Chapter 13

Refraction

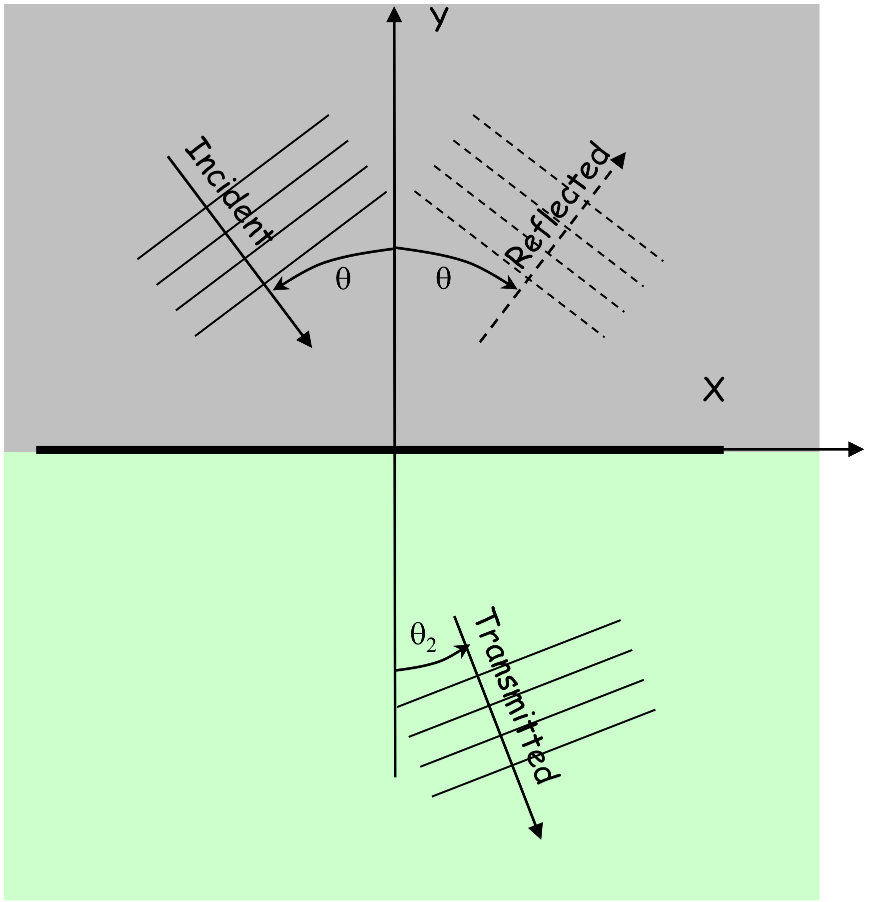
Jean-Louis Migeot

1. Snell-Descartes' law
2. Application to the middle-ear
3. Lifeguard analogy

Snell-Descartes' law

$$p_1(\vec{r},\omega) = I\,(\omega)\, e^{-ik_1 x \sin\theta_1 + ik_1 y \cos\theta_1} + R(\omega) e^{-ik_1 x \sin\theta_1 - ik_1 y \cos\theta_1}$$

$$p_2(\vec{r},\omega) = T(\omega) e^{-ik_2 x \sin\theta_2 + ik_2 y \cos\theta_2}$$



Pressure continuity:

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \longrightarrow \frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$$

$c_1 > c_2$ -> θ_2 real for all θ_1

$c_1 < c_2$ -> θ_2 real for all $\theta_1 < \theta_c$

$$\sin\theta_c = \frac{c_1}{c_2}$$

Transmitted and reflected fields

Pressure continuity:

$$T(\omega) = I(\omega) + R(\omega)$$

Velocity continuity:

$$\frac{I(\omega) - R(\omega)}{\rho_1 c_1} \cos\theta_1 = \frac{T(\omega)}{\rho_2 c_2} \cos\theta_2$$

$$R = \frac{\frac{\cos\theta_1}{\rho_1 c_1} - \frac{\cos\theta_2}{\rho_2 c_2}}{\frac{\cos\theta_1}{\rho_1 c_1} + \frac{\cos\theta_2}{\rho_2 c_2}} = \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - \frac{\cos\theta_2}{\cos\theta_1}}{\frac{\rho_2 c_2}{\rho_1 c_1} + \frac{\cos\theta_2}{\cos\theta_1}}$$

$$T = \frac{\frac{2\cos\theta_1}{\rho_1 c_1}}{\frac{\cos\theta_1}{\rho_1 c_1} + \frac{\cos\theta_2}{\rho_2 c_2}} = \frac{2\frac{\rho_2 c_2}{\rho_1 c_1}}{\frac{\rho_2 c_2}{\rho_1 c_1} + \frac{\cos\theta_2}{\cos\theta_1}}$$

Transmission coefficient τ

$$\left.\begin{aligned} I_{inc} &= \frac{|I|^2}{\rho_1 c_1} \\ I_{trans} &= \frac{|T|^2}{\rho_2 c_2} \end{aligned}\right\} \longrightarrow \tau = \frac{|T(\omega)|^2 \rho_1 c_1}{|I(\omega)|^2 \rho_2 c_2} = \frac{4\frac{\rho_2 c_2}{\rho_1 c_1}}{(\frac{\rho_2 c_2}{\rho_1 c_1} + \frac{\cos\theta_2}{\cos\theta_1})^2}$$

Middle ear: an impedance adapter

- Loss from air to endolymph (~ water)

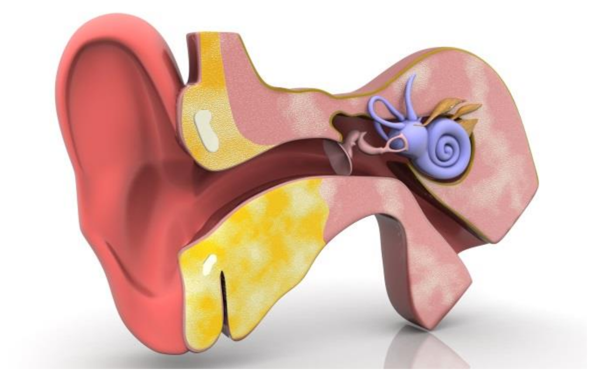
$$\tau = \frac{4\frac{\rho_2 c_2}{\rho_1 c_1}}{(1+\frac{\rho_2 c_2}{\rho_1 c_1})^2}$$

$$\frac{\rho_2 c_2}{\rho_1 c_1} = \frac{1000 \cdot 1500}{1.225 \cdot 340} = 3600$$

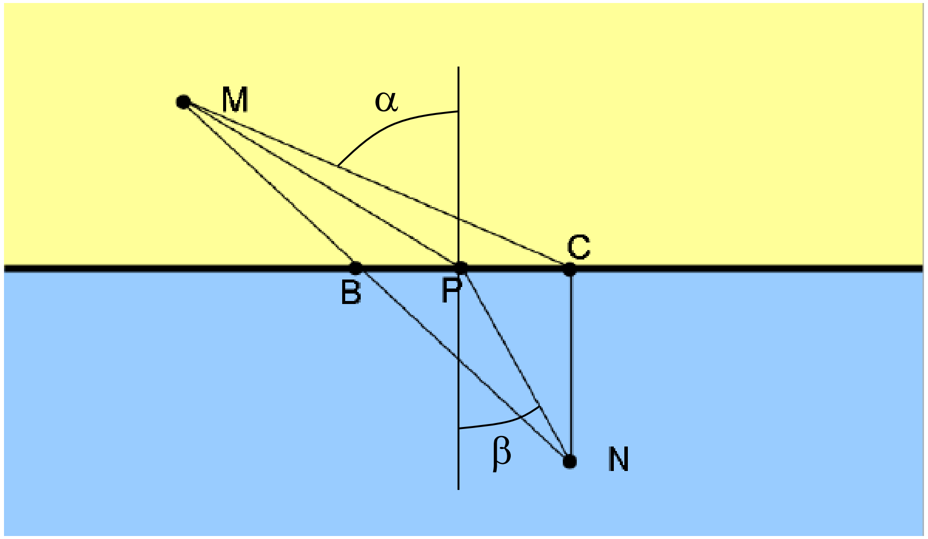
$$10 \log \tau = -29\ dB$$

- Is nearly compensated by two complementary mechanisms:
  - area ratio of oval vs. tympanic window: 20
  - mechanical amplification by the ossicle chain: 1.3

$$\Delta L = \underbrace{20 \log 20}_{\Delta L_1} + \underbrace{20 \log 1.3}_{\Delta L_2} \simeq 28\ dB$$



The *lifeguard* analogy



$$T = \frac{\sqrt{(x-x_M)^2+(y-y_M)^2}}{v_c} + \frac{\sqrt{(x_N-x)^2+(y_N-y)^2}}{v_n}$$

$$\frac{x-x_M}{v_c\sqrt{(x-x_M)^2+(y-y_M)^2}} - \frac{x_N-x}{v_n\sqrt{(x_N-x)^2+(y_N-y)^2}} = 0$$

$$\frac{\sin\alpha}{v_c} = \frac{\sin\beta}{v_n}$$

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