A classification of finite partial linear spaces with a primitive rank 3 automorphism group of grid type

Alice Devillers[*]

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Abstract

A *partial linear space* is a non-empty set of *points*, provided with a collection of subsets called *lines* such that any pair of points is contained in at most one line and every line contains at least two points. Graphs and linear spaces are particular cases of partial linear spaces. A partial linear space which is neither a graph nor a linear space is called proper. The aim of this paper is to classify the finite proper partial linear spaces with a primitive rank 3 automorphism group of grid type.

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1 Introduction

A *partial linear space* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a non-empty set $\mathcal{P}$ of *points*, provided with a collection $\mathcal{L}$ of subsets of $\mathcal{P}$ called *lines* such that any pair of points is contained in at most one line and every line contains at least two points. Partial linear spaces are a common generalization of graphs (where all lines have exactly two points) and of linear spaces (where any pair of points is contained in exactly one line). A partial linear space that is neither a graph nor a linear space will be called *proper*.

If a proper partial linear space admits a rank 3 group G as an automorphism group, then G is transitive on the ordered pairs of collinear points, as well as on the ordered pairs of non-collinear points. The non-trivial finite graphs with this property are exactly the rank 3 graphs, whose classification follows from the classification of finite primitive rank 3 groups. Our aim is to classify the finite proper partial linear spaces with this property. In [2], we have given a complete classification of all finite proper partial linear spaces with a primitive rank 3 automorphism group of almost simple type. In this paper we will classify the finite proper partial linear spaces with a primitive rank 3 automorphism group of grid type.

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[*] Chargé de Recherche FNRS

From now on, we will always consider proper partial linear spaces admitting a rank 3 automorphism group.

Let us recall some definitions. A *primitive* permutation group G acting on a set Ω is a transitive group admitting no non-trivial G-invariant equivalence relation on Ω. An *almost simple group* is a group G containing a non-abelian simple subgroup S such that $S \trianglelefteq G \leq Aut(S)$. A group of *grid type* is a permutation group G acting on n^2 points such that $S \times S \trianglelefteq G \leq S_0 wr 2$, where S_0 is 2-transitive on n points and almost simple with $S \trianglelefteq S_0 \leq Aut(S)$.

All finite primitive rank 3 groups have been classified (using the classification of finite simple groups). There are three types of such groups: the almost simple groups, the grid type groups, and the affine groups. This paper takes care of the second family. The classification of finite primitive rank 3 groups of grid type follows from the classification of finite 2-transitive almost simple groups (see [1, 3]).

We will prove (the notations will be explained below and in Section 3):

Theorem 1.1. *A finite proper partial linear space admitting a rank 3 automorphism group of grid type is one of the following:*
(i) $Grid(n)$ $(n \geq 5)$,
(ii) $Grid(PG(d,q))$ $(d \geq 2)$
(iii) $Grid(\overline{U(3,q)})$ $(q \geq 3)$,
(iv) $Grid(R(q))$ $(q = 3^{2e+1})$,
(v) $\overline{Grid}(PG(d,2))$,
(vi) $\overline{Grid}(R(3))_{Alt(4)}$,
(vii) $\overline{Grid}(R(3))_{Sym(4)}$.

2 Preliminary definitions and results

Let S be a finite proper partial linear space with a rank 3 automorphism group G. Since G is transitive on points, the number of lines through a point is a constant; since G is transitive on lines, all lines have the same size.

We will use Kantor's classification of finite non trivial linear spaces admitting a 2-transitive automorphism group [3] (non trivial means at least two lines and at least 3 points per line):

Theorem 2.1 (Kantor 1985). *Any finite non trivial linear space admitting a 2-transitive automorphism group is one of the following:*
(i) $PG(d,q)$,
(ii) $AG(d,q)$,
(iii) a unital with q^3+1 *points and lines of size* $q+1$ *associated with* $PSU(3,q)$ *or* ${}^2G_2(q)$,
(iv) one of two affine planes, having 3^4 *or* 3^6 *points,*
(v) one of two linear spaces with 3^6 *points and lines of size* 9.

Moreover, in cases (i) and (iii), the automorphism group is almost simple, while it is affine in the other cases.

We now recall the definitions of some well-known families of partial linear spaces.

A *generalized quadrangle* $GQ(s,t)$ of order (s,t) is a partial geometry with parameters $(s,t,1)$; it is also a particular type of polar space.

The $n \times n$ *grid*, denoted by $Grid(n)$, is the unique $GQ(n-1,1)$. We will denote its points by ordered pairs (a,b) (where a,b are elements of a given n-set Ω), two points (a,b) and (c,d) being collinear in the grid if and only if $a=b$ or $c=d$.

The projective space of dimension d over the field of q elements is denoted by $PG(d,q)$.

A *unital* is a linear space on q^3+1 points with lines of size $q+1$. We will describe two well-known classes of unitals. The hermitian unital $\overline{U(3,q)}$ is the set of absolute points and non-absolute lines of a unitary polarity in $PG(2,q^2)$. The Ree unital $R(q)$ (where $q=3^{2e+1}$) is the unital admitting the Ree group ${}^2G_2(q)$ as an automorphism group.

3 Results

A group of *grid type* is a permutation group G acting on $\Omega \times \Omega$ (Ω is a finite set of size n) such that $S \times S \trianglelefteq G \leq S_0 wr 2$, where S_0 is 2-transitive on Ω and almost simple with $S \trianglelefteq S_0 \leq Aut(S)$. We can visualize G as acting on the $n \times n$ grid. Note that there is no 2-transitive almost simple group of degree 4 or less, so we may assume $n \geq 5$. The finite 2-transitive almost simple groups have been classified [1, 3]. In every case (except for $S_0 = P\Gamma L(2,8)$ in its action on 28 points), the simple group S is 2-transitive.

In the exceptional case, we have $PSL(2,8) \times PSL(2,8) \trianglelefteq G \leq P\Gamma L(2,8) wr 2$. Since G has rank 3, the stabilizer of a point in G has an orbit of size 3^6, the size of a Sylow 3-subgroup of $P\Gamma L(2,8) wr 2$. Hence G must contain a Sylow 3-subgroup of $P\Gamma L(2,8) wr 2$. Therefore G must contain $P\Gamma L(2,8) \times P\Gamma L(2,8)$, but it must also contain an automorphism of the grid switching the two families of parallel lines. It follows that the only rank 3 group is $P\Gamma L(2,8) wr 2$ itself.

Our aim is to find all the proper partial linear spaces admitting a rank 3 primitive group G of grid type as an automorphism group. Let $\mathcal{S}$ be such a space, containing n^2 points. Then, by the above arguments, $H \times H \leq G$ where $H = S$ (a 2-transitive simple group of degree n) or $H = P\Gamma L(2,8)$ and $n = 28$, hence H is 2-transitive on Ω.

Suppose that the collinearity relation in $\mathcal{S}$ is the same as the collinearity relation in the grid, that is, suppose that (a,b) and (c,d) are collinear if and only if $a=b$ or $c=d$.

Of course $Grid(n)$ admits a rank 3 automorphism group, for instance $Sym(n) wr 2$.

Let $\mathbf{L}$ be a line of $\mathcal{S}$, containing two points x and y on a line l of $Grid(n)$. Since all the other points in $\mathbf{L}$ must be on a line of the grid together with both x and y, they are all on the line l. Assume $\mathbf{L} \neq l$, so that we are not in the case of $Grid(n)$. The stabilizer of l in G contains H, which acts 2-transitively on l, and the images of $\mathbf{L}$ under H must either intersect $\mathbf{L}$ in at most one point or coincide with $\mathbf{L}$, and so the points of l together with $\mathbf{L}^H$ form a linear space $LS(l)$, on which H acts 2-transitively. By Theorem 2.1, we know that $LS(l)$ is one of the following: $PG(d,q)$ $(d \geq 2)$ with $H = PSL(d+1,q)$ or $H = Alt(7)$ if $(d,q) = (3,2)$, a hermitian unital $\overline{U(3,q)}$ for $H = PSU(3,q)$ $(q \geq 3)$, or a Ree unital $R(q)$ for $H = {}^2G_2(q)$ if $q = 3^{2e+1} > 3$ or for $H = P\Gamma L(2,8) \simeq {}^2G_2(3)$ if $q = 3$ (in this exceptional case, the socle $PSL(2,8) \simeq {}^2G_2(3)'$ is not 2-transitive). If H is one of these groups and $\mathbf{L}$ is a line of the linear space $LS(l)$, it is easy to see that $\mathbf{L}^G$ is actually the line-set of a partial linear space. Since G has rank 3 on $\Omega \times \Omega$, this partial linear space has the required property and will be denoted by $Grid(\mathrm{LS(l)})$.

This ends the proof of the part "collinearity as in $Grid(n)$" of Theorem 1.1, giving the examples (i) to (iv).

Assume now that the collinearity relation in $\mathcal{S}$ is the same as the non-collinearity relation in $Grid(n)$, that is (a,b) and (c,d) are collinear if and only if $a \neq b$ and $c \neq d$.

Let us define two maps π_i $(i = 1,2)$, called *projections*, from $\Omega \times \Omega$ to Ω, by $\pi_i(x_1,x_2) = x_i$.

Lemma 3.1. *If $\mathcal{S}$ is a partial linear space admitting a rank 3 automorphism group of grid type, and whose collinearity relation is opposite to the one in $Grid(n)$, then the images of the lines of $\mathcal{S}$ by the projection π_i $(i = 1,2)$ form the line-set of a linear space (with point-set Ω).*

Proof. We prove it for $i = 1$, the proof being the same for $i = 2$. Suppose that the images of two lines $\mathbf{L_1}$ and $\mathbf{L_2}$ of $\mathcal{S}$ by π_1 are distinct sets intersecting in at least two points a and b of Ω. This means that $(a,x_1),(b,y_1) \in \mathbf{L_1}$, $(a,x_2),(b,y_2) \in \mathbf{L_2}$ and there exists a point (c,z_1) in $\mathbf{L_1}$ such that there is no point whose first coordinate is c in $\mathbf{L_2}$. Since H acts 2-transitively on Ω, there is an element $h \in H$ mapping x_1 onto x_2 and y_1 onto y_2. The element $Id \times h \in H \times H \leq G$ maps (a,x_1) onto (a,x_2) and (b,y_1) onto (b,y_2), hence it maps $\mathbf{L_1}$ onto $\mathbf{L_2}$. But it also maps (c,z_1) onto (c,z_1^h), contradicting the assumption that there is no point with first coordinate c in $\mathbf{L_2}$. This proves that two distinct images of lines of $\mathcal{S}$ by π_1 intersect in at most one point. Furthermore, any two points x,y of Ω are contained in the image of a line of $\mathcal{S}$ by π_1 (take for instance the line through (x,x) and (y,y)). This ends the proof of the Lemma. □

Let $\mathcal{S}_{\pi_i}$ be the two linear spaces with point-set Ω constructed in Lemma 3.1. Let g be an element of G fixing a point (ω,ω) and switching the two directions of parallel lines (g exists since G has rank 3). If we interpret the image of π_1 as $\Omega \times \{\omega\}$ and the image of π_2 as $\{\omega\} \times \Omega$, it is easy to see that g yields an isomorphism between $\mathcal{S}_{\pi_1}$ and $\mathcal{S}_{\pi_2}$.

Since $H \times H$ preserves the line-set of $\mathcal{S}$, H is an automorphism group of $\mathcal{S}_{\pi_1}$ acting 2-transitively on it. Using again Kantor's theorem 2.1, $\mathcal{S}_{\pi_1}$ must be a single line, $PG(d,q)$, $\overline{U(3,q)}$ or $R(q)$.

Let $\mathbf{L}$ be a line of $\mathcal{S}$. Suppose that $|\mathbf{L}| > 3$ and that some element $h \in H$ fixes two points a and b of $\mathbf{L}^{\pi_1}$ (hence fixes $\mathbf{L}^{\pi_1}$) but maps $c \in \mathbf{L}^{\pi_1}$ onto $d \in \mathbf{L}^{\pi_1}$, with $c \neq d$. Then $h \times Id \in H \times H \leq G$ fixes the points of $\mathbf{L}$ projecting on a and b (hence fixes $\mathbf{L}$) and maps the point projecting onto c onto a point collinear with it in $Grid(n)$, a contradiction. This shows that the stabilizer of two points in H must fix pointwise the whole line of $\mathcal{S}_{\pi_1}$ through them.

Since there is no sharply 2-transitive group in the list of finite 2-transitive almost simple groups, this rules out the possibility for $\mathcal{S}_{\pi_1}$ to be a single line. If $\mathcal{S}_{\pi_1} = PG(d,q)$, we must have $q = 2$. The case $\mathcal{S}_{\pi_1} = \overline{U(3,q)}$ is also ruled out since $q \geq 3$ and the stabilizer of a line in $H = PSU(3,q)$ acts as $PGL(2,q)$ on this line. If $\mathcal{S}_{\pi_1} = R(q)$, the stabilizer of two points in $H = {}^2G_2(q)$ is cyclic with no fixed point, except if $q = 3$ where it fixes the other two points of the line through them.

Thus we have just two cases to examine: $\mathcal{S}_{\pi_1} = PG(d,2)$ or $R(3)$.

If $\mathcal{S}_{\pi_1} = PG(d,2)$, then $H = PSL(d+1,2)$ or $Alt(7)$ if $d = 3$. There is only one possibility for the third point of a line containing (a_1,a_2) and (b_1,b_2) ($a_1 \neq b_1$ and $a_2 \neq b_2$), namely (c_1,c_2) where c_i is the third point of the line a_ib_i in $\mathcal{S}_{\pi_i}$ $(i = 1,2)$. This gives an example of a partial linear space admitting a rank 3 automorphism group of grid type. We will denote this space by $\overline{Grid}(PG(d,2))$.

If $\mathcal{S}_{\pi_1} = R(3)$, then $H = P\Gamma L(2,8)$ and, as proved above, G can only be $P\Gamma L(2,8)wr2$. There are two possibilities: the two remaining points of the line containing (a_1,a_2) and (b_1,b_2) ($a_1 \neq b_1$ and $a_2 \neq b_2$) can be either (c_1,c_2) and (d_1,d_2), or (c_1,d_2) and (d_1,c_2), where c_i and d_i are the remaining points of the line a_ib_i in $\mathcal{S}_{\pi_i}$ $(i = 1,2)$. Hence the line-set of $\mathcal{S}$ is either $\{(a,a),(b,b),(c,c),(d,d)\}^G$ or $\{(a,a),(b,b),(c,d),(d,c)\}^G$, where $\{a,b,c,d\}$ is a line of $R(3)$. Note that in the first case, the stabilizer of a line in G acts on this line as the stabilizer of a line of $R(3)$ in H, that is as $Alt(4)$. In the second case, the switching involution of G induces an odd permutation on the line, and so the stabilizer of a line acts on this line as $Sym(4)$. These give two new examples, which will be denoted by $\overline{Grid}(R(3))_{Alt(4)}$ and $\overline{Grid}(R(3))_{Sym(4)}$ respectively.

This ends the proof of the part "collinearity opposite to the one in the grid" of Theorem 1.1, giving the examples (v) to (vii).

References

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