Why do we want to measure vibrations?
Why do we want to measure vibrations?

- Better knowledge of the level of exposure to vibrations (fatigue, comfort)
- Identify the modal properties of the structure (experimental modal analysis)
- Check the validity of numerical models (model updating)
- Check the integrity of structures (structural health monitoring)
Physical principles for vibration sensors

Physical principles for measurements:
- Electrodynaminc sensor -> voltage % velocity
- Piezoelectric sensor -> charge % strain
- Piezoresistive sensors -> $\Delta R$ % strain
- Optical fiber sensor -> wavelength % strain
- Capacitive sensor -> Capacitance % distance

Most widely used sensors:
- Accelerometers (acceleration)
- Geophones (velocity)
- Strain gauges (strain)
- FBGS (Fiber Bragg Grating) optical fiber sensors (strain)

Measuring without contact:
- Laser vibrometer (full field velocity measurement)
### Electrodynaminc sensor

\[ V = B l v \]

- Voltage measurement on the voice coil (V) = measure of the relative velocity between the coil and the permanent magnet

- B = magnetic induction
- l = length of coil
- v = velocity of coil

---

### Piezoelectric sensors

- Electric charge is proportional to the strain applied to the piezoelectric element
- This effect can only be used in dynamics (static measurements are not possible)
Piezoresistive sensors

The change of resistance of the strain gauge is proportionnal to the applied strain.

FBG (Fiber Bragg Grating) optical fiber sensor

The change of reflected wavelength is proportionnal to the strain applied to the Bragg grating.
6. Measurement techniques

Capacitive sensor

The change of capacitance is a measure of the relative displacement between the two electrodes.

Mathematically, the capacitance is given by:

\[ C = \varepsilon S / h \]

where:
- \( C \) is the capacitance
- \( \varepsilon \) is the dielectric permittivity
- \( S \) is the surface area
- \( h \) is the distance between the electrodes

In practice, it is very difficult to have a fixed reference point for measurements.

Measuring without a reference point

- **Electrodynamic sensor**
- **Piezoelectric sensor**
- **Capacitive sensor**

In practice, it is very difficult to have a fixed reference point.
The accelerometer

\[ M \ddot{x}_r + c \dot{x}_r + k x_r = -M \ddot{x}_0 \]

In the frequency domain:

\[ \frac{x_r}{x_0} = \frac{-1}{-\omega^2 + \omega_n^2 + 2i\xi \omega \omega_n} \]

\[ \omega \ll \omega_n \quad \Rightarrow \quad \frac{x_r}{x_0} \approx \frac{-1}{\omega_n^2} \]

The accelerometer

\[ \frac{x_r}{x_0} \approx \frac{-1}{\omega_n^2} \]

Piezoelectric accelerometer

Longitudinal mode

Shear mode
**6. Measurement techniques**

The accelerometer

High resonant frequency
Low sensitivity

- 5 mV/g
- 1 → 10 000 Hz
- 3.1 gm

Low resonant frequency
High sensitivity

- 100 mV/g
- 1 → 4 000 Hz
- 27 gm

MEMs accelerometers

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6. Measurement techniques

**Piezoresistive MEMs accelerometers**

- Good linearity, wide freq range
- Piezoresistive element
- Cantilevers
- Proof Mass

**Capacitive MEMs accelerometers**

- Good temperature stability (no compensation needed)
- Double capacitor
6. Measurement techniques

The geophone

\[ M \ddot{x}_r + c \dot{x}_r + k x_r = -M \dot{x}_0 \]

In the frequency domain:

\[ \frac{x_r}{x_0} = \frac{\omega^2}{\omega^2 + \omega_n^2 + 2i \xi \omega \omega_n} \]

\( \omega >> \omega_n \) \quad \frac{x_r}{x_0} \approx -1

Geophone: practical example

Electrodynamical sensor to measure the relative velocity
6. Measurement techniques

FBG (Fiber Bragg Grating) optical fiber strain sensors

Advantages:
- Electromagnetic immunity
- Multiplexing capabilities
- Possibility to integrate inside structures

Drawbacks:
- Fragile
- No anti-aliasing filters
- Costly interrogation systems

Space telescope

[Integrated FOS in a fiber composite strut of the space telescope](http://www.spie.org)
6. Measurement techniques

Strain gauges

\[ \Delta R / R = k \cdot \Delta L / L \]

Wheatstone bridge

\[ \frac{V_{out}}{V_{in}} = \frac{R_3}{R_3 + R_4} - \frac{R_2}{R_2 + R_1} \]

Balanced bridge

\[ R_1 = R_2 = R_3 = R_4 = R_0 \quad \Rightarrow \quad \frac{V_{out}}{V_{in}} = 0 \]

Quarter bridge

\[ \frac{\Delta R_g}{R_g} = k \varepsilon \]

\[ \frac{V_{out}}{V_{in}} = - \frac{k \varepsilon}{4 \left(1 + \frac{k \varepsilon}{2}\right)} \approx - \frac{k \varepsilon}{4} \]
6. Measurement techniques

Strain gauges measurements

**Half bridge**

\[ R_4 = R_g + \Delta R_g \]

\[ R_3 = R_g - \Delta R_g \]  
(R\(_4\) is in tension, R\(_3\) in compression)

\[ \frac{\Delta R_g}{R_g} = k \varepsilon \]

\[ \frac{V_{out}}{V_{in}} = -\frac{k \varepsilon}{2} \]

---

**Full bridge**

\[ V_{out} = -k \varepsilon \]

\[ \frac{V_{out}}{V_{in}} = -k \varepsilon \]

Typical values:

\[ k \approx 2 \]

\[ \varepsilon = 50 - 300 \times 10^{-6} \]
Flat Piezoelectric sensors

PZT ceramic

- Large bandwidth
- Cheap

But brittle ...

Flat composite piezoelectric sensors

Macro Fiber Composite (MFC)
(from smart-material.com)

Aluminum bolted beam

(Park et al. 2007)
6. Measurement techniques

Measuring without contact

Scanning laser vibrometer (from Polytech)

Velocity measurement using Doppler effect (without contact)

Actuators for vibration measurements
6. Measurement techniques

**Impact hammer**

**Impulse excitation**

**Electrodynamic Shaker**

Excitation:
- Harmonic
- Periodic
- Random
- …
Electrodynamic Shaker

Piezoelectric actuators
- Large bandwidth
- Cheap
- Low forces
6. Measurement techniques

**Force sensors**

Piezoelectric force sensor

**Piezoelectric actuators**

APA100XXL
DFT and aliasing

Additional signal due to sampling

Part of the additional signal ‘falls back’ on the frequency band of interest.
The frequency of the fourth resonance is wrong due to aliasing
Anti-aliasing filters

Solution = anti-aliasing filter

Summary

• For a given $\Delta t = 1/f_s$, the useful band of the Fourier Transform is $f_{\text{max}} = 1/(2\Delta t) = f_s/2$

Example: $\Delta t = 1\text{ ms} \rightarrow f_{\text{max}} = 500\text{ Hz}$

• If the signal contains frequencies higher than $f = f_s/2$, they will ‘fall back’ on the useful band of the FFT at a ‘mirror’ frequency with respect to $f_s/2$

$\rightarrow$ Anti-aliasing filters should be used BEFORE sampling
Measuring FRFs

Principle

\[ H(\omega) = \frac{X(\omega)}{F(\omega)} \]
6. Measurement techniques

Types of excitations signals

- **Shaker**
- **Impact hammer**

**Periodic excitation**

Both signals have a period $T$ -> Discrete Fourier Transform

\[
x(t) = \sum_{n=-\infty}^{\infty} X_n e^{in\omega_0 t}
\]

\[
x_n = \frac{1}{T} \int_0^T x(t) e^{-in\omega_0 t} dt
\]

\[
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}
\]

\[
F_n = \frac{1}{T} \int_0^T f(t) e^{-in\omega_0 t} dt
\]

\[
H_n = \frac{X_n}{F_n}
\]

In practice, signals are sampled: integral $\rightarrow$ sum

Frequency interval: $[0, f_s/2]$
Periodic signals and averaging: examples

\[ f(t) = \sum_{i=0}^{n} \cos(\omega_0 t + \phi) \]

\( \omega_0 = 0.1 \text{rad/s}, T = 62.838 \text{s} \)

Periodic input signal

Output signal is periodic after transient.

Periodic signals: example

Signals sampled with \( \Delta t = T/128 \) (n=128 samples)

Discrete Fourier Transform

\( \Delta \omega = \frac{2\pi}{T} = 0.1 \text{rad/s} \)

\( \omega_{\text{max}} = 128 \times 0.1 \text{rad/s} = 12.8 \text{rad/s} \)
In reality, signals are noisy. Better estimates are obtained using averaging. In order to do that, signals are measured over several periods.

\[ H_{ni} = \frac{X_{ni}}{F_{ni}} \quad i = 1 \ldots m \]

**Average FRF**

\[ H_n = \frac{\frac{1}{m} \sum_{i=1}^{m} X_{ni}}{\frac{1}{m} \sum_{i=1}^{m} F_{ni}} \]

---

**Periodic signals and averaging : examples**

![Graphs showing signal averaging examples](image-url)
Effect of bad synchronisation on the period : leakage

- No leakage
- Small leakage
- Large leakage

Solutions to decrease leakage

The signal is multiplied by a windowing function also called window before the Fourier Transform.
Window types

- Rectangular (boxcar) window
- Hanning window
- Cosine-taper (Tukey) window

Effect on leakage

- No window (rectangle)
- Hanning
- Tukey (cosine tapper)
Transient signals

**Impulse**

**Sine burst**

**Chirp**

Signals limited in time

- repeated over several periods
- same processing as periodic signals (DFT + averaging)
6. Measurement techniques

Transient signals: example of impulse response

\[ k = 1 \text{ N/m} \]
\[ m = 1 \text{ kg} \]
\[ b = 0.05 \text{ Ns/m} \]

\[ \text{Impulse} = F \Delta t \]

FRF computed using DFT

Transient signals: effect of noise for impulse responses

Noise on the output

Exponential window

\[ x(t) \times w(t) \]
Transient signals: effect of noise for impulse responses

Without exponential window

With exponential window

FRF computed using DFT

Noise on the input (force sensor)
**6. Measurement techniques**

### Transient signals: effect of noise for impulse responses

Hanning should never be used with impulse excitations!

![Hanning window](image)

![FRF computed using DFT](image)

**Random signals: Definitions**

Random signals cannot be treated like periodic signals. It is necessary to introduce the following functions:

**The Autocorrelation and Cross-correlation functions**

\[
R_{ff}(\tau) = E[f(t)f(t+\tau)]
\]
\[
R_{xx}(\tau) = E[x(t)x(t+\tau)]
\]

**Expected (average) value**

\[
E[x(t)] = \int_{-\infty}^{\infty} x(t)dt
\]

**The Power Spectral Density and Cross Spectral Density**

\[
S_{ff}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\tau)e^{-i\omega \tau}d\tau
\]
\[
S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega \tau}d\tau
\]

\[
S_{xf}(\omega) = S_{fx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xf}(\tau)e^{-i\omega \tau}d\tau
\]

\[
S_{sx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{sx}(\tau)e^{-i\omega \tau}d\tau
\]
Random signals: Definitions

\[ k = 1 \text{ N/m} \]
\[ m = 1 \text{ kg} \]
\[ b = 0.05 \text{ Ns/m} \]

Output is random (no periodicity …)
Random signals: Properties and FRF computation

\[
\begin{align*}
S_{ff}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\tau)e^{-i\omega \tau} \, d\tau \\
S_{fx}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{fx}(\tau)e^{-i\omega \tau} \, d\tau \\
S_{xx}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega \tau} \, d\tau \\
x(t) &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau) \, d\tau = f(t) * h(t)
\end{align*}
\]

Duhamel’s integral

\[
S_{xx}(\omega) = H^*(\omega)S_{fx}(\omega)
\]

\[
S_{fx}(\omega) = H(\omega)S_{ff}(\omega)
\]

\[
S_{fx}(\omega) = S_{xf}^*(\omega)
\]

\[
S_{xx}(\omega) = |H(\omega)|^2S_{ff}(\omega)
\]

\[
\gamma^2(\omega) = \frac{H_1(\omega)}{H_2(\omega)} = \frac{|S_{fx}(\omega)|^2}{S_{ff}(\omega)S_{xx}(\omega)}
\]

\[
\begin{align*}
H(\omega) &= \frac{S_{fx}(\omega)}{S_{ff}(\omega)} = H_1(\omega) \\
H(\omega) &= \frac{S_{xx}(\omega)}{S_{xf}(\omega)} = H_2(\omega) \\
\gamma^2(\omega) &= \frac{H_1(\omega)}{H_2(\omega)} = \frac{|S_{fx}(\omega)|^2}{S_{ff}(\omega)S_{xx}(\omega)}
\end{align*}
\]

Coherence function

Coherence is =1 if measurement is perfect.

Coherence usually drops:
- around eigenfrequencies
- when measurements are very noisy
- when the structure is non-linear
Random signals: Practical computation of PSD and CSD

\[ S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau \]
\[ R_{xx}(\tau) = E[x(t) x(t + \tau)] \]
\[ E[x(t) x(t + \tau)] = \int_{-\infty}^{\infty} x(t) x(t + \tau) dt = x(t) * x(-t) \]

In the same way:

\[ S_{yy}(\omega) = X(\omega) X^*(\omega) \]
\[ S_{yx}(\omega) = X(\omega) F^*(\omega) \]
\[ S_{xy}(\omega) = X^*(\omega) F^*(\omega) \]

These FRF estimates can be computed for any type of signal, including random signals, impulse excitation, …