A remarkable Mathieu graph tower*

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Abstract

Two different constructions are given of a rank 8 arc-transitive graph with 165 vertices and valency 8, whose automorphism group is M_{11}. One involves 3-subsets of an 11-set while the other involves 4-subsets of a 12-set, and the constructions are linked with the Witt designs on 11, 12 and 24 points. Four different constructions are given of a rank 9 arc-transitive graph with 55 vertices and valency 6 whose automorphism group is $\mathrm{PSL}(2, 11)$. This graph occurs as a subgraph of the M_{11} graph, and two of the constructions involve 2-subsets of an 11-set while the remaining two involve 3-subsets of an 11-set. The $\mathrm{PSL}(2, 11)$ and M_{11} graphs occur as the second and third members of a tower of graphs defined on a conjugacy class of involutions of

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the simple groups A_5, $\mathrm{PSL}(2,11)$, M_{11} and M_{12} with two involutions adjacent if they generate a special S_3. The first graph in the tower is the line graph of the Petersen graph while the fourth is the Johnson graph $J(12,4)$.

1 Introduction

The Mathieu groups have a rich history going back to their initial discovery by Mathieu in 1861 and 1873. They are five of the 26 sporadic finite simple groups and provide the only four 4-transitive permutation groups other than the alternating or symmetric groups. Their structure underpins many interesting mathematical objects such as the Witt designs, Golay codes and Leech Lattice (see for example [5, 9, 13]). In light of the extensive investigations of the combinatorial and geometrical structures associated with these groups it seemed to us surprising that we would unexpectedly uncover an M_{11}-graph that seems not to have been previously studied. This paper is devoted to an exposition of such a graph and its associated structures.

We construct an arc-transitive graph of valency 8 with 165 vertices and automorphism group M_{11}. It turns out that the graph has connections with three of the Witt designs, namely those on 11, 12 and 24 points. The graph does not appear to have been previously studied but is the collinearity graph of the partial linear space numbered 2.6 in [7]. During our search for appearances of this graph in the literature, we discovered that a graph on 165 vertices with automorphism group M_{11} has been identified in [10] as the smallest half-arc-transitive graph with a primitive automorphism group, but this graph has valency 48. (A graph is called *half-arc-transitive* if it is vertex-transitive and edge-transitive, but not arc-transitive.)

We discovered the M_{11} graph when studying transitive decompositions of the Johnson graphs [8]. Given a partition $\mathcal{P}$ of the edge set of a graph Γ, we say that $(\Gamma, \mathcal{P})$ is a *G-transitive decomposition* if there exists a group of automorphisms G of Γ such that G preserves $\mathcal{P}$ and $G^{\mathcal{P}}$ is transitive. The Johnson graph $J(n,k)$ is the graph with vertices the set of k-subsets of an n-set and two k-sets are adjacent if they have $k-1$ points in common. We found an M_{12}-transitive decomposition of $J(12,4)$ such that the stabiliser of a part is isomorphic to M_{11} and each part is isomorphic to the M_{11} graph. This provided us with our original definition of the graph as a graph with vertex set a certain collection of 165 4-subsets of a 12-set. The group M_{11} in its 3-transitive action on 12 points has two orbits on the set of 4-subsets, one of size 165 and one of size 330. Since $165 = \binom{11}{3}$ and the stabiliser of a vertex is the stabiliser in M_{11} of a 3-subset of an 11-set, we searched for and found another description of the graph in terms of 3-subsets of an 11-set. A *Steiner system* $S(t,k,v)$ is a collection of k-subsets (called *blocks*) of a v-set such that each t-subset is contained in a unique block. There is, up to isomorphism a unique $S(4,5,11)$ with point set X of size 11. It is the smallest of the Witt designs we consider. We call the blocks in $S(4,5,11)$ *pentads*.

Theorem 1.1. *There exists an arc-transitive graph* Γ *of order* 165*, valency* 8*, girth* 3 *and diameter* 4 *such that* $G = \mathrm{Aut}(\Gamma) = M_{11}$ *and given adjacent vertices* v, w *we have* $G_v = 2^{\cdot}S_4$ *and* $G_{vw} = S_3$*. Moreover,* Γ *can be defined in the following ways.*

- $V\Gamma$ *is the orbit of length* 165 *of* M_{11} *on the set of* 4*-subsets of a* 12*-set such that two vertices are adjacent if and only if they intersect in a* 3*-set.*
- $V\Gamma$ *is the set of* 3*-subsets of an* 11*-set forming the point set of an* $S(4,5,11)$ *such that two vertices are adjacent if and only if they are disjoint and the complement of their union is a pentad.*

The analysis for Theorem 1.1 is in Section 2.

In [8], we also found an M_{11}-transitive decomposition of $J(11,3)$ where each part is a graph on 55 vertices and the stabiliser of a part is PSL(2, 11). The group PSL(2, 11) in its 2-transitive action on 11 points has two orbits on 3-subsets, one of length 55 and the other of length 110. The orbit of length 55 gives a $2-(11,3,3)$ design known as the *Petersen design* (see [2]). Our PSL(2, 11) graph is the collinearity graph of the partial linear space numbered 6.1.2 in [2], where the distribution diagram for the PSL(2, 11) graph is given. Since $55 = \binom{11}{2}$ we also expect a 2-subset definition. In fact this PSL(2, 11)-graph has 4 equivalent definitions.

Theorem 1.2. *There exists an arc-transitive graph* Π *of order* 55*, valency* 6*, girth* 3 *and diameter* 3 *such that* $G = \mathrm{Aut}(\Pi) = \mathrm{PSL}(2,11)$ *and for adjacent vertices* v, w *we have* $G_v = D_{12}$ *and* $G_{vw} = C_2$*. Moreover,* Π *can be identified in the following ways.*

- $V\Pi$ *is the set of blocks of the Petersen design and two vertices are adjacent if and only if they intersect in a* 2*-set.*
- $V\Pi$ *is the set of blocks of the Petersen design and two vertices are adjacent if and only if they are disjoint and the complement of their union is a pentad.*
- $V\Pi$ *is the set of* 2*-subsets of an* 11*-set and two vertices are adjacent if and only if they meet in one point and their union is a block of the Petersen design.*

- *$V\Pi$ is the set of 2-subsets of an 11-set and two vertices are adjacent if and only if they are disjoint and their union contains no blocks of the Petersen design.*

The analysis for Theorem 1.2 is in Section 3.

When trying to understand the interplay of the two definitions of Γ we realised that the vertices in the 4-subset definition were the sets of fixed points of the involutions of M_{11} in its action on 12 points, while any 3-subset occurs as the set of fixed points of an involution in the M_{11}-action on 11 points. This led to our discovery of an interplay between the two definitions provided by considering the two vertex sets as certain subsets of complementary dodecads in the Witt design on 24 points (Theorem 2.5). It also led to the discovery of a tower of graphs defined on the involutions of the simple groups M_{12}, M_{11}, $\mathrm{PSL}(2,11)$ and A_5 with the middle two graphs being Γ and Π from Theorems 1.1 and 1.2 respectively. The definition of the graphs in the tower is in the spirit of the investigations by Fischer [11] of groups generated by a conjugacy class of *3-transpositions*, that is a conjugacy class of involutions such that any two either commute or generate an S_3.

Theorem 1.3. *There is a tower of graphs defined on a conjugacy class of involutions of $A_5, \mathrm{PSL}(2,11), M_{11}$ and M_{12} with two involutions adjacent if they generate an S_3 from a certain conjugacy class such that the graphs are the line graph of the Petersen graph, Π, Γ and $J(12,4)$.*

We define the tower and prove Theorem 1.3 in Section 4.

A *partial linear space* $(\mathcal{P}, \mathcal{L})$ is a set of points $\mathcal{P}$ and a set of subsets $\mathcal{L}$ of $\mathcal{P}$, called *lines*, such that any pair of points is contained in at most one line. Our results can be interpreted in the language of partial linear spaces as seen in Section 5 and the following theorem.

Theorem 1.4. *There is a tower of partial linear spaces defined on a conjugacy class of involutions of $A_5, \mathrm{PSL}(2,11), M_{11}$ and M_{12} with lines corresponding to subgroups S_3 from a certain conjugacy class such that the tower of collinearity graphs is the one given by Theorem 1.3.*

The partial linear spaces in Theorem 1.4 are reminiscent of the Fischer spaces of [1] whose point set is a conjugacy class of 3-transpositions, and whose lines are those triples of 3-transpositions contained in a given S_3. See also [6].

2 The M_{11} graph

We refer to [9, Section 6] for the following facts about the Witt design $S(5,8,24)$. It has automorphism group M_{24} and the blocks in $S(5,8,24)$

are called *octads*.

(a) Two distinct octads meet in 0, 2, or 4 points.

(b) Every 4-set is contained in 5 octads. Moreover, the symmetric difference $O_1 \ominus O_2$ of any two octads O_1, O_2 intersecting in a 4-subset is also an octad [9, Lemma 6.8A].

(c) If O_1 and O_2 are two disjoint octads, then the complement of $O_1 \cup O_2$ is also an octad [13, p 78].

(d) The symmetric difference of two octads that intersect in a 2-subset is called a *dodecad*.

(e) An octad intersects a dodecad in exactly 2, 4, or 6 points (consequence of [9, Lemma 6.8C]).

(f) The complement of a dodecad is also a dodecad ([9, p.208]).

(g) Let D be a dodecad. We obtain a Steiner system $S(5,6,12)$ on D by taking as 6-subsets (called *hexads*) the intersections of size 6 of octads with D [9, p. 207]. Moreover, complements of hexads are hexads and their corresponding octads have two points in common (consequence of [9, Lemma 6.8C(ii)]).

(h) Let D be a dodecad and $\alpha \in D$. We obtain a Steiner system $S(4,5,11)$ on $D \backslash \{\alpha\}$ by taking as pentads the 5-sets which together with α form a hexad of the $S(5,6,12)$ defined on D in (g).

(i) Let D and D^* be two complementary dodecads in $S(5,8,24)$ and let α be a point in D. The stabiliser of D in M_{24} is isomorphic to M_{12}, and the stabiliser of D and α in M_{24} is isomorphic to M_{11}. The group $(M_{24})_{D,\alpha}$ has the usual action of M_{11} on $D \setminus \{\alpha\}$ and acts 3-transitively on D^* (see p.208 of [9]).

We have the following lemma.

Lemma 2.1. *Let D and D^* be two complementary dodecads in $S(5,8,24)$ and let α be a point in D. Then for all 3-subsets w of $D \backslash \{\alpha\}$, there exists a unique octad O_w such that $O_w \cap D = w \cup \{\alpha\}$.*

Proof. Let w be a 3-subset in $D \backslash \{\alpha\}$. By property (b), there are five octads containing w and α. Since an octad intersects D in exactly 2, 4, or 6 points (property (e)), and since there is a unique octad through any 5 points, there are four octads containing w and α, and intersecting D in 6 points. Thus there is a unique octad, say O_w, which intersects D in 4 points. □

This allows us to construct a $3-(12,4,3)$ design, which was also constructed in [12].

Lemma 2.2. *Let* D *and* D^* *be two complementary dodecads in* $S(5,8,24)$ *with* $\alpha \in D$ *and let* $\mathcal{B} = \{O_w \cap D^* \mid w \text{ a 3-subset of } D\backslash\{\alpha\}\}$. *Then* $(D^*, \mathcal{B})$ *is a* $3-(12,4,3)$ *design.*

Proof. Let $\{\beta,\gamma,\delta\}$ be a 3-subset of D^*. Then $\{\alpha,\beta,\gamma,\delta\}$ is contained in 5 octads. By property (e) these octads meet D in 2 or 4 points. Since every 5-subset is contained in a unique octad, the 11 points of $D\backslash\{\alpha\}$ can be partitioned according to the octad containing $\{\alpha,\beta,\gamma,\delta\}$ which they belong to. The only possibility is that two of the five octads contain one point of $D\backslash\{\alpha\}$ and the remaining three octads meet $D\backslash\{\alpha\}$ in a 3-set. These three octads are of the form O_w for a 3-subset w of $D\backslash\{\alpha\}$ and so each 3-set of D^* is contained in 3 blocks of $\mathcal{B}$. Hence $(D^*, \mathcal{B})$ is a $3-(12,4,3)$ design. □

We now define two graphs.

Definition 2.3. Let Γ_1 be the graph whose vertex-set is the M_{11}-orbit of size 165 on 4-subsets of a 12-set, two such subsets being adjacent if and only if they meet in a 3-subset.

Definition 2.4. Let Γ_2 be the graph whose vertices are the 3-subsets of an 11-set forming the point set of an $S(4,5,11)$, two 3-sets being adjacent if and only if the complement of their union is a pentad.

To show that Γ_1 and Γ_2 are isomorphic we first need to set up a framework.

Let D and D^* be two complementary dodecads in $S(5,8,24)$ and let α be a point in D. Set $X = D\backslash\{\alpha\}$ and $Y = D^*$. For a set W and integer $k \leq |W|$ define $\binom{W}{k}$ to be the set of all k-subsets of W. Define

$$\begin{array}{rccc} i: & \binom{X}{3} & \rightarrow & \binom{Y}{4} \\ & w & \mapsto & Y \cap O_w \end{array} \tag{1}$$

Theorem 2.5. *The map* i *defined in (1) induces an isomorphism from* Γ_2 *onto* Γ_1.

Proof. By Lemma 2.1, the map i is well-defined. Let $w_1, w_2 \in \binom{X}{3}$ and suppose $Y \cap O_{w_1} = Y \cap O_{w_2}$. Then O_{w_1} and O_{w_2} have 5 points in common and hence are equal. Thus i is one-to-one. Let v be a 4-subset of Y. Since $|v \cup \{\alpha\}| = 5$, there is a unique octad containing v and α. By property (e), the three remaining points of that octad can either all be in D, or be one in D and two in D^*. This divides the 4-subsets of Y into two types. Let V be the set of all 4-subsets of the first type, and let $v \in V$. Then the unique octad

$O^{(v)}$ containing $v \cup \{\alpha\}$ is of the form $O^{(v)} = v \cup \{\alpha\} \cup w$ where w is a 3-subset of X. Moreover, $O_w = O^{(v)}$ and so $i(w) = v$. Hence i is a bijection from $\binom{X}{3}$ to V and we may define $i^{-1}(v) = O^{(v)} \cap X = w$. Thus $|V| = \binom{11}{3} = 165$. Since $G_{D,\alpha} = M_{11}$ is 3-transitive on X, fixes Y and preserves octads, it follows that V is an orbit of M_{11} on 4-subsets of Y. Hence i is a bijection from $V\Gamma_2$ to $V\Gamma_1$.

Let w_1 and w_2 be two 3-subsets of X corresponding to adjacent vertices of Γ_2, that is $X \setminus (w_1 \cup w_2)$ is a pentad of $S(4, 5, 11)$, in other words, w_1 and w_2 are disjoint and, by (g) and (h), $w_1 \cup w_2$ is a hexad of D, contained in an octad O meeting D in 6 points. The octads O_{w_1} and O are distinct and meet in at least the 3-subset w_1. Thus by property (a) they meet in 4 points, which means that $i(w_1)$ contains exactly one point of O. Moreover by (b), $O \ominus O_{w_1}$ is an octad containing w_2 and α and intersecting Y in 4 points. By Lemma 2.1, $O \ominus O_{w_1}$ is equal to O_{w_2}. Thus $i(w_1)$ and $i(w_2)$ meet in 3 points, and so they are adjacent vertices of Γ_1.

Conversely, let v_1 and v_2 be two 4-subsets of Y corresponding to adjacent vertices of Γ_1, that is they are in the M_{11}-orbit of size 165 and they meet in a 3-subset. We have that $O_1 = v_1 \cup \{\alpha\} \cup i^{-1}(v_1)$ and $O_2 = v_2 \cup \{\alpha\} \cup i^{-1}(v_2)$ are octads containing the 4-subset $\{\alpha\} \cup (v_1 \cap v_2)$. Since $v_1 \neq v_2$, the octads O_1, O_2 are distinct and so by property (a) we have $|O_1 \cap O_2| = 4$. Hence $i^{-1}(v_1)$ and $i^{-1}(v_2)$ are disjoint. Moreover, by (b) the symmetric difference of O_1 and O_2 is an octad, which intersects D in $i^{-1}(v_1) \cup i^{-1}(v_2)$. Hence $i^{-1}(v_1) \cup i^{-1}(v_2)$ is a hexad in D, and so by properties (g) and (h), its complement in X is a pentad. Thus $i^{-1}(v_1)$ and $i^{-1}(v_2)$ are adjacent vertices of Γ_2. □

The two definitions of Γ given in Theorem 1.1 are Γ_1 and Γ_2, which we have just shown to be isomorphic.

Figure 1 gives the distance diagram of the graph $\Gamma \cong \Gamma_2$ according to the orbits of a vertex stabiliser. There is one orbit of vertices at distance 1, one of vertices at distance 2, four of vertices at distance 3, and one of vertices at distance 4. Here is what the different orbits correspond to, in terms of the fixed 3-subset s_0 of the graph Γ_2:

$s \in$	$\lvert s \cap s_0 \rvert$	Extra condition
A	0	$X \setminus (s \cup s_0)$ is a pentad.
B	1	$s = \{x, a, b\}$ with $x \in s_0$, the pentads P_a and P_b containing respectively $s_0 \cup \{a\}$ and $s_0 \cup \{b\}$ are distinct, and $(P_a \cup P_b) \setminus (s_0 \setminus \{x\})$ is not a pentad.
C	0	$s \cup s_0$ contains a pentad with 3 points in s_0.
D	2	
E	0	$s \cup s_0$ contains a pentad with 3 points in s.
F	1	$s \cup s_0$ is a pentad.
G	1	$s = \{x, a, b\}$ with $x \in s_0$, the pentads P_a and P_b containing respectively $s_0 \cup \{a\}$ and $s_0 \cup \{b\}$ are distinct, and $(P_a \cup P_b) \setminus (s_0 \setminus \{x\})$ is a pentad.

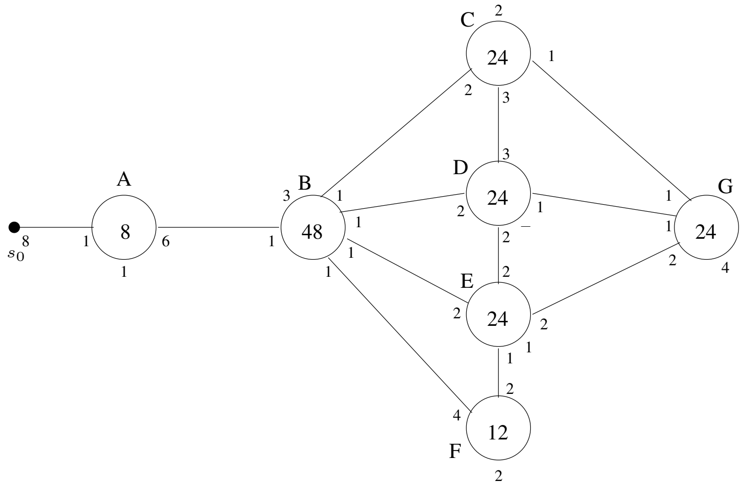

Figure 1: Distribution diagram of Γ

The half-arc-transitive graph in [10] with 165 vertices and valency 48 has the set of neighbours of s_0 being $C \cup E$. It can be determined via Magma [3] that two vertices are adjacent in Γ if and only if in the valency 48 graph they are at distance two and there are precisely 9 vertices adjacent to both.

To complete the proof of Theorem 1.1 it remains to determine some properties of Γ.

Theorem 2.6. *The graph* Γ *has* 165 *vertices, valency* 8 *and diameter 4. Its maximal cliques are of size* 3 *and two cliques meet in at most one vertex. Its full automorphism group is* M_{11}*, which is arc-transitive and vertex-primitive, with vertex stabiliser* $M_8 \rtimes S_3 \cong 2^{\cdot}S_4$*, and arc stabiliser* S_3*. Moreover, it is a rank* 8 *graph.*

Proof. The number of vertices is $\binom{11}{3} = 165$. Using the 4-subset definition of Γ, Lemma 2.2 implies that the vertices of Γ form a $3-(12,4,3)$ design. A vertex v contains four 3-subsets and each 3-subset is contained in 2 blocks other than v. Hence Γ has valency 8. Moreover, the maximal cliques have size 3 and since a clique is defined by the 3-subset in common with all three vertices, two cliques meet in at most one vertex. It follows from Figure 1 that Γ has diameter 4.

Let $A = \mathrm{Aut}(\Gamma)$. A MAGMA [3] calculation shows that $A = M_{11}$. Alternatively, one can use the results of [14] on overgroups of primitive groups (which require the Classification of Finite Simple Groups) to show that $A = M_{11}$.

Using the 3-subset definition again, the stabiliser of a vertex w in A is $M_8 \rtimes S_3 \cong 2{}^{\cdot}S_4$, the stabiliser in M_{11} of a 3-subset. By [4], the stabiliser in M_{11} of a pentad is S_5 and this acts 3-transitively (as $\mathrm{PGL}(2,5)$) on the 6 points not in the pentad. Hence the stabiliser in A of w acts transitively on the set of 8 pentads having an empty intersection with the 3-subset w. Thus A_w acts transitively on the set of neighbours of w and so A is arc-transitive on Γ_2 with arc stabiliser S_3. Looking at the permutation character of the action of M_{11} on 3-subsets of an 11-set in [4, p 18], we conclude that the graph has rank 8. □

3 The $\mathrm{PSL}(2,11)$ graph

Let D and D^* be complementary dodecads in $S(5,8,24)$ and let $\alpha \in D$ and $\beta \in D^*$. From property (h), the stabiliser in M_{24} of D and α is M_{11} and acts 3-transitively on D^*. Furthermore, the stabiliser in M_{11} of β is $\mathrm{PSL}(2,11)$ which acts 2-transitively on $D\backslash\{\alpha\}$ and $D^*\backslash\{\beta\}$.

Construction 3.1. Let $(D^*, \mathcal{B})$ be the $3-(12,4,3)$ design given by Lemma 2.2. Let $\beta \in D^*$. Since $|\mathcal{B}| = 165$, it follows that there are 55 blocks of $\mathcal{B}$ containing β. Let $\overline{\mathcal{B}}$ be the set of 3-subsets v of $D^*\backslash\{\beta\}$ such that $v \cup \{\beta\}$ is a block of $\mathcal{B}$. Then $(D^*\backslash\{\beta\}, \overline{\mathcal{B}})$ is a $2-(11,3,3)$ design. This design is known as the Petersen design [2]. We note that $\overline{\mathcal{B}}$ is the unique orbit of length 55 of $\mathrm{PSL}(2,11)$ on the set of 3-subsets of $D^*\backslash\{\beta\}$. The other orbit has length 110.

We now give our first definition of the graph Π.

Definition 3.2. Let X be a set of size 11 forming the point set of a $2-(11,3,3)$ Petersen design. Let Π_1 be the graph with vertex set the set of blocks of the design such that two blocks are adjacent if they have two points in common.

The graph Π_1 is the collinearity graph of the geometry 6.1.2 of [2] and the distance diagram is given there and reproduced in Figure 2, where it is seen that Π_1 has diameter 3.

Theorem 3.3. *The graph* Π_1 *has* 55 *vertices and valency* 6. *Its maximal cliques are of size* 3 *and two cliques meet in at most one vertex. Its full automorphism group is* $\mathrm{PSL}(2,11)$, *which is arc-transitive and vertex-primitive with vertex stabiliser* D_{12} *and arc stabiliser* C_2. *Moreover,* Π_1 *is a Cayley graph for the group* $C_{11} \rtimes C_5$ *and is a rank* 9 *graph.*

Proof. Let v be a vertex of Π_1. Then v contains three 2-subsets and since the vertex set of Π_1 forms a $2-(11,3,3)$ design, each 2-subset of v is contained in 3 blocks. Hence v is adjacent to 6 other vertices. Moreover, the maximal cliques have size 3 and meet in at most one vertex.

A MAGMA [3] calculation shows that $\mathrm{Aut}(\Pi_1) = \mathrm{PSL}(2,11)$. Alternatively, the results of [14] on overgroups of primitive groups (which rely on the Classification of Finite Simple Groups) can be used to determine $\mathrm{Aut}(\Pi_1)$.

Let $L = \mathrm{PSL}(2,11)$. Then $L_v = D_{12}$ and this induces S_3 on the 3 points of v. Thus the kernel $L_{(v)}$ of L_v on v has order 2 and, by the character table of [4, p 7], fixes the three points of v and is fixed point free on the remaining 8 points. Note that L_v preserves a partition $\mathcal{P}$ of the set of neighbours of v into three sets of size 2 given by the three possible choices of intersections with v in a 2-set. Then $L_{(v)}$ acts trivially on $\mathcal{P}$, and since L_v acts transitively on $\mathcal{P}$, either $L_{(v)}$ acts trivially on each part of $\mathcal{P}$ or interchanges the two vertices in each part. If we have the first case then $L_{(v)}$ acts trivially on the set of neighbours of v in Π_1. Since $L_{(v)}$ fixes each of the points contained in v, it follows that $L_{(v)}$ fixes each of the points which together with a 2-set of v form a vertex. This contradicts $L_{(v)}$ being fixed point free on the set of points not in v. Hence $L_{(v)}$ interchanges the two vertices in each part of the partition. It follows that L_v acts transitively on the set of neighbours of v and so L is arc-transitive. Moreover, the arc stabiliser is C_2.

Note that $\mathrm{PSL}(2,11)$ has a subgroup $C_{11} \rtimes C_5$ which intersects trivially with D_{12} and so by comparing orders we have $\mathrm{PSL}(2,11) = D_{12}(C_{11} \rtimes C_5)$. Hence $C_{11} \rtimes C_5$ acts regularly on the vertex set of Π_1 and so Π_1 is a Cayley graph of $C_{11} \rtimes C_5$. By the permutation character in [4], $\mathrm{PSL}(2,11)$ has rank 9 in its action on vertices. □

Thinking of $\overline{\mathcal{B}}$ as a subset of $\mathcal{B}$, we see that Π_1 is as a subgraph of Γ_1. The map i defined in (1) defines an isomorphism from Γ_2 to Γ_1, hence the preimage of Π_1 in Γ_2 is isomorphic to Π_1. Note that the preimages of the vertices of Π_1 are 3-sets w in $D \setminus \{\alpha\}$ such that $\beta \in i(w)$. If we take these 3-sets as blocks, we find an isomorphic $2-(11,3,3)$ design, since this block set and $\overline{\mathcal{B}}$ are interchanged by elements of $\mathrm{PGL}(2,11)$ not in $\mathrm{PSL}(2,11)$. This gives the following definition of a graph isomorphic to Π_1.

Definition 3.4. Let X be an 11-set forming the point set of a $2-(11,3,3)$ Petersen design preserved by a group $H = \mathrm{PSL}(2,11)$ and also forming the point set of an $S(4,5,11)$ preserved by H. Let Π_2 be the graph with vertex set the set of blocks of the Petersen design such that blocks v_1 and v_2 are joined by an edge if $v_1 \cap v_2 = \varnothing$ and $X\backslash(v_1 \cup v_2)$ is a pentad.

Theorem 3.5. $\Pi_1 \cong \Pi_2$.

Proof. The isomorphism is given by the map i from (1). □

Figure 2 gives the distance diagram of the graph $\Pi \cong \Pi_2$ according to the orbits of a vertex stabiliser. There is one orbit of vertices at distance 1, two of vertices at distance 2, and 5 of vertices at distance 3.

By Theorem 3.3, the stabiliser of a vertex is D_{12} which according to [4, p 7] also fixes a 2-subset of the 11-set. Thus the stabiliser in L of the vertex s_0 has one orbit $\{\epsilon_1, \epsilon_2\}$ of size 2. Here is what the different orbits correspond to, in terms of the fixed 3-subset s_0 of the graph Π_2:

$s \in$	$\lvert s \cap s_0\rvert$	Extra condition
A	0	$X \setminus (s \cup s_0)$ is a pentad.
B	1	$s \cup s_0$ is not a pentad and $\lvert s \cap \{\epsilon_1, \epsilon_2\}\rvert = 1$.
C	1	$s \cup s_0$ is not a pentad and $\lvert s \cap \{\epsilon_1, \epsilon_2\}\rvert = 0$.
D	1	$s \cup s_0$ is a pentad disjoint from $\{\epsilon_1, \epsilon_2\}$.
E	0	$s \cup s_0$ contains a pentad with 3 points in s.
F	2	
G	1	$s \cup s_0$ contains a pentad with 3 points in s_0.
H	1	$s \cup s_0$ is a pentad containing $\{\epsilon_1, \epsilon_2\}$.

Note that $55 = \binom{11}{2}$ and the stabiliser of a block of the Petersen design is also the stabiliser of a 2-set. This suggests a 2-subset definition of Π_1.

Definition 3.6. Let X be an 11-set forming the point set of a $2-(11,3,3)$ Petersen design. Let Π_3 be the graph with vertex set the set of all 2-subsets of X such that two vertices are adjacent if they have one point in common and their union is a block of the Petersen design.

Lemma 3.7. Π_3 *is isomorphic to the graph whose vertices are the triangles of* Π_1 *and two triangles are adjacent if they have a common vertex.*

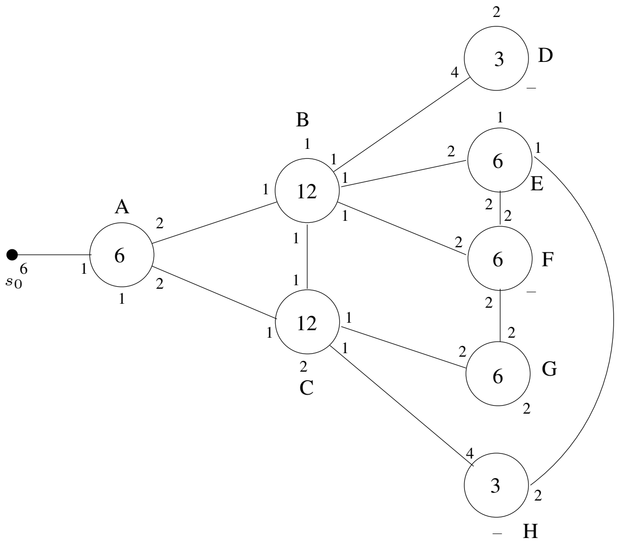

Figure 2: Distribution diagram of Π

Proof. Since the Petersen design is a $2-(11,3,3)$ design, each 2-subset is contained in three blocks and these three blocks form a triangle in Π_1. Conversely, each triangle in Π_1 is described by a unique 2-subset, this being the 2-subset in common with each of the three blocks which are vertices in the triangle. Thus there is a bijection between the vertices of Π_3 and the set of triangles of Π_1. Moreover, two triangles of Π_1 have a vertex in common if and only if the union of the two 2-sets corresponding to the two triangles is the block of the Petersen design given by the common vertex. Hence the result follows. □

We also have the following definition of a graph on the set of 2-subsets of an 11-set.

Definition 3.8. Let X be an 11-set equipped with a $2-(11,3,3)$ Petersen design. Let Π_4 be the graph with vertex set the set of all 2-subsets of X such that two vertices x_1, x_2 are adjacent if they are disjoint and $x_1 \cup x_2$ contains no blocks of the Petersen design.

We have the following lemma.

Lemma 3.9. Π_4 *is isomorphic to the graph whose vertices are the triangles of* Π_2 *such that two triangles are adjacent if they have a common vertex.*

Proof. A triangle $T = \{v_1, v_2, v_3\}$ in Π_2 is a set of three disjoint blocks of the $2-(11,3,3)$ Petersen design such that the complement of the union of any two is a pentad. Note that the complement of a pentad is a hexad and so it follows that $v_i \cup v_j$ is a hexad for any $i \neq j$. Thus there are two remaining points ϵ_1, ϵ_2 of X such that $\{\epsilon_1, \epsilon_2\} \cup v_i$ is a pentad for $i = 1, 2, 3$. It follows that we can define a map ϕ from the set of triangles of Π_2 to the vertex set of Π_4 by mapping a triangle T to its associated 2-subset $\{\epsilon_1, \epsilon_2\}$. Let $L = \mathrm{Aut}(\Pi_2) \cong \mathrm{PSL}(2, 11)$. Then L maps triangles to triangles and so preserves the set of images of ϕ. Since L acts 2-transitively on X, it follows that ϕ is a bijection. Each triangle of Π_2 is adjacent to 6 other triangles and using MAGMA [3] we see that Π_4 has valency 6. Thus it remains to show that ϕ maps adjacent triangles of Π_2 to adjacent vertices of Π_4.

The stabiliser in L of the triangle T is $L_T \cong D_{12}$ (since $\Pi_1 \cong \Pi_2$ and a triangle of Π_1 corresponds to a 2-subset) and this fixes the subset $\{\epsilon_1, \epsilon_2\}$ while inducing S_3 on the three vertices of T. The 2-set $\{\epsilon_1, \epsilon_2\}$ is contained in three blocks of the Petersen design and so defines a subset $\{\gamma_1, \gamma_2, \gamma_3\}$ of X such that $\{\epsilon_1, \epsilon_2, \gamma_i\}$ is a block for each $i = 1, 2, 3$. This 3-set must be fixed setwise by L_T and the remaining 6 points of X form an orbit of L_T (with the usual action of D_{12}). Thus we may assume that $\gamma_i \in v_i$ for $i = 1, 2, 3$.

Let T_2 be a triangle with one vertex v_1 in common with T and let $\phi(T_2) = \{\delta_1, \delta_2\}$. Then $\{\delta_1, \delta_2\} \cap \{\epsilon_1, \epsilon_2\} = \varnothing$ as the union of v_1 and each of the pairs is a pentad and no two pentads have 4 points in common. Moreover, if $\{\delta_1, \delta_2\} \subset v_i$ for $i = 2, 3$, then the pentad $v_1 \cup \{\delta_1, \delta_2\}$ is contained in the hexad $v_i \cup v_1$. This contradicts the fact that pentads are not contained in hexads, otherwise the Steiner system $S(5, 6, 12)$ would have two distinct hexads with 5 points in common. Hence we may suppose that $\delta_1 \in v_2$ and $\delta_2 \in v_3$. Similarly, if T_3 is the third triangle containing the vertex v_1 then $\phi(T_3) = \{\mu_1, \mu_2\}$ with $\mu_1 \in v_2 \backslash \{\delta_1\}$ and $\mu_2 \in v_3 \backslash \{\delta_2\}$. Now $L_{v_1} \cong D_{12}$ and acts transitively on $\{T, T_2, T_3\}$. Since v_1 is the only orbit of length three of L_{v_1}, it follows that $\{\epsilon_1, \epsilon_2, \delta_1, \delta_2, \mu_1, \mu_2\}$ is the unique orbit of length 6 of L_{v_1}. Moreover, $\{\delta_1, \delta_2, \mu_1, \mu_2\}$ is an orbit of L_{T,v_1}. Also L_{T,v_1} fixes γ_1 and interchanges γ_2 and γ_3. Thus $\{\gamma_2, \gamma_3\}$ is disjoint from $\{\mu_1, \mu_2, \delta_1, \delta_2\}$. Hence $\{\epsilon_1, \epsilon_2, \delta_1, \delta_2\}$ does not contain any of the blocks of the Petersen design containing $\{\epsilon_1, \epsilon_2\}$. Analysing the action of L_{T_2,v_1} in a similar fashion we see that $\{\epsilon_1, \epsilon_2, \delta_1, \delta_2\}$ also contains no blocks containing $\{\delta_1, \delta_2\}$ and so $\{\epsilon_1, \epsilon_2, \delta_1, \delta_2\}$ contains no blocks of the Petersen design. Thus ϕ maps adjacent triangles of Π_2 to adjacent vertices of Π_4 and the proof is complete. □

Corollary 3.10. $\Pi_3 \cong \Pi_4$

Proof. Follows from the fact that $\Pi_1 \cong \Pi_2$ and Π_3 is the triangle graph of Π_1 while Π_4 is the triangle graph of Π_2. □

We wish to show that Π_3 is isomorphic to Π_1. To do this we first construct the following setup. Let D and D^* be two complementary dodecads in $S(5,8,24)$ and let $\alpha \in D$ and $\beta \in D^*$. Let H be the stabiliser in M_{24} of D, α and β. Then $H \cong \mathrm{PSL}(2,11)$. Recall from (1) the map i from the set of 3-subsets of $D\backslash\{\alpha\}$ into the set of 4-subsets of D^* such that the image $\mathcal{B}$ of i is a $3-(12,4,3)$ design and the set of blocks of $\mathcal{B}$ containing β yields a $2-(11,3,3)$ Petersen design on $D^*\backslash\{\beta\}$ with blocks $\overline{\mathcal{B}}$.

Let $v \in \overline{\mathcal{B}}$. Since $(D^*, \mathcal{B})$ is a $3-(12,4,3)$ design there are three blocks of $\mathcal{B}$ containing v, one of which is $v \cup \{\beta\}$. Let $\{\epsilon_1, \epsilon_2\}$ be the two points such that $v \cup \{\epsilon_i\}$ are also blocks of $\mathcal{B}$. This allows us to define a map

$$\begin{array}{rccc} j: & \overline{\mathcal{B}} & \rightarrow & \binom{D^*\backslash\{\beta\}}{2} \\ & v & \mapsto & \{\epsilon_1, \epsilon_2\} \end{array} \tag{2}$$

Moreover, since $v \cup \{\beta\}, v \cup \{\epsilon_1\}, v \cup \{\epsilon_2\}$ are mutually adjacent in Γ_1, Theorem 2.5 implies that we can partition D as $\{\alpha, \delta, \mu\} \cup w_1 \cup w_2 \cup w_3$ such that $i(w_1) = v \cup \{\beta\}$, $i(w_2) = v \cup \{\epsilon_1\}$ and $i(w_3) = v \cup \{\epsilon_2\}$, and for $i = 1,2,3$ the set $\{\delta, \mu\} \cup w_i$ is a pentad. Thus we can also define the map

$$\begin{array}{rccc} k: & \overline{\mathcal{B}} & \rightarrow & \binom{D\backslash\{\alpha\}}{2} \\ & v & \mapsto & \{\delta, \mu\} \end{array} \tag{3}$$

The maps j and k are linked in the following way.

Lemma 3.11. *Let $v \in \overline{\mathcal{B}}$ and let j, k be the maps defined in (2), (3) respectively. Then $\{\beta\} \cup v \cup j(v) \cup k(v)$ is an octad.*

Proof. Let $j(v) = \{\epsilon_1, \epsilon_2\}$. By the definition of the map i in (1) we have the three octads $O_{w_1} = \{\alpha, \beta\} \cup v \cup w_1$, $O_{w_2} = \{\alpha, \epsilon_1\} \cup v \cup w_2$ and $O_{w_3} = \{\alpha, \epsilon_2\} \cup v \cup w_3$. By property (b), $O_{w_2} \ominus O_{w_3} = \{\epsilon_1, \epsilon_2\} \cup w_2 \cup w_3$ is an octad disjoint to O_{w_1}. Thus by property (c), $k(v) \cup (D^*\backslash(\{\beta, \epsilon_1, \epsilon_2\} \cup v))$ is also an octad. Thus by property (g) it follows that $k(v) \cup v \cup j(v) \cup \{\beta\}$ is an octad. □

We now show that Π_1 and Π_3 are isomorphic.

Theorem 3.12. *The map k defined in (3) induces an isomorphism from Π_1 to Π_3.*

Proof. We first construct Π_1 by using the Petersen design $(D^*\backslash\{\beta\}, \overline{\mathcal{B}})$. Thus the vertices of Π_1 are the blocks of $\overline{\mathcal{B}}$ and two vertices are adjacent if they intersect in a 2-set. Now $H = \mathrm{PSL}(2,11)$ preserves the set of octads containing $\{\alpha, \beta\}$ and meeting D in 4 points. There are 55 such octads and so the set $\mathcal{C} = \{w \mid \beta \in O_w\}$ is an orbit of length 55 of H on 3-subsets of $D\backslash\{\alpha\}$.

Thus $(D\backslash\{\alpha\},\mathcal{C})$ is a $2-(11,3,3)$ Petersen design and we can construct Π_3 from this design, that is the vertices of Π_3 are the 2-subsets of $D\backslash\{\alpha\}$ and two vertices are adjacent if they have one point in common and their union is a block of $\overline{\mathcal{B}}$.

Since $\mathrm{PSL}(2,11)$ preserves $\overline{\mathcal{B}}$ and $\mathcal{C}$, it also preserves the set of images of k. Then as $\mathrm{PSL}(2,11)$ acts 2-transitively on $D\backslash\{\alpha\}$, it follows that k is onto. Moreover, as $|\overline{\mathcal{B}}|=|\mathcal{C}|=55=\binom{11}{2}$ it follows that k is a bijection.

We saw in Theorem 3.3 that Π_1 has 55 vertices and is of valency 6. Now Π_3 has $\binom{11}{2}=55$ vertices. Moreover, since each 2-set is contained in 3 blocks of $\mathcal{C}$, it follows that Π_3 also has valency 6. Thus we only need to show that if v_1 and v_2 are adjacent in Π_1 then $k(v_1)$ and $k(v_2)$ are adjacent in Π_3.

Let $v_1,v_2\in\overline{\mathcal{B}}$ such that $|v_1\cap v_2|=2$, that is, v_1,v_2 are adjacent vertices in Π_1. Suppose that $j(v_1)$ and $j(v_2)$ have a point ϵ in common. Then $v_1\cup\{\beta\},v_2\cup\{\beta\},v_2\cup\{\epsilon\}$ and $v_1\cup\{\epsilon\}$ are blocks of $\mathcal{B}$ forming a 4-cycle in Γ_1. However, looking at the distance diagram for Γ_1 in Figure 1, we see that Γ_1 does not contain 4-cycles. Thus if $|v_1\cap v_2|=2$ then $j(v_1)\cap j(v_2)=\varnothing$. Moreover, by Lemma 3.11, $O_1=v_1\cup j(v_1)\cup k(v_1)\cup\{\beta\}$ and $O_2=v_2\cup j(v_2)\cup k(v_2)\cup\{\beta\}$ are octads containing the three points $(v_1\cap v_2)\cup\{\beta\}$. Hence by property (a) they have four points in common and so $|k(v_1)\cap k(v_2)|=1$.

Let $k(v_1)=\{\delta_1,\mu\}$ and $j(v_1)=\{\epsilon_1,\epsilon_2\}$, and let w_1,w_2,w_3 be 3-subsets of $D\backslash\{\alpha\}$ such that $O_{w_1}=\{\alpha,\beta\}\cup w_1\cup v_1$, $O_{w_2}=\{\alpha,\epsilon_1\}\cup w_2\cup v_1$ and $O_{w_3}=\{\alpha,\epsilon_2\}\cup w_3\cup v_1$ are octads. Similarly, let $k(v_2)=\{\delta_2,\mu\}$ and $j(v_2)=\{\epsilon_1',\epsilon_2'\}$, and let w_1',w_2',w_3' be 3-subsets of $D\backslash\{\alpha\}$ such that $O_{w_1'}=\{\alpha,\beta\}\cup w_1'\cup v_2$, $O_{w_2'}=\{\alpha,\epsilon_1'\}\cup w_2'\cup v_2$ and $O_{w_3'}=\{\alpha,\epsilon_2'\}\cup w_3'\cup v_2$ are octads. Recall the octads $O_1=v_1\cup j(v_1)\cup k(v_1)\cup\{\beta\}$ and $O_2=v_2\cup j(v_2)\cup k(v_2)\cup\{\beta\}$, which have the four points $(v_1\cap v_2)\cup\{\beta,\mu\}$ in common. Hence by property (b),

$$O_1\ominus O_2=\{\delta_1,\delta_2\}\cup(v_1\ominus v_2)\cup\{\epsilon_1,\epsilon_2,\epsilon_1',\epsilon_2'\}$$

is an octad. Since $O_1\ominus O_2$ contains one point of v_2 it has at least one point in common with $O_{w_1'}$ and so by property (a) it follows that $\delta_1\in w_1'$. Similarly, comparing $O_1\ominus O_2$ with O_{w_1} we see that $\delta_2\in w_1$. Thus $O_1\ominus O_{w_1}=w_1\cup\{\alpha,\delta_1,\mu,\epsilon_1,\epsilon_2\}$ and $O_2\ominus O_{w_1'}=w_1'\cup\{\alpha,\delta_2,\mu,\epsilon_1',\epsilon_2'\}$ are octads with the 4-set $w=\{\delta_1,\delta_2,\mu,\alpha\}$ in common. By property (b), the 4-set w is contained in 5 octads, which by Lemma 2.1 are $O_{\{\delta_1,\delta_2,\mu\}}$ and four octads R_1,R_2,R_3,R_4 meeting D in 6 points. The octads $R_1:=O_1\ominus O_{w_1}$ and $R_2:=O_2\ominus O_{w_1'}$ are two of these four octads. Moreover, the 2-sets $(R_i\cap D)\backslash w$ for $i=1,2,3,4$, partition the set $D\backslash w$ of eight points. The octads R_1,R_2 account for the four points of $(w_1\cup w_1')\backslash\{\delta_1,\delta_2\}$ and so for $i=3$ or 4, the two points of $(R_i\cap D)\backslash x$ are in $w_2\cup w_3$ but not in w_1'. If $\beta\in R_i$ for $i=3,4$, then R_i will have 5 points in common with either $O_1\ominus O_{w_2}=w_2\cup\{\alpha,\beta,\delta_1,\mu,\epsilon_2\}$ or

$O_1 \ominus O_{w_3} = w_3 \cup \{\alpha, \beta, \delta_1, \mu, \epsilon_1\}$. Since these two octads do not contain δ_2, it follows that neither of them are R_i, and so by property (a), β is not in any of the R_i. Now R_1, R_2, R_3, R_4 define eight points of D^* and $i(\{\delta_1, \delta_2, \mu\})$ is the set of 4 points of D^* not in any R_i. For $i \neq j$ the sets $R_i \cap D^*$ and $R_j \cap D^*$ are disjoint and have size 2. Hence $R_1, \ldots, R_4$ provide us with 8 points of $D^* \backslash \{\beta\}$. Thus $\beta \in i(\{\delta_1, \delta_2, \mu\})$ and so $\{\delta_1, \delta_2, \mu\} \in \mathcal{C}$. Hence $\{\delta_1, \delta_2, \mu\}$ is a block of the Petersen design $(D \backslash \{\alpha\}, \mathcal{C})$ preserved by PSL(2, 11). Thus $k(v_1)$ is adjacent to $k(v_2)$ in Π_3 and so $\Pi_1 \cong \Pi_3$. □

Corollary 3.13. *$\Pi_1 \cong \Pi_2 \cong \Pi_3 \cong \Pi_4$.*

This completes the proof of Theorem 1.2. We note here that it can be shown that the map j induces an isomorphism from Π_1 to Π_4.

4 A tower of graphs

The Johnson graph $J(12, 4)$ can be defined in terms of one of the two conjugacy classes of involutions of M_{12}.

Construction 4.1. Let $G = M_{12}$. Let Δ_1 be the graph with vertex set the set of involutions of G from the conjugacy class $2B$ with 4 fixed points (in an M_{12} action of degree 12) [4, p 33] such that two involutions are joined by an edge if and only if they generate an S_3 which has 3 fixed points.

Lemma 4.2. *$\Delta_1 \cong J(12, 4)$.*

Proof. Let Y be a set of size 12 such that G acts 5-transitively on Y. Then for each 4-subset A of Y, there is an involution g whose set of fixed points is A. Since the class $2B$ has 495 elements and $\binom{12}{4} = 495$ it follows that g is unique. Hence we can define a bijection ϕ between the vertices of Δ_1 and the vertices of $J(12, 4)$ which maps each involution g to $\mathrm{Fix}(g)$.

Let g_1 and g_2 be two adjacent vertices of Δ_1. Then $\langle g_1, g_2 \rangle \cong S_3$ fixes 3 points. These 3 points are in $\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$, which shows that $\phi(g_1)$ and $\phi(g_2)$ are adjacent in $J(12, 4)$.

Now let v_1 and v_2 be two adjacent vertices of $J(12, 4)$, that is, they are 4-sets of Y intersecting in a 3-set. The pointwise stabiliser in G of this 3-set is $L \cong M_9 \cong C_3^2 \rtimes Q_8$, which has a unique Sylow 3-subgroup S and a unique conjugacy class of 9 subgroups isomorphic to Q_8 (the Sylow 2-subgroups). Let $g \in L$ be an involution. Then g is the unique involution of some Q_8 and so inverts each nontrivial element of S. Thus g lies in a unique Q_8 and so L contains 9 involutions. These are the elements sg, for $s \in S$. The two involutions fixing respectively v_1 and v_2 (that is $\phi^{-1}(v_1)$ and $\phi^{-1}(v_2)$) are in

L so they can be written as s_1g and as s_2g. Their product is the element $s_1s_2^{-1}$ of order 3, and so they generate an S_3. Moreover, this S_3 lies inside L, and so fixes 3 points. It follows that $\phi^{-1}(v_1)$ and $\phi^{-1}(v_2)$ are adjacent in Δ_1. □

Let Y be a set of size 12 on which the group $G = M_{12}$ acts 5-transitively. Now G contains two classes of subgroups isomorphic to M_{11} and these are interchanged by an outer automorphism. One of these two classes corresponds to the stabilisers in G of a point of Y. Involutions in G lie in one of two conjugacy classes: fixed point free involutions and the class $2B$ where they each fix 4 points [4, p 33]. Thus involutions in the subgroups M_{11} of this class are from the class $2B$ and as outer automorphisms of M_{12} fix the class $2B$ setwise, it follows that involutions in the other class of M_{11} subgroups are also from the class $2B$. Now M_{11} has 165 involutions and they form a single conjugacy class. Moreover, the stabiliser in M_{11} of 3 points of the 12-set Y is S_3 and since M_{11} is 3-transitive, the set of subgroups S_3 fixing 3 points forms a single conjugacy class. Since the normaliser of such an S_3 fixes the 3-set of fixed points setwise, each such S_3 subgroup has normaliser $S_3 \times S_3$. If H is an M_{11} in G which is not the stabiliser of a point of Y then H is transitive on Y and the set of fixed point subsets of the involutions of H is the orbit of H of size 165 on 4-subsets.

This suggests a connection with Γ_1.

Definition 4.3. Let $H = M_{11}$. Let Δ_2 be the graph with vertex set the set of involutions of H such that two involutions are adjacent if and only if they generate an S_3 with normaliser $S_3 \times S_3$.

Lemma 4.4. $\Delta_2 \cong \Gamma_1$.

Proof. In the action of M_{11} on the 12-set Y, the subgroups isomorphic to S_3 with normaliser $S_3 \times S_3$ are those which fix precisely 3 points. Let $\phi : V\Delta_2 \to Y$ such that $\phi(g) = \mathrm{Fix}(g)$. Then ϕ is a bijection from the vertex set of Δ_2 to the vertex set of Γ_1. Suppose that $g_1, g_2 \in V\Delta_2$ are adjacent. Then $\langle g_1, g_2\rangle \cong S_3$ and fixes 3 points. Hence $|\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)| = 3$ and so $\phi(g_1)$ is adjacent to $\phi(g_2)$ in Γ_1. Conversely, suppose that $\phi(g_1)$ is adjacent to $\phi(g_2)$ in Γ_1. Then $|\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)| = 3$ and so $\langle g_1, g_2\rangle$ fixes 3 points. Since the pointwise stabiliser in M_{11} of 3 points of Y is isomorphic to S_3, it follows that $\langle g_1, g_2\rangle \cong S_3$ and has normaliser $S_3 \times S_3$. Thus g_1 is adjacent to g_2 in Γ_1 and so $\Delta_2 \cong \Gamma_1$. □

We have just exhibited an embedding of the graph Δ_2 in $J(12, 4)$. It is proved in [8] that in fact $J(12, 4)$ decomposes into 12 pairwise disjoint copies of Δ_2 and these 12 copies are transitively permuted by M_{12}.

Next we look at a subgroup $K = \mathrm{PSL}(2,11)$ of $H = M_{11}$. This occurs as the stabiliser in H of a point $\beta \in Y$. Hence K contains a class of subgroups S_3 which fix 3 points of Y, one of which is β. Since K is 2-transitive on the 11 points of $Y \backslash \{\beta\}$ it follows that all such subgroups S_3 are conjugate in K. Thus we can use this conjugacy class of S_3 subgroups to define a graph on the set of involutions of K, and this graph will be a subgraph of Δ_2.

Construction 4.5. Let Δ_3 be the graph whose vertex set is the set of involutions of $\mathrm{PSL}(2,11)$ such that two involutions are adjacent if and only if they generate an S_3 which fixes 2 points in the action on 11 points.

Lemma 4.6. $\Delta_3 \cong \Pi_1$

Proof. By [4, p 7] there are 55 involutions in $\mathrm{PSL}(2,11)$ and each fixes 3 points in the action on a set of size 11. Let $\phi : V\Delta_3 \to V\Pi_1$ such that $\phi(g) = \mathrm{Fix}(g)$. Since $\mathrm{PSL}(2,11) < M_{11}$ as the stabiliser of a point in the action on a set of size 12, and no two involutions in M_{11} have the same fixed point set, it follows that no two involutions in $\mathrm{PSL}(2,11)$ have the same fixed point set. Thus ϕ is one-to-one. Moreover, $\phi(V\Delta_3)$ gives an orbit of $\mathrm{PSL}(2,11)$ on 3-subsets of size 55 and so $\phi(V\Delta_3)$ must be the set of blocks of a $2-(11,3,3)$ Petersen design (as defined in Section 3. Thus ϕ is a bijection. The fact that ϕ is a graph isomorphism follows in the same way as in the proof of Lemma 4.4. □

Note that $\mathrm{PSL}(2,11)$ contains two classes of subgroups S_3 and these are fused in $\mathrm{PGL}(2,11)$. Hence, the graph formed on the set of involutions of $\mathrm{PSL}(2,11)$, with two involutions joined by an edge if they generate an S_3 in the class other than the one used to define Δ_3, is isomorphic to Δ_3.

Since the stabiliser in $\mathrm{PSL}(2,11)$ of an arc of Δ_3 is C_2 (Theorem 3.3) while the stabiliser of an arc of Δ_2 in M_{11} is S_3 (Theorem 2.6), it follows that if we spin the subgraph Δ_3 of Δ_2 under M_{11} we do not obtain a decomposition of Δ_3. Instead we obtain a cover of Δ_2 with each edge occurring in precisely 3 of the 12 subgraphs isomorphic to Δ_3, that is, we obtain a transitive 3-cover.

Inside K we can choose a subgroup $L \cong A_5$. Note that there are two conjugacy classes of such subgroups that are interchanged by $\mathrm{PGL}(2,11)$. Each such subgroup L contains 15 involutions and these form a single L-conjugacy class.

Construction 4.7. Let Δ_4 be the graph with vertex set the 15 involutions of A_5 such that two involutions are adjacent if and only if they generate an S_3.

Lemma 4.8. Δ_4 *is the line graph of the Petersen graph.*

Proof. When written in the usual permutation representation of A_5 on 5 points, the involutions of A_5 are of the form $(a,b)(c,d)$. Let τ be the map which takes each involution $(a,b)(c,d)$ in A_5 to the edge $\{\{a,b\},\{c,d\}\}$ of the Petersen graph. This is clearly a bijection. There are precisely four involutions of A_5 which generate an S_3 with $(a,b)(c,d)$. If e is the fifth point of the set then these involutions are $(a,b)(c,e)$, $(a,b)(d,e)$, $(a,e)(c,d)$ and $(b,e)(c,d)$. Under τ, these are mapped to the 4 edges incident with the edge $\{\{a,b\},\{c,d\}\}$ of the Petersen graph. Hence τ is an isomorphism. $\square$

Since the stabiliser in A_5 of an arc of Δ_4 is C_2 and this is the stabiliser in $\mathrm{PSL}(2,11)$ of an arc in Δ_3, it follows that Δ_3 decomposes into 11 copies of Δ_4 and these copies are transitively permuted by $\mathrm{PSL}(2,11)$.

5 A tower of partial linear spaces

We can construct a partial linear space from the graph Γ of Theorem 1.1.

Construction 5.1. Let $\tilde{\mathcal{S}}_2 = (V\Gamma, \mathcal{M}_2)$ be the incidence structure where $\mathcal{M}_2$ is the set of maximal cliques of Γ, and incidence is inclusion.

We call a point-line geometry a $(0,1)$-*geometry* if given a line L and a point p not on L, then p is collinear with 0 or 1 points of L.

Theorem 5.2. *The incidence structure $\tilde{\mathcal{S}}_2$ is a partial linear space with* 165 *points,* 220 *lines of size* 3*, and each point is on* 4 *lines. It is a* $(0,1)$*-geometry. Its full automorphism group is M_{11}, which acts flag-transitively on $\tilde{\mathcal{S}}_2$.*

Proof. Two collinear points of $\tilde{\mathcal{S}}_2$ are adjacent vertices in Γ. Since two adjacent vertices are contained in a unique maximal clique of Γ, by Theorem 2.6, we have that $\tilde{\mathcal{S}}_2$ is indeed a partial linear space. Also, by Theorem 2.6 the maximal cliques have size 3. Using the 4-subset definition of Γ, each 4-set contains four 3-subsets, and each 3-subset determines a maximal clique in Γ. Hence each point is on 4 lines and there are $\binom{12}{3} = 220$ lines.

A point p of $\tilde{\mathcal{S}}_2$ not on a line $L \in \mathcal{M}_2$ cannot be collinear with 2 or 3 points of L, hence it is a $(0,1)$-geometry. It follows from the definition of $\tilde{\mathcal{S}}_2$ that $\mathrm{Aut}(M_{11}) \leqslant \mathrm{Aut}(\tilde{\mathcal{S}}_2)$.

We can construct $\tilde{\mathcal{S}}_2$ from Γ by taking the cliques to be the lines. Also Γ is the collinearity graph of $\tilde{\mathcal{S}}_2$. Thus $\mathrm{Aut}(\tilde{\mathcal{S}}_2) = \mathrm{Aut}(\Gamma) = M_{11}$. The flag-transitivity is a consequence of the arc-transitivity of Γ. $\square$

We note that $\tilde{\mathcal{S}}_2$ is the partial linear space numbered 2.6 of [7].

We can make a similar construction with Π.

Construction 5.3. Let $\tilde{\mathcal{S}_3} = (V\Pi, \mathcal{M}_3)$ be the incidence structure where $\mathcal{M}_3$ is the set of maximal cliques of Π, and incidence is inclusion.

Theorem 5.4. *The incidence structure $\tilde{\mathcal{S}_3}$ is a partial linear space with 55 points, 55 lines of size 3 and each point is on 3 lines. Moreover $\tilde{\mathcal{S}_3}$ is self-dual, a $(0,1)$-geometry, and its full automorphism group is $\mathrm{PSL}(2,11)$, which acts flag-transitively.*

Proof. By Theorem 3.3, the maximal cliques of Π have size 3 and meet in at most one vertex. Hence $\tilde{\mathcal{S}_3}$ is indeed a partial linear space. Using the model Π_1 of Π, each 3-set contains three 2-subsets, and each 2-subset determines a line. Hence there are 3 lines on a point and $\binom{11}{2}$ lines. We proved in Lemma 3.9 that Π is isomorphic to its triangle graph, and triangles represent precisely the lines of $\tilde{\mathcal{S}_3}$, hence the self-duality.

A point p of $\tilde{\mathcal{S}_3}$ outside a line L cannot be collinear with 2 or 3 points of L, hence it is a $(0,1)$-geometry. It follows from the definition of $\tilde{\mathcal{S}_3}$ that $\mathrm{Aut}(\Pi) \leqslant \mathrm{Aut}(\tilde{\mathcal{S}_3})$. Also Π is the collinearity graph of $\tilde{\mathcal{S}_3}$. Thus $\mathrm{Aut}(\tilde{\mathcal{S}_3}) = \mathrm{Aut}(\Pi) = \mathrm{PSL}(2,11)$. The flag-transitivity is a consequence of the arc-transitivity of Π. $\square$

We note that $\tilde{\mathcal{S}_3}$ is the partial linear space numbered 6.1.2 of [2].

Since we have presented several definitions of Γ and Π, there are many equivalent definitions of the partial linear spaces $\tilde{\mathcal{S}_2}$ and $\tilde{\mathcal{S}_3}$. In fact, the involution definitions given in Section 4 lead to a natural definition of a tower of partial linear spaces.

Construction 5.5. Let $G_1 = M_{12}, G_2 = M_{11}, G_3 = \mathrm{PSL}(2,11)$ and $G_4 = A_5$. Let Y be a set of size 12 on which G_1 acts 5-transitively, G_2 is 3-transitive, $G_3 = (G_2)_\alpha$ and $G_4 = (G_2)_{\alpha,\beta}$ for distinct $\alpha, \beta \in Y$. Let $\mathcal{S}_1$ be the partial linear space with point set the conjugacy class $2B$ of involutions of G_1, and for $i = 2,3,4$, let $\mathcal{S}_i$ be the partial linear space with point set the unique conjugacy class of involutions of G_i. We define the line sets of each $\mathcal{S}_i$ as follows:

- For $i = 1$ let the line set be the 3-sets of involutions such that any two generate an S_3 which has 3 fixed points in the action of G_1 on Y.
- For $i = 2$ let the line set be the 3-sets of involutions such that any two generate an S_3 which has 3 fixed points in the action of G_2 on Y.
- For $i = 3$ let the line set be the 3-sets of involutions such that any two generate an S_3 which fixes two points in the action on $Y \backslash \{\alpha\}$.

- For $i = 4$ let the line set be the 3-sets of involutions such that any two generate an S_3.

Since any pair of involutions generating an S_3 uniquely determines a third involution which generates S_3 with either of the original pair the $\mathcal{S}_i$ are indeed partial linear spaces.

These four partial linear spaces are embedded in each other, in the sense that we take a subset of the point set and the induced lines. They provide the tower of partial linear spaces for Theorem 1.4.

Lemma 5.6. *For $i = 1, 2, 3, 4$, the partial linear space $\mathcal{S}_i$ has collinearity graph Δ_i. Moreover, $\mathcal{S}_2 \cong \tilde{\mathcal{S}}_2$ and $\mathcal{S}_3 \cong \tilde{\mathcal{S}}_3$.*

Proof. The first part follows from the definitions of the $\mathcal{S}_i$ and Δ_i. By Lemma 4.4, $\Delta_2 \cong \Gamma$ and by Lemma 4.6, $\Delta_3 \cong \Pi$ and so the second part holds. $\square$

The partial linear space $\mathcal{S}_1$ is different from the other three in the tower as it is not induced from the set of maximal cliques of Δ_1. However we have the following geometric construction.

Construction 5.7. Let X be a 12-set forming the point set of an $S(5, 6, 12)$ design. Let $\tilde{\mathcal{S}}_1$ be the partial linear space with points the 4-sets of X and with lines the triples $\{v_1, v_2, v_3\}$ such that $v_1 \cap v_2 \cap v_3$ is a 3-set and $v_1 \cup v_2 \cup v_3$ is a hexad.

Lemma 5.8. $\mathcal{S}_1 \cong \tilde{\mathcal{S}}_1$.

Proof. Take $\phi : 2B \longrightarrow \binom{X}{4}$ as in the proof of Lemma 4.2. It was proved there that ϕ is a bijection and so it remains to prove that ϕ maps the line set of $\mathcal{S}_1$ onto the line set of $\tilde{\mathcal{S}}_1$.

Let $\{g_1, g_2, g_3\}$ be a line of $\mathcal{S}_1$. Then all the g_i lie in a subgroup S_3 fixing a 3-set T, so $\phi(g_1) \cap \phi(g_2) \cap \phi(g_3)$ is T. Let $\{a_i\} = \phi(g_i) \setminus T$ for $i = 1, 2, 3$. We claim that g_i fixes a_i and interchanges a_j and a_k whenever $\{i, j, k\} = \{1, 2, 3\}$. Since $g_k \in \langle g_i, g_j \rangle \cong S_3$, $g_k = g_i g_j g_i$, so we have $a_j^{g_i} = a_j^{g_j g_i} = a_j^{g_i g_k}$. Thus $a_j^{g_i}$ is fixed by g_k but is not in T, so it is a_k. By symmetry, $a_k^{g_i} = a_j$, so the claim is proved. Hence g_i fixes setwise the 5-set $T \cup \{a_j, a_k\}$, and so g_i must fix also the sixth point of the hexad containing that 5-set. Since a_i is the only other point fixed by g_i it follows that $T \cup \{a_1, a_2, a_3\}$ is the hexad. Since $T \cup \{a_1, a_2, a_3\} = \phi(g_1) \cup \phi(g_2) \cup \phi(g_3)$ we have $\phi(\{g_1, g_2, g_3\})$ is a line of $\tilde{\mathcal{S}}_1$. The number of lines of $\tilde{\mathcal{S}}_1$ is equal to the number of hexads times the number of 3-sets in a hexad. Hence there are $132.20 = 2640$ lines. The number of lines of $\mathcal{S}_1$ is the number of 3-subsets of a 12-set times the number of S_3 subgroups fixing a given 3 points. Thus there are $220.12 = 2640$ lines in $\mathcal{S}_1$ and so ϕ maps the set of lines of $\mathcal{S}_1$ onto the set of lines of $\tilde{\mathcal{S}}_1$. $\square$

We now determine the properties of this partial linear space.

Theorem 5.9. *The incidence structure* $\tilde{\mathcal{S}_1}$ *is a partial linear space with* 495 *points,* 2640 *lines of size* 3 *and* 16 *lines on a point. Its full automorphism group is* M_{12}*, which acts flag-transitively on* $\tilde{\mathcal{S}_1}$*.*

Proof. Each 4-set is contained in 4 hexads and contains 4 subsets of size 3. Hence each point is on 16 lines. The number of lines was computed in the proof of Lemma 5.8.

We can construct $\tilde{\mathcal{S}_1}$ from $S(5,6,12)$ as explained above, so $M_{12} \leqslant \mathrm{Aut}(\tilde{\mathcal{S}_1})$. Moreover, the the collinearity graph of $\tilde{\mathcal{S}_1}$ is $\Delta_1 \cong J(12,4)$ and so $\mathrm{Aut}(\tilde{\mathcal{S}_1}) \leqslant \mathrm{Aut}(J(12,4)) = S_{12}$. Since automorphisms map lines to lines, and lines correspond to a set of three 4-sets whose union is a hexad, it follows that automorphisms of the partial linear space are permutations of S_{12} which map hexads to hexads. Hence $\mathrm{Aut}(\tilde{\mathcal{S}_1}) = M_{12}$. The flag-transitivity is a consequence of the 5-transitivity of M_{12}. □

It remains to determine the properties of $\mathcal{S}_4$.

Theorem 5.10. *The incidence structure* $\mathcal{S}_4$ *is a partial linear space with* 15 *points,* 10 *lines of size* 3*, and each point is on* 2 *lines. It is a* $(0,1)$*-geometry. Its full automorphism group is* S_5*, which acts flag-transitively on* $\mathcal{S}_4$*.*

Proof. The numbers come from easy counting arguments. Since the collinearity graph is the line graph of the Petersen graph, which has automorphism group S_5, it follows that $\mathcal{S}_4$ has automorphism group S_5. Flag-transitivity then follows by examining the Petersen graph. □

References

[1] F. Buekenhout, La geometrie des groupes de Fischer, unpublished notes, Free University of Brussels (1974).

[2] F. Buekenhout, P. Cara, K. Vanmeerbeek, Geometries of the group PSL(2, 11). Geometriae Dedicata 83 (2000) 169–206.

[3] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, J. Symb. Comp. 24 3/4 (1997) 235-265. Also see the MAGMA home page at http://www.maths.usyd.edu.au:8000/u/magma/.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, Atlas of finite groups, Clarendon Press, Oxford, 1985.

[5] J. H. Conway and N. J. A. Sloane, Sphere packings, lattices and groups. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov, Grundlehren der Mathematischen Wissenschaften 290, Springer, New York, NY, 1999.

[6] H. Cuypers and J. I. Hall, The 3-transposition groups with trivial centre, J. Algebra 178 (1995) 149–193.

[7] M. Dehon, D. Leemans and X. Miller, The residually weakly primitive and $(IP)_2$ geometries of M_{11}, unpublished. Available at http://cso.ulb.ac.be/˜dleemans/abstracts/m11rwpri.html

[8] A. Devillers, M. Giudici, C. H. Li and C. E. Praeger, Primitive decompositions of Johnson graphs, in preparation.

[9] J. D. Dixon and B. Mortimer, Permutation Groups, Graduate Texts in Mathematics 163, Springer, New York, 1996.

[10] S. F. Du and M. Y. Xu, Vertex-primitive $\frac{1}{2}$-arc-transitive graphs of smallest order. Comm. Algebra 27 (1999), 163–171.

[11] B. Fischer, Finite groups generated by 3-transpositions. I. Invent. Math. 13 (1971), 232–246.

[12] D. R. Hughes, On t-designs and groups, Amer. J. Math. 87 (1965), 761–778.

[13] A. A. Ivanov, Geometry of sporadic groups I: Petersen and tilde geometries, Cambridge University Press, Cambridge, 1999.

[14] M. W. Liebeck, C. E. Praeger and J. Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups. J. Algebra 111 (1987), 365–383.