{i\omega t} = F e^{i\omega t} \]

\[ X = \frac{F}{k - \omega^2 m} \]

\[ X = \frac{F}{k - \omega^2 m} \quad X_0 = \frac{F}{k} \quad (\omega = 0) \]

\[ \omega_n = \sqrt{\frac{k}{m}} \]

- Positive if \( \omega < \omega_n \)
- Infinite if \( \omega = \omega_n \)
- Negative if \( \omega > \omega_n \)
Equation of motion: particular solution for harmonic excitation

\[ \frac{X}{X_0} = \frac{1}{1 - \frac{\omega^2}{\omega_{11}^2}} \]

Dynamic amplification factor

Bode diagram (linear frequency axis)

Resonance
How resonance can lead to failure

Wine glass
Is stiffer stronger?

Forced excitation videos
Building resonance

Damped vibrations under free and forced harmonic excitation
Effect of damping on a building

Damped equation of motion

Damping force: \( F_b = -b\ddot{x} \)  
viscous damping

\[ m\ddot{x} + b\dot{x} + kx = f \]
Damped equation of motion: general solution

\[ m \ddot{x} + b \dot{x} + kx = f \]
\[ \ddot{x} + 2\xi \omega_n \dot{x} + \omega_n^2 x = \frac{f}{m} \]

**General solution**

\[ x = A e^{rt} \]

**Characteristic equation:**

\[ r^2 + 2\xi \omega_n r + \omega_n^2 = 0 \]
\[ r = -\xi \omega_n \pm i\omega_n \sqrt{1 - \xi^2} = -\xi \omega_n \pm i\omega_d \]

Initial conditions:

\[ x(t) = e^{-\xi \omega_n t} \left( A \cos \omega_d t + B \sin \omega_d t \right) \]

**Displacement** \( x_0 \)  \quad **Velocity** \( \dot{x}_0 \)

\[ x(t) = e^{-\xi \omega_n t} \left( x_0 \cos \omega_d t + \frac{\dot{x}_0 + \omega_n \xi x_0}{\omega_d} \sin \omega_d t \right) \]
Damped equation of motion: general solution

Number of oscillations after which the vibration amplitude is reduced by one half
2. 1DOF system

Damped equation of motion: general solution

\[ \xi > 1 \quad x(t) = e^{-\xi \omega_n t} \left( x_0 \cosh \mu t + \frac{x_0 + \omega_n \xi x_0}{\mu} \sinh \mu t \right) \]

\[ \mu = \omega_n \sqrt{\xi^2 - 1} \]

\[ \xi = 1 \quad x(t) = e^{-\omega_n t} \left( (x_0 + \omega_n x_0 \Delta t) t + x_0 \right) \]

Critical damping

Impulse response

Impulse = \( F \Delta t \)

\[ m \ddot{x} + b \dot{x} + kx = f \quad x_0 = 0, \dot{x}_0 = 0 \]

\[ mx_0|_{\Delta t} = F \Delta t - \int_0^{\Delta t} k \dot{x} dt - \int_0^{\Delta t} k x dt \]

\[ \dot{x}_0|_{\Delta t} = \frac{F \Delta t}{m} \]

Equivalent to initial velocity at \( \Delta t \)
### Impulse response

For an initial velocity, the response of the system is:

\[ x(t) = \frac{e^{-\xi\omega_n t} \dot{x}_0}{\omega_d} \sin(\omega_d t) \]

with \( \dot{x}_0 = \frac{F \Delta t}{m} \)

For a unit impulse \( F\Delta t = 1 \), we define the impulse response \( h(t) \):

\[ h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d} \sin(\omega_d t) \]

\( \omega_n = 1, \xi = 0.01 \)

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### Damped equation of motion: particular solution

\[ \ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = \frac{f}{m} \]

\[ x(t) = X e^{i\omega t} \]

\[ f(t) = F e^{i\omega t} \]

\[ (\omega_n^2 + 2i\xi\omega_n - \omega^2)X = \frac{F}{m} \]

\[ X = \frac{F}{m} \left( \frac{1}{\omega_n^2 + 2i\xi\omega_n - \omega^2} \right) = \frac{F}{k} \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi \frac{\omega}{\omega_n}} \right) \]

\[ = X_0 \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\xi \frac{\omega}{\omega_n}} \right) \]
Damped equation of motion : particular solution

\[ X_r = X_0 \frac{1 - \frac{\omega^2}{\omega_n^2}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2} \]

\[ X_i = X_0 \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2} \]

\[ \frac{X}{X_0} = \sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \]

\[ \tan \phi = \frac{-2\xi \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}} \]

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Damped equation of motion : particular solution

Maximum

\[ \frac{\omega}{\omega_n} = \sqrt{1 - 2\xi^2} \]

Max ampl. \[ |\frac{X}{X_0}| = \frac{1}{2\xi \sqrt{1 - \xi^2}} \]

For \( \xi \) small

\[ \frac{\omega}{\omega_n} = 1 \]

\[ |\frac{X}{X_0}| = \frac{1}{2\xi} \]
2.1DOF system

**Bode diagram**

- **Displacement**
  \[ u(t) = Ae^{i\omega t} \]
- **Velocity**
  \[ v(t) = \frac{du(t)}{dt} = i\omega Ae^{i\omega t} = i\omega u(t) \]
- **Acceleration**
  \[ a(t) = \frac{dv(t)}{dt} = -\omega^2 Ae^{i\omega t} = -\omega^2 u(t) \]

Dynamic response for arbitrary force input
Time domain response using Duhamel’s integral

\[ f(\tau) \, d\tau \, h(t - \tau) \]

(h(t) is the impulse response)

The total contribution is therefore:

\[ x(t) = \int_{0}^{t} f(\tau) h(t - \tau) \, d\tau \]

We have \( h(t) = 0 \) and \( f(t) = 0 \) for \( t < 0 \) so that we can write

\[ x(t) = \int_{-\infty}^{\infty} f(\tau) h(t - \tau) \, d\tau = f(t) * h(t) \]

Convolution integral

The convolution integral of two time functions \( x(t) \) and \( h(t) \) yields a new time function \( y(t) \) defined as:

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau \]

\[ y(t) = x(t) * h(t) \]

- Take the two functions \( x(t) \) and \( h(t) \) and replace \( t \) by the dummy variable \( \tau \)
- Mirror the function \( h(\tau) \) against the ordinate, this yields \( h(-\tau) \)
- Shift the function \( h(-\tau) \) with a quantity \( t \)
- Determine for each value of \( t \) the product of \( x(\tau) \) with \( h(t-\tau) \)
- Compute the integral of the product \( y(t) \)
- Let \( t \) vary from \(-\infty\) (or a value small enough to make the product zero) to \( \infty \) (or a value of \( t \) that is big enough)
2. 1DOF system

Convolution integral

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

1st function

\[ x(\tau) \]

Mirrored 2nd function

\[ h(-\tau) \]

Shift \( h(\tau) \):

\[ \frac{1}{2} \]

Convolution integral

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

Property:

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) d\tau \]
2. 1DOF system

Harmonic excitation below resonance

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \frac{\omega}{\omega_n} = 0.1 \]

\[ \varepsilon_n = 1, \xi = 0.01 \]

Harmonic excitation near resonance

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \frac{\omega}{\omega_n} = 0.95 \]

\[ \varepsilon_n = 1, \xi = 0.01 \]
2.1DOF system

Harmonic excitation at resonance

\[ f(t) = \sin(\omega t) \]

\[ \frac{\omega}{\omega_n} = 1, \xi = 0.01 \]

\[ \frac{\omega}{\omega_n} = 1, \xi = 0.01 \]

Harmonic excitation at resonace

\[ \frac{\omega}{\omega_n} = 1, \xi = 0.05 \]
2. 1DOF system

Harmonic excitation above resonance

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \frac{\omega}{\omega_m} = 2 \]

\[ \omega_m = 1, \xi = 0.01 \]

Sine sweep excitation

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \omega_m = 1, \xi = 0.01 \]
Base excitation

\[ m \ddot{x} = -k(x - x_0) + b(\dot{x} - \dot{x}_0) \]

\[ x_r = x - x_0 \]

\[ m \ddot{x}_r + b \dot{x}_r + k x_r = -m \dddot{x}_0 \]

Excitation
Santa Cruz earthquake base excitation

\[ f(t) = -m \ddot{x}_0 \]

\[ h(t) \]

\[
\begin{align*}
\text{Natural frequency} & = 6 \text{ Hz} \\
\xi & = 0.01
\end{align*}
\]

Response to earthquake excitation

Illustration

Mass on a Spring
Application example: the accelerometer

In the frequency domain:

\[ M\ddot{x}_r + c\dot{x}_r + kx_r = -M\dot{x}_0 \]

In the frequency domain:

\[ \frac{x_r}{x_0} = \frac{-1}{-\omega^2 + \omega_n^2 + 2i\xi\omega_n} \]

\[ \omega \ll \omega_n \quad \Rightarrow \quad \frac{x_r}{x_0} \approx \frac{-1}{\omega_n^2} \]
2. 1DOF system

Application example: the accelerometer

Piezoelectric accelerometer

\[
\frac{x'}{x(0)} \approx \frac{-1}{\omega_n^2}
\]

Longitudinal mode

Shear mode

Application example: the accelerometer

High resonant frequency
Low sensitivity

Low resonant frequency
High sensitivity