3. Basics of signal processing

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A test-case based learning of vibrations in mechanical engineering

Case study 3: Passenger comfort in cars (engine vibration)

- Source of excitation
- Effects
- Design methodology
- Remedial measures
Engine vibration: analysis of the vibration source

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Engine vibration: analysis of the vibration source (2)

[mide.com]
Discrete Fourier transform

Harmonic analysis: the Fourier transform

Let $u(t)$ be a periodic function of period $T$

$u(t)$ can be decomposed into a discrete Fourier series of the form

$$u(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

$$\omega_0 = \frac{2\pi}{T}$$
Harmonic analysis: the Fourier transform

\[ u(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \]

\[ a_0 = \frac{1}{T} \int_{0}^{T} u(t)dt \]

\[ a_n = \frac{2}{T} \int_{0}^{T} u(t) \cos(n\omega_0 t)dt \quad b_n = \frac{2}{T} \int_{0}^{T} u(t) \sin(n\omega_0 t)dt \]

Alternative formulation

\[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \]

\[ a_0 = \frac{2}{T} \int_{0}^{T} u(t)dt \]
3. Signal Processing

**Harmonic analysis: the Fourier transform**

\[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \]

Amplitudes and phases

\[ u(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t - \phi_n) \]

\[ d_0 = \frac{a_0}{2} \]

\[ d_n = \sqrt{a_n^2 + b_n^2} \quad \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \]

**Complex formulation**

\[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \]

\[ \cos(n\omega_0 t) = \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} \quad \sin(n\omega_0 t) = \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i} \]

\[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} + b_n \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i} \right] \]

\[ = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{a_n - ib_n}{2} e^{in\omega_0 t} + \frac{a_n + ib_n}{2} e^{-in\omega_0 t} \right] \]

\[ = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \]

\[ c_0 = \frac{a_0}{2} \]

\[ c_n = \frac{a_n - ib_n}{2} \quad c_{-n} = \frac{a_n + ib_n}{2} \]
Complex formulation

\[
\sum_{n=-\infty}^{\infty} c_n e^{i n \omega_0 t} = \sum_{n=-\infty}^{\infty} a_n e^{i n \omega_0 t} + i b_n e^{i n \omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{i n \omega_0 t} = \sum_{n=-\infty}^{\infty} \left( \frac{a_n}{2} - i \frac{b_n}{2} \right) e^{i n \omega_0 t} = \sum_{n=-\infty}^{\infty} \left( \frac{a_n}{2} + i \frac{b_n}{2} \right) e^{-i n \omega_0 t}
\]

\[
c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{0}^{T} u(t) dt
\]

\[
c_n = \frac{a_n - i b_n}{2} = \frac{1}{T} \int_{0}^{T} u(t) \left( \cos(n \omega_0 t) - i \sin(n \omega_0 t) \right) dt = \frac{1}{T} \int_{0}^{T} u(t) e^{-i n \omega_0 t} dt
\]

\[
c_{-n} = \frac{a_n + i b_n}{2} = \frac{1}{T} \int_{0}^{T} u(t) \left( \cos(n \omega_0 t) + i \sin(n \omega_0 t) \right) dt = \frac{1}{T} \int_{0}^{T} u(t) e^{i n \omega_0 t} dt
\]

\[
u(t) = \sum_{n=-\infty}^{\infty} c_n e^{i n \omega_0 t}
\]

\[
u(t) = \sum_{n=-\infty}^{\infty} a_n e^{i n \omega_0 t} + i b_n e^{i n \omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{i n \omega_0 t} = \sum_{n=-\infty}^{\infty} \left( \frac{a_n}{2} - i \frac{b_n}{2} \right) e^{i n \omega_0 t} = \sum_{n=-\infty}^{\infty} \left( \frac{a_n}{2} + i \frac{b_n}{2} \right) e^{-i n \omega_0 t}
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\[
c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{0}^{T} u(t) dt
\]

\[
c_n = \frac{1}{T} \int_{0}^{T} u(t) dt
\]

\[
c_{-n} = \frac{1}{T} \int_{0}^{T} u(t) e^{-i n \omega_0 t} dt
\]
### Complex formulation: alternative formulation

\[
 u(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{in\omega_0 t}
\]

\[
 c_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t) dt
\]

\[
 c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t) e^{-in\omega_0 t} dt
\]

### Examples of Fourier transform of periodic signals

- **Time signal**
- **Amplitude of Fourier coefficients**

\[
d_n = \sqrt{a_n^2 + b_n^2}
\]
Interest of frequency representation: many signals are concentrated in given frequency bands. The response needs to be computed only in this frequency band.

Examples of Fourier transform of periodic signals

\[ d_n = \sqrt{a_n^2 + b_n^2} \]

Continuous Fourier Transform
Non periodic signals

Continuous Fourier transform for non-periodic signals

- Period $T$ tends to $\infty$
- Discrete frequency step $\Delta \omega = \omega_0$ tends to $d\omega$
- Discrete frequency $n\omega_0$ tends to $\omega$

\[
c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)e^{-in\omega_0 t} \, dt
\]

\[
\lim_{T \to \infty} Tc_n = \lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)e^{-in\omega_0 t} \, dt = \int_{-\infty}^{\infty} u(t)e^{-i\omega t} \, dt = U(\omega)
\]

Continuous Fourier Transform of $u(t)$
Continuous Fourier transform for non-periodic signals

\[ U(\omega) = \lim_{T \to \infty} T c_n = \int_{-\infty}^{\infty} u(t)e^{-i\omega t} \, dt = U(\omega) \]

\[ u(t) = \lim_{T \to \infty} \sum_{n=-\infty}^{n=\infty} c_n e^{i\omega_n t} = \lim_{T \to \infty} \sum_{n=-\infty}^{n=\infty} c_n \frac{T}{T} e^{i\omega_n t} \]

\[ = \lim_{T \to \infty} \sum_{n=-\infty}^{n=\infty} (c_n T) \frac{\omega_n}{2\pi} e^{i\omega_n t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega)e^{i\omega t} \, d\omega \]

\[ U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t} \, dt \quad \text{Continuous Fourier Transform} \]

\[ u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega)e^{i\omega t} \, d\omega \quad \text{Continuous Inverse Fourier Transform} \]

Alternative formulation

<table>
<thead>
<tr>
<th>Pulsation (rad/s)</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U(\omega) )</td>
<td>( U(f) )</td>
</tr>
<tr>
<td>( u(t) )</td>
<td>( u(t) )</td>
</tr>
</tbody>
</table>

\[ \omega = 2\pi f \Rightarrow d\omega = 2\pi df \]

\[ u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega)e^{i\omega t} \, d\omega \quad \Rightarrow \quad u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft} \, df \]
Link between impulse response and transfer function

Duhamel’s integral with harmonic excitation 
\[ f(t) = F e^{i\omega t} \quad x(t) = X e^{i\omega t} \]

\[ x(t) = X e^{i\omega t} = \int_{-\infty}^{\infty} F e^{i\omega \tau} h(t-\tau) d\tau = \int_{-\infty}^{\infty} F e^{i\omega (t-\tau)} h(\tau) d\tau \]

\[ = F e^{i\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-i\omega \tau} d\tau = F e^{i\omega t} H(\omega) \]

\[ H(\omega) = \frac{X(\omega)}{F(\omega)} \quad \text{The Fourier transform of } h(t) \text{ is the transfer function} \]

For a single degree of freedom, we have:

\[ h(t) = \frac{e^{-\xi \omega_n t}}{m\omega_d} \sin(\omega_d t) \rightarrow \frac{X(\omega)}{F(\omega)} = \frac{1}{m} \left( \frac{1}{\omega_n^2 + 2i\xi\omega_n - \omega^2} \right) \]

Examples of continuous Fourier transform

<table>
<thead>
<tr>
<th>( u(t) )</th>
<th>( U(f) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\delta(t)} )</td>
<td>( \frac{\delta(f)}{\delta(f - f_0) + \delta(f + f_0)} )</td>
</tr>
<tr>
<td>( \cos(2\pi f_0 t) )</td>
<td>( \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta(f - n) )</td>
</tr>
<tr>
<td>( \sin(2\pi f_0 t) )</td>
<td>( \frac{1}{2i} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T}) )</td>
</tr>
</tbody>
</table>
Examples of continuous Fourier transform

\[ u(t) = H(t + a) - H(t - a) = \begin{cases} 
1 & -a < t < 0 \\
0 & |t| > a 
\end{cases} \]

\[ U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi ft} \, dt \]
\[ = \int_{-a}^{a} e^{-i2\pi ft} \, dt \]
\[ = \frac{-1}{i2\pi f} \left[ e^{-i2\pi ft} \right]_{-a}^{a} \]
\[ = \frac{(e^{i2\pi fa} - e^{-i2\pi fa})}{2i\pi f} \]
\[ = \frac{2 \sin(2\pi fa)}{2\pi f} \]

\[ 2 \sin(2\pi fa) \]
\[ \frac{2\pi f}{2\pi f} = 2\text{sinc}(2fa) \]

\[ \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \]
3. Signal Processing

Examples of continuous Fourier transform

Examples of continuous Fourier transform
Examples of continuous Fourier transform

Properties of the Fourier transform

<table>
<thead>
<tr>
<th>Time domain function</th>
<th>Frequency domain function</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a f(t) + b g(t)$</td>
<td>$a F(f) + b G(f)$</td>
<td>Linearity</td>
</tr>
<tr>
<td>$f(bt)$</td>
<td>$F \left( \frac{f}{b} \right)$</td>
<td>Time Scaling</td>
</tr>
<tr>
<td>$f(t) e^{j \omega_0 t}$</td>
<td>$F(f - \omega_0)$</td>
<td>Frequency shifting</td>
</tr>
<tr>
<td>$f(t)$ real even function</td>
<td>$F(f)$ real even function</td>
<td></td>
</tr>
<tr>
<td>$f(t)$ real odd function</td>
<td>$F(f)$ imag odd function</td>
<td></td>
</tr>
<tr>
<td>$f(t)$ real</td>
<td>$F(-f) = F(f)^*$</td>
<td></td>
</tr>
</tbody>
</table>

Even function = $f(-t)=f(t)$  Odd function = $f(t)=-f(-t)$
The convolution theorem

Convolution in the time domain corresponds with a multiplication in the frequency domain:

\[ y(t) = x(t) * h(t) \]
\[ Y(f) = X(f) \cdot H(f) \]

Proof:

\[
Y(f) = \int_{-\infty}^{\infty} y(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \right] e^{-i2\pi ft}dt \\
= \int_{-\infty}^{\infty} h(\tau) \left[ \int_{-\infty}^{\infty} x(t-\tau)e^{-i2\pi ft}d\tau \right]d\tau \\
u = t - \tau \\
du = dt \\
Y(f) = \int_{-\infty}^{\infty} h(\tau) \left[ \int_{-\infty}^{\infty} x(u)e^{-i2\pi f(u+\tau)}du \right]d\tau \\
Y(f) = \int_{-\infty}^{\infty} h(\tau)e^{-i2\pi f\tau}d\tau \left[ \int_{-\infty}^{\infty} x(u)e^{-i2\pi f\tau}du \right] = H(f) \cdot X(f)
\]

The convolution theorem

In the same way, one can prove that:

\[ y(t) = x(t) \cdot h(t) \]
\[ Y(f) = X(f) \ast H(f) \]

A multiplication in the time domain is a convolution in the frequency domain
Signal processing in practice:
DFT, aliasing and FFT

Sampling and aliasing

- In practice, vibration signals are recorded on computers at discrete time steps $\Delta t$. This is called sampling.
- Sampling at time intervals $\Delta t$ can be seen as multiplying the continuous function by a Dirac comb with spacing $\Delta t$. 

\[ X(t) \rightarrow \sum_{n=-\infty}^{\infty} x(n\Delta t) \delta(t-n\Delta t) \]
Sampling and aliasing

Continuous Fourier Transform of a sampled signal using the convolution theorem

\[ y(t) = x(t) * h(t) \]
\[ Y(f) = X(f) * H(f) \]

Example:

\[ x(t) = \sin(2\pi f_0 t) \rightarrow X(f) = \frac{\delta(f-f_0) + \delta(f+f_0)}{2i} \]  
Continuous function

\[ h(t) = \sum_{n=-\infty}^{\infty} \delta(t-n\Delta t) \rightarrow H(f) = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{\Delta t}) \]  
Sampling function
3. Signal Processing

Sampling and aliasing

\[ f_0 < \frac{f_s}{2} \]

Sampling and aliasing

\[ f_0 > \frac{f_s}{2} \]
3. Signal Processing

### Sampling and aliasing

Additional signal due to sampling

Fourier transform of the continuous signal

Sine at frequency $f_s$ appears at frequency $(f_s - f_0)$ (mirror frequency with respect to $f_s/2$) due to too low sampling frequency. This is **aliasing**

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### Discrete Fourier Transform applied to sampled signals

Basic principle:

- Time interval $[0 \ T]$, $N$ samples, sampling time $\Delta t$

- Fourier transform computed at $N$ regular frequency intervals $\rightarrow$ Discrete FT

\[
\Delta \omega = \frac{2\pi}{N \Delta t} = \frac{2\pi}{T}
\]

resulting in a frequency band $[0 \ f_{\text{max}}]$ with

\[
f_{\text{max}} = N \frac{\Delta \omega}{2\pi} = \frac{N}{T} = \frac{1}{\Delta t} = f_s
\]

and only the part below $\frac{f_s}{2}$ is useful.
Discrete Fourier Transform applied to sampled signals

The discrete Fourier transform is defined as:

\[ c_n = \frac{1}{T} \int_{0}^{T} u(t) e^{-i\omega_0 t} dt \]

For a signal \( u(t) \) sampled at \( N \) regular time intervals (\( \Delta t \)):

\[ c_n \approx \frac{1}{T} \sum_{j=0}^{N} \Delta t \cdot u(j \Delta t) e^{-i\omega_0 j \Delta t} \]

\[ = \frac{\Delta t}{T} \sum_{j=0}^{N} u(j \Delta t) e^{-i\omega_0 j \Delta t} = \frac{1}{N} \text{fft}(u) \]

Result of \text{fft} function in Matlab

Remarks:

* Taking a time interval \([0, T]\) can therefore be seen as implicitly assuming that the period of the signal is \( T \).

* The DFT is exact only if the signal is periodic of period \( T \), or if the signal is zero before \( t=0 \) and after \( t=T \).

* Due to the periodicity of the Fourier transform of a sampled signal, the DFT needs to be computed only at \( N \) frequencies, and the useful part of the spectrum contains only \( N/2 \) points and ranges from 0 to \( f_s/2 \).

* In practice, the DFT is computed using the FFT (Fast Fourier Transform) algorithm.
Link between FFT and continuous Fourier transform

The continuous Fourier transform is defined as:

\[ U(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft} dt = \lim_{T \to \infty} T \epsilon_n \]

\[ = \lim_{T \to \infty} \frac{T}{N} \text{fft}(u) = (\Delta t) \text{fft}(u) \]

Result of fft function in Matlab

Fourier transform applied to sampled signals: summary

Summary:

- For a signal \( u(t) \) sampled at frequency \( f_s = 1/\Delta t \) over time interval \([0 \ T]\), the number of samples in the time domain is \( N = T f_s \).

- The DFT is computed at \( N \) frequencies with spacing \( \Delta f = 1/T \) in the frequency interval \([0 \ 1/\Delta t] = [0 \ f_s]\).

- The useful part of the DFT is in the frequency interval \([0 \ f_s/2]\).

*The frequency resolution only depends on \( T \). A better resolution in the DFT is achieved by increasing the length of the measurements.*

- If \( \Delta t \) is kept constant, it results in more data to be stored, but the same frequency range.

- If \( \Delta t \) is changed in order to keep the same number of measurement points, there is no increase in the data, but the frequency band is reduced.
Example 1: FFT of impulse response using Matlab

\[ k = 1 N/m, \quad m = 1 kg, \quad b = 0.01 Ns/m \]

\[ \Delta t = 0.1534s \rightarrow f_s = 6.519 Hz \]
\[ n = 4096 \]
\[ T = 628s \rightarrow \Delta f = 0.001592 Hz \]

Approximation of the transfer function in the frequency band \([0 f_s/2]\]

Example 3: FFT of engine vibration in idle mode

\[ \Delta t = 50 \mu s \rightarrow f_s = 20 kHz \]
\[ n = 100001 \]
\[ T = 5s \rightarrow \Delta f = 0.2 Hz \]
Example 3: FFT of random signals using Matlab

\[ k = 1 \text{N/m}, m = 1 \text{kg}, b = 0.01 \text{Ns/m} \]

\[ \Delta t = 0.01534s \rightarrow f_s = 65.19 \text{Hz} \]

\[ n = 45056 \]

\[ T = 691s \rightarrow \Delta f = 0.00144 \text{Hz} \]

Determining the natural frequency of an object with your phone
Engine vibration: analysis of the vibration source - spectrogram

[mide.com]