2. Vibrations: single degree of freedom system

Arnaud Deraemaeker (aderaema@ulb.ac.be)

One degree of freedom systems in real life
2. 1DOF system

Reduction of a system to a one dof system

Example 1:

Reduction of a system to a one dof system

Example 2:
Example 3:

Reduction of a system to a one dof system

Example 4:

Reduction of a system to a one dof system

Offshore platform
Harmonic motion

\[ x = A \sin \omega t \]

\[ \omega = \sqrt{\frac{k}{m}} \]
Harmonic signals

A periodic vibration of which the amplitude can be described by a sinusoidal function:

\[ u(t) = a \cos(\omega t + \phi) \]
\[ u(t) = a \sin(\omega t + \phi) \]

is called a harmonic vibration with:

- amplitude \( a \)
- angular frequency \( \omega = 2\pi f \)
- frequency \( f \)
- period \( T = 1/f \) or \( f = 1/T \)
- phase angle \( \phi \) at \( t=0 \)
- total phase angle \( \omega t + \phi \)

Representation in the complex plane:

\[ u(t) = ae^{i(\omega t + \phi)} \]
\[ = a \cos(\omega t + \phi) + ia \sin(\omega t + \phi) \]

\[ A = \frac{ae^{i\phi}e^{i\omega t}}{a \cos \phi + ia \sin \phi} = Ae^{i\omega t} \]

Independent of time

Projection of the rotating vector on the real axis is a cosine
Projection of the rotating vector on the imaginary axis is a sine
Harmonic signals

- **Displacement**
  \[ u(t) = Ae^{i\omega t} \]

- **Velocity**
  \[ v(t) = \frac{du(t)}{dt} = i\omega Ae^{i\omega t} = i\omega u(t) \]

- **Acceleration**
  \[ a(t) = \frac{dv(t)}{dt} = -\omega^2 Ae^{i\omega t} = -\omega^2 u(t) \]

The phase angle of \( u(t) \) is 90° behind \( v(t) \)
The phase angle of \( v(t) \) is 90° behind \( a(t) \)

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2. 1DOF system

Undamped vibrations under free and forced harmonic excitations

Conservative system: equation of motion

- Spring force: $-kx$
- External force $f$ acting on the mass.

Newton's law:

$$ m\ddot{x} = \sum F_x $$

$$ m\ddot{x} + kx = f $$
2. 1DOF system

Dynamics of Structures 2019-2020

Conservative system: equation of motion

What about the effect of gravity?

\[ l_0 \]
\[ k \]
\[ l_0 + \Delta l \]
\[ \Delta l \]
\[ m \]
\[ m \]
\[ mg \]

The displacement \( x \) is defined with respect to the equilibrium position of the mass subjected to gravity. The effect of gravity should therefore not be taken into account in the equation of motion of the system.

Equation of motion: general solution

\[ m\ddot{x} + kx = 0 \]

\[ x = A e^{rt} \]

Characteristic equation:

\[ mr^2 + k = 0 \]

\[ r = \pm i \sqrt{k/m} \]

\[ x = A \cos \omega_n t + B \sin \omega_n t \]

\[ \omega_n = \sqrt{k/m} \]

- In the absence of external excitation force, the motion is oscillatory. The natural angular frequency \( \omega_n \) is defined by the values of \( k \) and \( m \)
- The motion is initialized by imposing initial conditions on the displacement and the velocity
Equation of motion: general solution

\[ x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \]

Alternative representation:

\[ x(t) = a \cos (\omega_n t + \phi) \]

\[ x_0 = a \cos \phi \quad \tan \phi = \frac{\dot{x}_0}{\omega_n x_0} \]

\[ \frac{\dot{x}_0}{\omega_n} = a \sin \phi \]

The motion can be described by a cosine function with a phase. The phase is a function of the initial conditions.
Equation of motion: particular solution for harmonic excitation

\[ f(t) = F e^{i\omega t} \quad x(t) = X e^{i\omega t} \]

\[ m\ddot{x} + kx = f \]

\[ (k - \omega^2 m) X e^{i\omega t} = F e^{i\omega t} \]

\[ X = \frac{F}{k - \omega^2 m} \]

\[ X = \frac{F}{k - \omega^2 m} \quad X_0 = \frac{F}{k} \quad (\omega = 0) \]

\[ \omega_n = \sqrt{\frac{k}{m}} \]

\[ \frac{X}{X_0} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2}} \]

- Positive if \( \omega < \omega_n \)
- Infinite if \( \omega = \omega_n \)
- Negative if \( \omega > \omega_n \)
2. 1DOF system

Equation of motion: particular solution for harmonic excitation

\[
\frac{X}{X_0} = \frac{1}{1 - \frac{\omega^2}{\omega_0^2}}
\]

Dynamic amplification factor

Bode diagram (linear frequency axis)

Resonance
How resonance can lead to failure

Wine glass
Is stiffer stronger?

Forced excitation videos
2. 1DOF system

Building resonance

Damped vibrations under free and forced harmonic excitation
Effect of damping on a building

Damped equation of motion

Damping force: \( F_b = -b \dot{x} \)  

\[ m\ddot{x} + b\dot{x} + kx = f \]
Damped equation of motion: general solution

\[ m\ddot{x} + b\dot{x} + kx = f \]

Characteristic equation:
\[ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f/m \]

General solution

\[ x = A e^{r_1 t} \]

Characteristic equation:
\[ r^2 + 2\xi\omega_n r + \omega_n^2 = 0 \]

Initial conditions:

\[ x(t) = e^{-\xi\omega_n t} \left( A \cos \omega_d t + B \sin \omega_d t \right) \]

Initial conditions:

\[ x(t) = e^{-\xi\omega_n t} \left( x_0 \cos \omega_d t + \frac{\dot{x}_0 + \omega_n \xi x_0}{\omega_d} \sin \omega_d t \right) \]
Damped equation of motion: general solution

Number of oscillations after which the vibration amplitude is reduced by one half
Damped equation of motion: general solution

\[ x(t) = e^{-\xi \omega_n t} \left( x_0 \cosh \mu t + \frac{x_0 + \omega_n x_0}{\mu} \sinh \mu t \right) \]

where

\[ \mu = \omega_n \sqrt{\xi^2 - 1} \]

- \( \xi > 1 \)
- \( \xi = 1 \) \text{ Critical damping}

\[ x(t) = e^{-\omega_n t} \left( x_0 + \omega_n t \right) + x_0 \]

\[ \xi = 0.1 \]

Impulse response

\[ \text{Impulse} = F \Delta t \]

\[ m \ddot{x} + b \dot{x} + k x = f \]

\[ x_0 = 0, \dot{x}_0 = 0 \]

\[ m \ddot{x}_0|_{\Delta t} = F \Delta t - \int_0^{\Delta t} k x dt - \int_0^{\Delta t} k \dot{x} dt \]

\[ \dot{x}_0|_{\Delta t} = \frac{F \Delta t}{m} \]

Equivalent to initial velocity at \( \Delta t \)
Impulse response

For an initial velocity, the response of the system is:

\[ x(t) = \frac{e^{-\xi\omega_n t} \dot{x}_0}{\omega_d} \sin(\omega_d t) \]

with \( \dot{x}_0 = \frac{F \Delta t}{m} \)

For a unit impulse \( F \Delta t = 1 \), we define the impulse response \( h(t) \)

\[ h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d} \sin(\omega_d t) \]

\[ \omega_n = 1, \xi = 0.01 \]

---

Damped equation of motion: particular solution

\[ \ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = \frac{f}{m} \]

\[ x(t) = X e^{i\omega t} \]

\[ f(t) = F e^{i\omega t} \]

\( (\omega_n^2 + 2i\xi\omega_n - \omega^2)X = \frac{F}{m} \)

\[ X = \frac{F}{m} \left( \frac{1}{\omega_n^2 + 2i\xi\omega_n - \omega^2} \right) = \frac{F}{k} \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\frac{\xi}{\omega_n}} \right) \]

\[ = X_0 \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2i\frac{\xi}{\omega_n}} \right) \]
Damped equation of motion: particular solution

\[
X_r = X_0 \frac{1 - \frac{\omega^2}{\omega_n^2}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}
\]

\[
X_i = X_0 \frac{-2\xi \frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}
\]

\[
\frac{|X|}{X_0} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}
\]

\[
\tan \phi = -\frac{2\xi \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}
\]

Maximum

\[
\frac{\omega}{\omega_n} = \sqrt{1 - 2\xi^2}
\]

Max ampl.

\[
|X|/X_0 = \frac{1}{2\xi \sqrt{1 - \xi^2}}
\]

For \(\xi\) small

\[
\frac{\omega}{\omega_n} = 1
\]

\[
|X|/X_0 = \frac{1}{2\xi}
\]
Bode diagram

Displacement \[ u(t) = Ae^{i\omega t} \]

Velocity \[ v(t) = \frac{du(t)}{dt} = i\omega Ae^{i\omega t} = i\omega u(t) \]

Acceleration \[ a(t) = \frac{dv(t)}{dt} = -\omega^2 Ae^{i\omega t} = -\omega^2 u(t) \]

Dynamic response for arbitrary force input
Time domain response using Duhamel’s integral

- \( f(t) \) is decomposed into a series of short impulses at time \( \tau \)
- The contribution of one impulse \( f(\tau) d\tau \) to the response of the system is given by:
  \[
  f(\tau) d\tau h(t - \tau)
  \]
- (\( h(t) \) is the impulse response)
- The total contribution is therefore:
  \[
  x(t) = \int_0^t f(\tau) h(t - \tau) d\tau
  \]
  
We have \( h(t) = 0 \) and \( f(t) = 0 \) for \( t < 0 \) so that we can write

\[
 x(t) = \int_{-\infty}^\infty f(\tau) h(t - \tau) d\tau = f(t) * h(t)
\]

Convolution integral

- The **convolution integral** of two time functions \( x(t) \) and \( h(t) \) yields a new time function \( y(t) \) defined as:
  \[
  y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau
  \]
  
- \( y(t) = x(t) * h(t) \)

- Take the two functions \( x(t) \) and \( h(t) \) and replace \( t \) by the dummy variable \( \tau \)
- Mirror the function \( h(\tau) \) against the ordinate, this yields \( h(-\tau) \)
- Shift the function \( h(-\tau) \) with a quantity \( t \)
- Determine for each value of \( t \) the product of \( x(\tau) \) with \( h(t-\tau) \)
- Compute the integral of the product \( y(t) \)
- Let \( t \) vary from \(-\infty\) (or a value small enough to make the product zero) to \( \infty \) (or a value of \( t \) that is big enough)
Convolution integral

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau \]

1st function

Mirrored 2nd function

Property:

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau = \int_{-\infty}^{\infty} x(t - \tau)h(\tau) d\tau \]
2. 1DOF system

Harmonic excitation below resonance

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \frac{\omega}{\omega_0} = 0.1 \]
\[ \xi = 0.01 \]

Harmonic excitation near resonance

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \frac{\omega}{\omega_0} = 0.95 \]
\[ \xi = 1, \xi = 0.01 \]

Beating
2. 1DOF system

Harmonic excitation at resonance

\[ f(t) = \sin(\omega t) \]

\[ \frac{\omega}{\omega_n} = 1 \]

\[ \omega_n = 1, \xi = 0.01 \]

Steady-state regime

Transient regime

Harmonic excitation at resonance

\[ \frac{\omega}{\omega_n} = 1 \]

\[ \omega_n = 1, \xi = 0.005 \]
2. 1DOF system

Harmonic excitation above resonance

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \frac{\omega}{\omega_n} = 2 \]
\[ \omega_n = 1, \xi = 0.01 \]

Sine sweep excitation

\[ f(t) = \sin(\omega t) \]

\[ h(t) \]

\[ \omega_m = 1, \xi = 0.01 \]

Resonance
2. 1DOF system

Base excitation

\[
\begin{align*}
  x_0 &= 0 \\
  x &= 0 \\
  x_r &= x - x_0
\end{align*}
\]

\[
\begin{align*}
  m\ddot{x} &= -k(x - x_0) - b(\dot{x} - \dot{x}_0) \\
  m\ddot{x}_r + b\dot{x}_r + kx_r &= -m\ddot{x}_0
\end{align*}
\]
Illustration

Application example: the accelerometer

\[
M \ddot{x}_r + c \dot{x}_r + k x_r = -M \ddot{x}_0
\]

In the frequency domain:

\[
\frac{x_r}{x_0} = \frac{-1}{-\omega^2 + \omega_n^2 + 2i\xi \omega \omega_n}
\]

\[\omega \ll \omega_n \quad \rightarrow \quad \frac{x_r}{x_0} \approx \frac{-1}{\omega_n^2}\]
Application example: the accelerometer

Piezoelectric accelerometer

\[ \frac{x_r}{x_r^0} \approx -\frac{1}{\omega_n^2} \]

Longitudinal mode

Shear mode

Application example: the accelerometer

High resonant frequency
Low sensitivity

Low resonant frequency
High sensitivity
Response of a 1DOF system subjected to ground acceleration

\[ m\ddot{x} = -k(x - x_0) - b(\dot{x} - \ddot{x}_0) \]
\[ x_r = x - x_0 \]
\[ m\ddot{x}_r + b\dot{x}_r + kx_r = -m\ddot{x}_0 \]

Earthquake base acceleration

Duhamel’s integral

\[ x(t) = \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau = f(t) * h(t) \]

Impulse response

\[ h(t) = \frac{e^{-\xi\omega_n t}}{m \omega_d} \sin(\omega_d t) \]

Response to earthquake excitation

\[ x_r(t) = \int_{-\infty}^{\infty} -\ddot{x}_0(t) \frac{e^{-\xi\omega_n t}}{\omega_d} \sin(\omega_d t) \]
Response spectrum

\[ x_r(t) = \int_{-\infty}^{\infty} -\ddot{x}_0(t) e^{-\xi \omega_n t} \frac{\sin(\omega_d t)}{\omega_d} \, dt \]

Relative displacement spectrum

\[ S_d(\xi, \omega_n) = |x_r(t)|_{max} \]

Relative velocity spectrum

\[ S_v(\xi, \omega_n) = |\dot{x}_r(t)|_{max} \]

Absolute acceleration spectrum

\[ S_a(\xi, \omega_n) = |\ddot{x}_0(t) + \ddot{x}_r(t)|_{max} \]

Pseudo velocity response spectrum

\[ S_{pv}(\xi, \omega_n) = \omega_n S_d(\xi, \omega_n) \]

Pseudo acceleration response spectrum

\[ S_{pa}(\xi, \omega_n) = \omega_n^2 S_d(\xi, \omega_n) \]

Example of response spectrum

Figure 4.2 Displacement response spectra of El-Centro, 1940 earthquake ground motion.
Example of response spectrum

Figure 4.3 Velocity response spectra of El-Centro, 1940 earthquake ground motion.

Example of response spectrum

Figure 4.4 Acceleration response spectra of El-Centro, 1940 earthquake ground motion.
For an undamped one dof system:

\[ m\ddot{x}_r + kx_r = -m\ddot{x}_0 \]
\[ \dot{x}_r + \omega^2_n x_r = -\dot{x}_0 \]
\[ \ddot{x}_r + \ddot{x}_0 = -\omega^2_n x_r \]

\[ S_{pa} = S_c = \omega^2_n S_d \approx S_d \]

The response spectrum specified in the Eurocode is an approximation of the total acceleration, valid when damping is small.
Response spectra in the Eurocode 8

![Response spectra diagram](image)

### Table 3.2: Values of the parameters describing the recommended Type 1 elastic response spectra

<table>
<thead>
<tr>
<th>Ground type</th>
<th>( T_a )</th>
<th>( T_1 ) (s)</th>
<th>( T_2 ) (s)</th>
<th>( T_c ) (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.0</td>
<td>0.15</td>
<td>0.4</td>
<td>2.0</td>
</tr>
<tr>
<td>B</td>
<td>1.2</td>
<td>0.15</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>C</td>
<td>1.5</td>
<td>0.20</td>
<td>0.4</td>
<td>2.0</td>
</tr>
<tr>
<td>D</td>
<td>1.32</td>
<td>0.20</td>
<td>0.8</td>
<td>2.0</td>
</tr>
<tr>
<td>E</td>
<td>1.6</td>
<td>0.15</td>
<td>0.5</td>
<td>2.0</td>
</tr>
</tbody>
</table>

**Figure 3.3:** Shape of the elastic response spectrum

- \( T_a \): The value of the periods \( T_a, T_1, \) and \( T_2 \) and of the soil factor \( S \) describing the shape of the elastic response spectrum depend upon the ground type.

### Table 3.3: Values of the parameters describing the recommended Type 2 elastic response spectra

<table>
<thead>
<tr>
<th>Ground type</th>
<th>( S )</th>
<th>( T_1 ) (s)</th>
<th>( T_2 ) (s)</th>
<th>( T_c ) (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.0</td>
<td>0.05</td>
<td>0.29</td>
<td>1.2</td>
</tr>
<tr>
<td>B</td>
<td>1.55</td>
<td>0.05</td>
<td>0.25</td>
<td>1.2</td>
</tr>
<tr>
<td>C</td>
<td>1.3</td>
<td>0.10</td>
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<td>1.2</td>
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<tr>
<td>D</td>
<td>1.8</td>
<td>0.10</td>
<td>0.30</td>
<td>1.2</td>
</tr>
<tr>
<td>E</td>
<td>1.8</td>
<td>0.05</td>
<td>0.25</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Response spectra in the Eurocode

![Response spectra diagram](image)

**Figure 3.3:** Recommended Type 2 elastic response spectra for ground types A to E (5% damping)