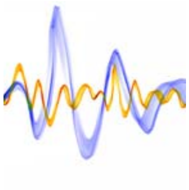

Vibrations and acoustics

7. Experimental Modal Analysis

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Outline of the chapter

*Principle of EMA

*Measuring FRFs

*SDOF Identification

*MDOF Identification

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Principle of EMA

1. Measure FRFs

2. Estimate poles (natural frequencies)

3. Identify mode shapes

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Principle of EMA

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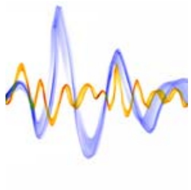

Measuring FRFs

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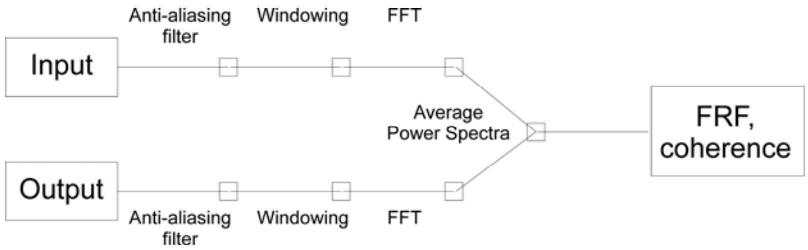
Measuring FRFs : summary



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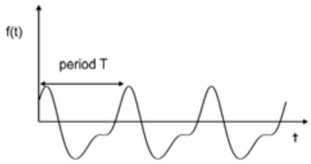
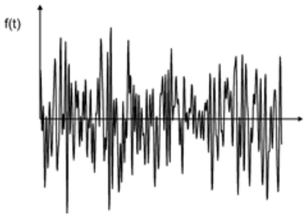
Use of windows : summary



Periodic — No window (synchronisation !)

Transient — Exponential window to reduce effect of noise (output) + force window (input)



Random — Hanning window to reduce leakage

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Measuring FRFs

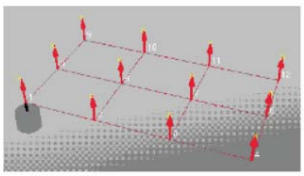
Shaker excitation

One column

- Single Fixed Excitation (reference)
- Single Roving Response **SISO**

or

- Multiple (Roving) Responses **SIMO**
  Multiple-Output: Optimize **data consistency**



$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & \dots & H_{1n} \\ H_{21} & H_{22} & H_{23} & \dots & H_{2n} \\ H_{31} & H_{32} & H_{33} & \dots & H_{3n} \\ & \vdots & & & \\ H_{n1} & H_{n2} & H_{n3} & \dots & H_{nn} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix}$$

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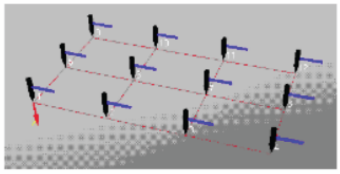
Measuring FRFs

Roving hammer excitation

One row

- One Roving Excitation
- One Fixed Response (reference)

SISO



$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & \dots H_{1n} \\ H_{21} & H_{22} & H_{23} & \dots H_{2n} \\ H_{31} & H_{32} & H_{33} & \dots H_{3n} \\ & & \vdots & \\ H_{n1} & H_{n2} & H_{n3} & \dots H_{nn} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix}$$

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Measuring FRFs

Reciprocity

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & \dots H_{1n} \\ H_{21} & H_{22} & H_{23} & \dots H_{2n} \\ H_{31} & H_{32} & H_{33} & \dots H_{3n} \\ & \vdots & & \\ H_{n1} & H_{n2} & H_{n3} & \dots H_{nn} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & \dots H_{1n} \\ H_{21} & H_{22} & H_{23} & \dots H_{2n} \\ H_{31} & H_{32} & H_{33} & \dots H_{3n} \\ & & \vdots & \\ H_{n1} & H_{n2} & H_{n3} & \dots H_{nn} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix}$$

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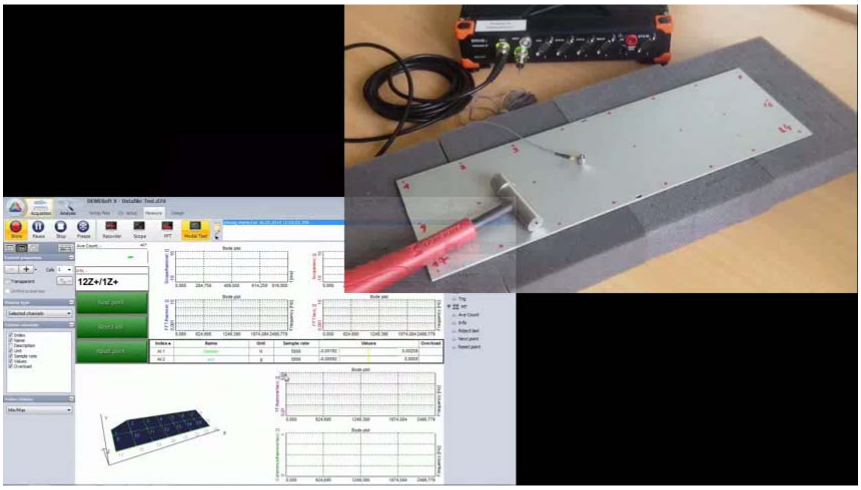
Measuring FRFs in practice



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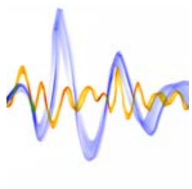

SDOF identification

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The pole-residue model in the frequency domain

The frequency response function of a one dof system is :

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{m}\frac{1}{\omega_n^2 - \omega^2 + 2j\xi\omega\omega_n}$$

The Pole-residue model in the frequency domain is :

$$H(\omega) = \frac{R}{j\omega - \lambda} + \frac{R^*}{j\omega - \lambda^*}$$

$$\lambda = -\xi\omega_n + j\omega_d$$

$$R = -\frac{j}{2m\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

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The pole-residue model in the time domain

The impulse response function of a one DOF system is :

$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d} sin(\omega_d t)$$

The Pole-residue model in the time domain is :

$$h(t) = Re^{\lambda t} + R^* e^{\lambda^* t}$$

$$\lambda = -\xi\omega_n + j\omega_d$$

$$R = -\frac{j}{2m\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

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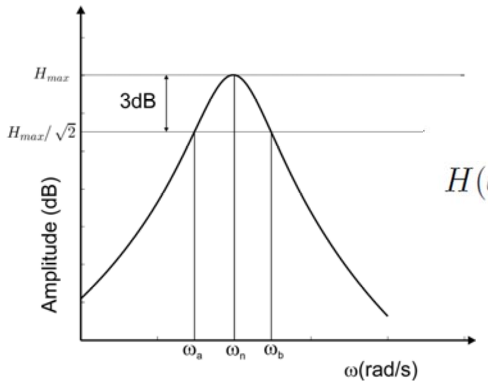
Estimating parameters : Peak-picking method



$$H(\omega) = \frac{1}{m}\frac{1}{\omega_n^2 - \omega^2 + 2j\xi\omega\omega_n}$$

ω_n	Natural frequency (ratio k/m)
$2\xi \simeq \frac{\omega_b - \omega_a}{\omega_n}$	Damping coefficient (b)
$m = \frac{1}{2\xi\omega_n^2 H_{max}}$	Mass (m)

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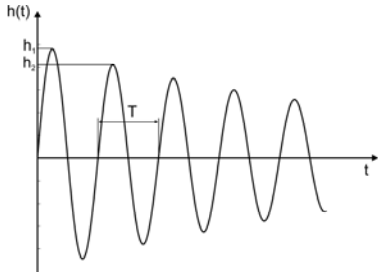
Estimating parameters : logarithmic decrement method



$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d} sin(\omega_d t)$$

$\xi = \frac{1}{2\pi N} ln\left(\frac{h(t)}{h(t+NT)}\right)$	Damping coefficient (b)
$\omega_d = \omega_n\sqrt{1-\xi^2} = \frac{2\pi}{T}$	Natural frequency (ratio k/m)
$m = \frac{e^{-\xi\omega_n \frac{T}{2}}}{\omega_d h_1}$	Mass (m)

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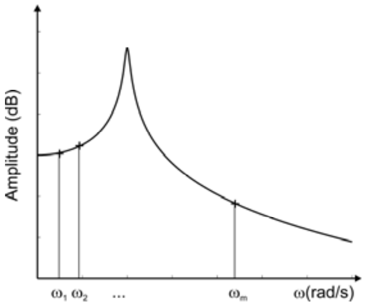
Estimating parameters : curve fitting

$$(k - \omega^2 m + j\omega b)X(\omega) = F(\omega)$$



$$\begin{bmatrix} -\omega_1^2 X(\omega_1) & j\omega_1 X(\omega_1) & X(\omega_1) \\ -\omega_2^2 X(\omega_2) & j\omega_2 X(\omega_2) & X(\omega_2) \\ \dots & & \\ -\omega_m^2 X(\omega_m) & j\omega_m X(\omega_m) & X(\omega_m) \end{bmatrix} \begin{Bmatrix} m \\ b \\ k \end{Bmatrix} = \begin{Bmatrix} F(\omega_1) \\ F(\omega_2) \\ \dots \\ F(\omega_m) \end{Bmatrix}$$

$$[A]\{\theta\} = \{F\}$$

More equations than unknowns

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Estimating parameters : curve fitting

$$[A]\{\theta\} = \{F\}$$

More equations than unknowns

$$[A]^*[A]\{\theta\} = [A]^*\{F\}$$

$$\{\theta\} = ([A]^*[A])^{-1}[A]^*\{F\} = A^\dagger\{F\}$$

$$A^\dagger = ([A]^*[A])^{-1}[A]^*$$

Moore-Penrose pseudo-inverse : Least-squares solution

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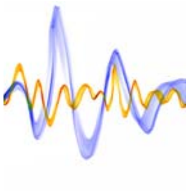

MDOF identification

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The pole-residue model in the frequency domain

The FRF matrix of a MDOF system is :

$$X(\omega) = H(\omega)F(\omega) \qquad H(\omega) = \sum_{i=1}^{N} \frac{\psi_i^T \psi_i}{\mu_i(\omega_i^2 + 2j\xi\omega\omega_i - \omega^2)}$$

The Pole-residue model in the frequency domain is :

$$H(\omega) = \sum_{i=1}^{N} \frac{R_i}{j\omega - \lambda_i} + \frac{R_i^*}{j\omega - \lambda_i^*}$$

R_i is a matrix

$$\lambda_i = -\xi_i\omega_i + j\omega_i\sqrt{1-\xi_i^2}$$
$$R_i = \frac{-j\psi_i^T\psi_i}{2\mu_i\omega_i\sqrt{1-\xi_i^2}}$$

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The pole-residue model in the time domain

$$H(\omega) = \sum_{i=1}^{N} \frac{R_i}{j\omega - \lambda_i} + \frac{R_i^*}{j\omega - \lambda_i^*}$$

Inverse Fourier transform

The Pole-residue model in the time domain is :

$$h(t) = \sum_{i=1}^{N} R_i e^{\lambda_i t} + R_i^* e^{\lambda_i^* t}$$

R_i is a matrix

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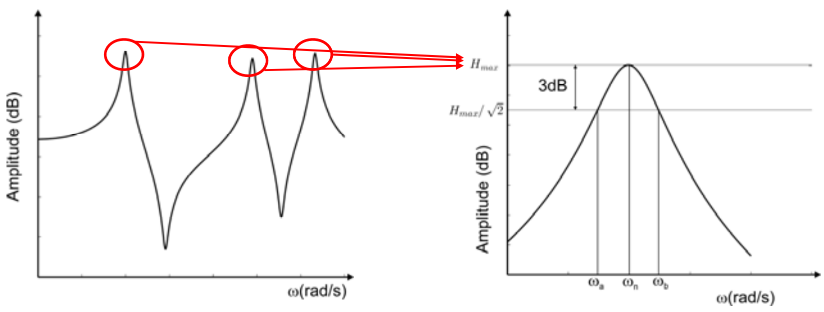
Estimating the poles : peak picking method



$\omega_i, \xi_i \quad i = 1..N$ Natural frequencies and modal damping

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Estimating the mode shapes : curve fitting in the frequency domain

$$\omega_i, \xi_i \quad i = 1..N \longrightarrow \lambda_i = -\xi_i\omega_i + j\omega_i\sqrt{1-\xi_i^2}$$

For each single FRF :

$$H_{kl}(\omega) = \sum_{i=1}^{N} \frac{R_{i,kl}}{j\omega - \lambda_i} + \frac{R^*_{i,kl}}{j\omega - \lambda^*_i}$$

l=input, k=output

$$\begin{bmatrix} \frac{1}{j\omega_1-\lambda_1} & \frac{1}{j\omega_1-\lambda_1^*} & \frac{1}{j\omega_1-\lambda_2} & \frac{1}{j\omega_1-\lambda_2^*} & \cdots & \cdots & \frac{1}{j\omega_1-\lambda_N} & \frac{1}{j\omega_1-\lambda_N^*} \\ \frac{1}{j\omega_2-\lambda_1} & \frac{1}{j\omega_2-\lambda_1^*} & \frac{1}{j\omega_2-\lambda_2} & \frac{1}{j\omega_2-\lambda_2^*} & \cdots & \cdots & \frac{1}{j\omega_2-\lambda_N} & \frac{1}{j\omega_2-\lambda_N^*} \\ \cdots & \cdots & & & & & \cdots & \cdots \\ \frac{1}{j\omega_m-\lambda_1} & \frac{1}{j\omega_m-\lambda_1^*} & \frac{1}{j\omega_m-\lambda_2} & \frac{1}{j\omega_m-\lambda_2^*} & \cdots & \cdots & \frac{1}{j\omega_m-\lambda_N} & \frac{1}{j\omega_m-\lambda_N^*} \end{bmatrix} \begin{Bmatrix} R_{1,kl} \\ R^*_{1,kl} \\ R_{2,kl} \\ R^*_{2,kl} \\ \cdots \\ \cdots \\ R_{N,kl} \\ R^*_{N,kl} \end{Bmatrix} = \begin{Bmatrix} H_{kl}(\omega_1) \\ H_{kl}(\omega_2) \\ \cdots \\ H_{kl}(\omega_m) \end{Bmatrix}$$

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Estimating the mode shapes

$$[A]\{\theta\} = \{F\} \longrightarrow \{\theta\} = ([A]^*[A])^{-1}[A]^*\{F\} = A^\dagger\{F\}$$

$$R_{i,kl} = \frac{-j\psi_{i,k}\psi_{i,l}}{2\mu_i\omega_i\sqrt{1-\xi_i^2}}$$

↓ Keep l fixed, vary k

$\psi_{i,k}$ Mode shapes

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Estimating natural frequencies and mode shapes : the complex exponential method

$$h(t) = \sum_{i=1}^{N} R_i e^{\lambda_i t} + R_i^* e^{\lambda_i^* t}$$

$$h(t)_{kl} = \sum_{i=1}^{N} R_{i,kl} e^{\lambda_i t} + R_{i,kl}^* e^{\lambda_i^* t}$$

l=input, k=output

$$h(t)_{kl} = \sum_{i=1}^{2N} R_{i,kl} e^{\lambda_i t}$$

$$R_{i+N,kl} = R_{i,kl}^*$$
$$\lambda_{i+N} = \lambda_i^*$$

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Estimating natural frequencies and mode shapes : the complex exponential method

$$\begin{aligned} h_0 &= h(0)_{kl} = \sum_{i=1}^{2N} R_{i,kl} \\ h_1 &= h(\Delta t)_{kl} = \sum_{i=1}^{2N} R_{i,kl} e^{\lambda_i(\Delta t)} \\ \dots & \quad \dots \\ h_L &= h(L\Delta t)_{kl} = \sum_{i=1}^{2N} R_{i,kl} e^{\lambda_i(L\Delta t)} \end{aligned}$$

$V_i = e^{\lambda_i(\Delta t)}$ →

$$\begin{aligned} h_0 &= \sum_{i=1}^{2N} R_{i,kl} \\ h_1 &= \sum_{i=1}^{2N} R_{i,kl} V_i \\ \dots & \quad \dots \\ h_L &= \sum_{i=1}^{2N} R_{i,kl} V_i^L \end{aligned}$$

Because the roots appear as complex conjugate pairs, so do the V_i, therefore there exists real coefficients β such that :

$$\beta_0 + \beta_1 V_i + \beta_2 V_i^2 + \dots + \beta_L V_i^L = 0$$

→

$$\sum_{j=0}^{L} \beta_j h_j = \sum_{i=1}^{2N} \left(R_{i,kl} \sum_{j=0}^{L} \beta_j V_i^j \right) = 0 \quad \forall V_i$$

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Estimating natural frequencies and mode shapes : the complex exponential method

$$\sum_{j=0}^{L} \beta_j h_j = 0$$

$$\longrightarrow \sum_{j=0}^{L} \beta_j h_j V_i^n = \sum_{j=0}^{L} \beta_j h_{j+n} = 0$$

$$\downarrow \quad L = 2N, \quad \beta_{2N} = 1$$

$$\begin{bmatrix} h_0 & h_1 & h_2 & \dots & h_{2N-1} \\ h_1 & h_2 & h_3 & \dots & h_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ h_{2N-1} & h_{2N} & h_{2N+1} & \dots & h_{4N-2} \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_{2N-1} \end{Bmatrix} = - \begin{Bmatrix} h_{2N} \\ h_{2N+1} \\ \dots \\ h_{4N-1} \end{Bmatrix}$$

$$\beta_0 + \beta_1 V_i + \beta_2 V_i^2 + \dots + \beta_L V_i^L = 0 \longrightarrow V_i = e^{\lambda_i(\Delta t)} \longrightarrow \lambda_i = \frac{ln(V_i)}{\Delta t}$$

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Estimating natural frequencies and mode shapes : the complex exponential method

$$\begin{aligned} h_0 &= \sum_{i=1}^{2N} R_{i,kl} \\ h_1 &= \sum_{i=1}^{2N} R_{i,kl} V_i \\ \dots & \quad \dots \\ h_L &= \sum_{i=1}^{2N} R_{i,kl} V_i^L \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 & \dots & 1 \\ V_1 & V_2 & \dots & V_{2N} \\ V_1^2 & V_2^2 & \dots & V_{2N}^2 \\ \dots & \dots & \dots & \dots \\ V_1^{2N-1} & V_2^{2N-1} & \dots & V_{2N}^{2N-1} \end{bmatrix} \begin{Bmatrix} R_{1,kl} \\ R_{2,kl} \\ R_{3,kl} \\ \dots \\ R_{2N,kl} \end{Bmatrix} = \begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \dots \\ h_{2N-1} \end{Bmatrix}$$

$$\downarrow$$

$$R_{i,kl} = \frac{-j\psi_{i,k}\psi_{i,l}}{2\mu_i\omega_i\sqrt{1-\xi_i^2}}$$

$$\downarrow$$

$$\psi_{i,k}$$

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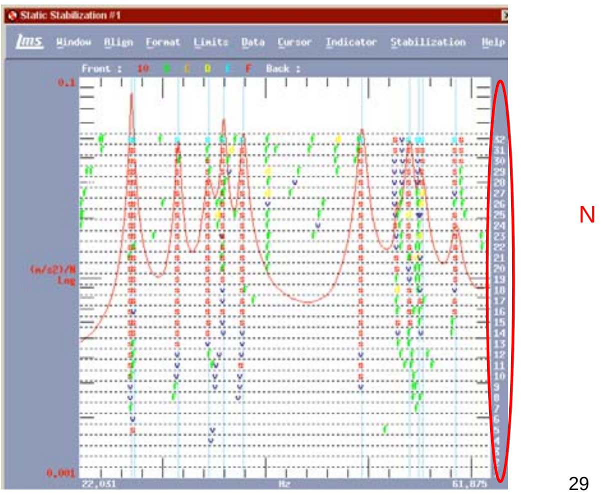
Model order and stabilisation diagram

$$h(t)_{kl} = \sum_{i=1}^{2N} R_{i,kl} e^{\lambda_i t}$$ → N ?

s:stable in both freq and damping

d: stable in damping

f: stable in frequency



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