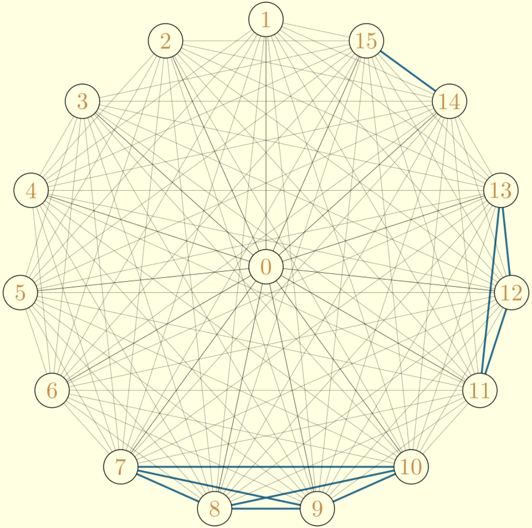

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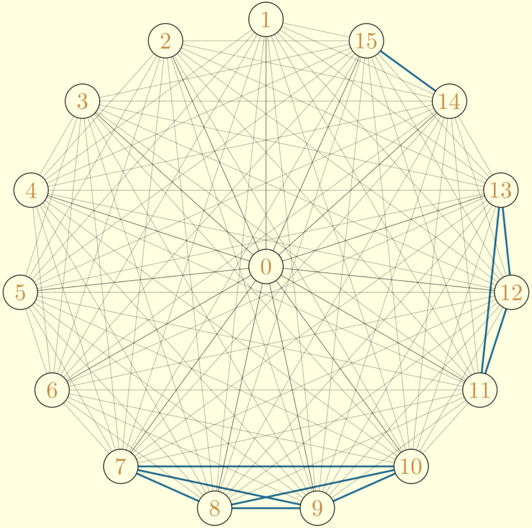
A sandpile model for the shuffle theorem



Michele D'Adderio | Mark Dukes | Alessandro Iraci | Alexander Lazar | Yvan Le Borgne | Anna Vanden Wyngaerd

In this talk

In this talk

A new combinatorial model for the symmetric function ∇e_n

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$\mathbb{K} = \mathbb{Q}(q,t)$

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Schur positivity:

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* Haiman 2002

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Δ is a linear operator on $\Lambda_{\mathbb{Q}(q,t)}$ with as eigenvectors the Macdonald basis

This result is closely related to the Macdonald positivity theorem (See Garsia-Haiman 1993 and Haiman 2001)

* Haiman 2002 $\quad\Delta$ Bergeron-Garsia 1999

Shuffle theorem

The shuffle theorem* is an explicit formula for ∇e_n as a generating function of labelled Dyck paths

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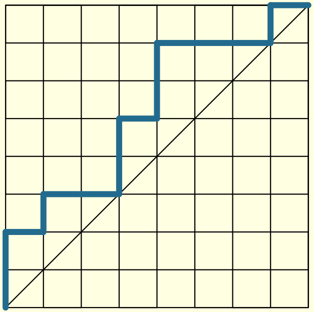

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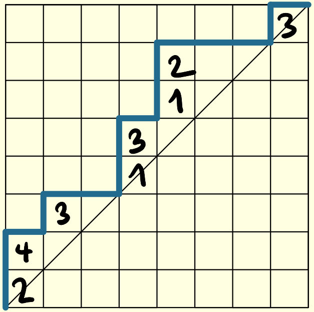

A Dyck path of size 8

Positive integer labels on vertical steps,
must be strictly increasing ↑

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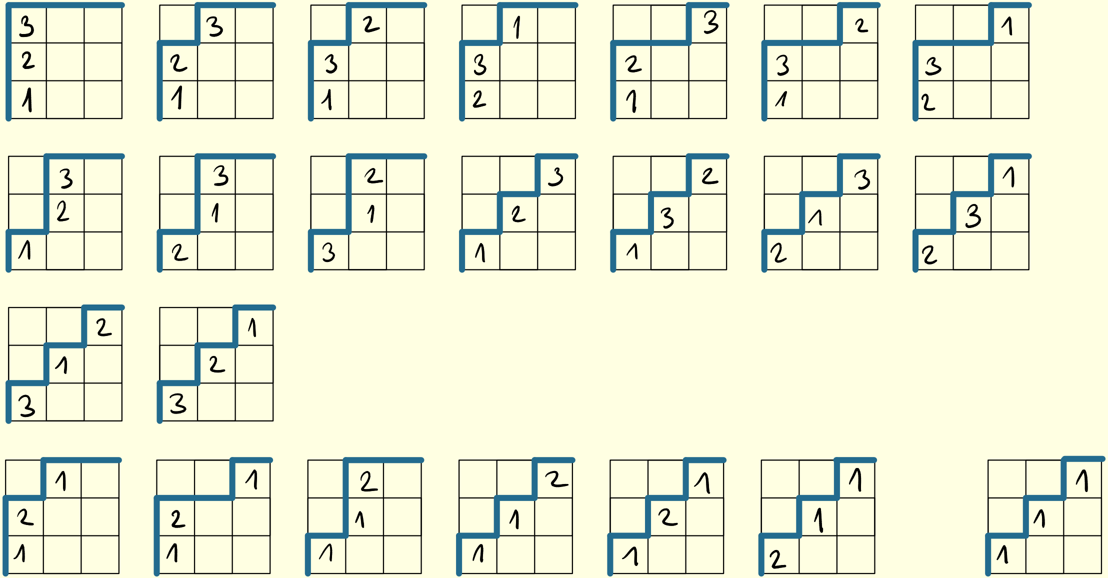

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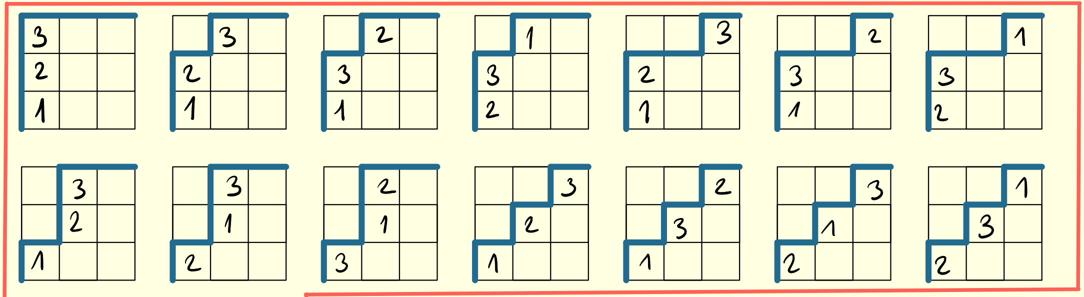
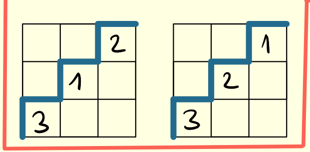
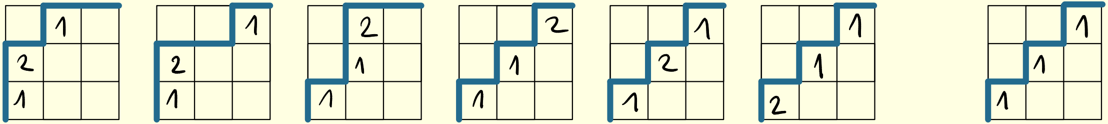




labels encode the multiplicities in the x-monomial



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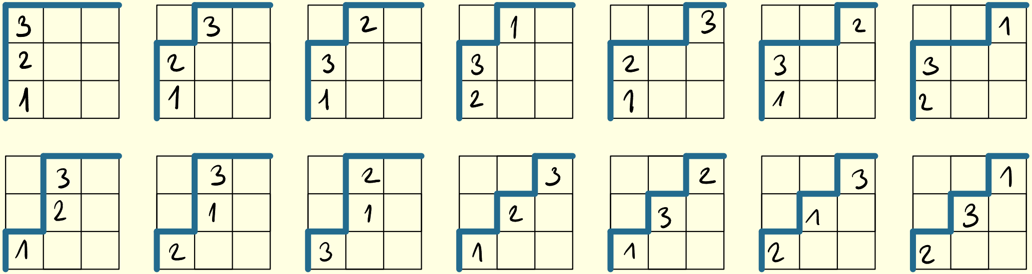

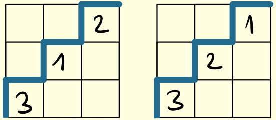

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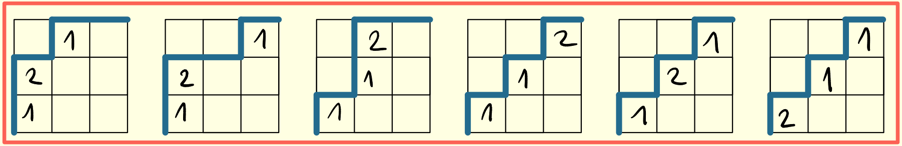

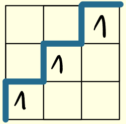

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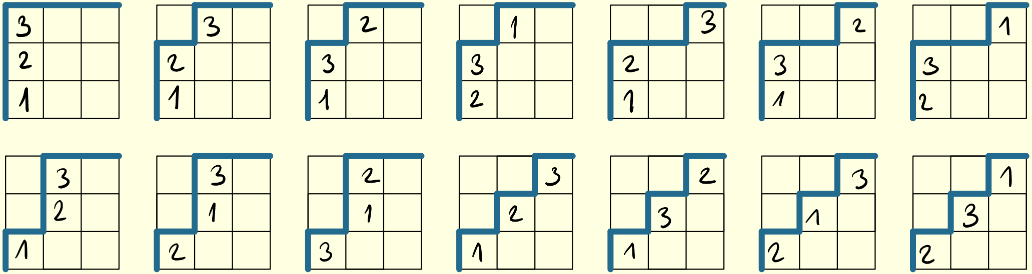

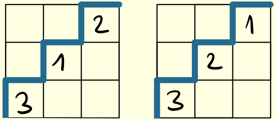

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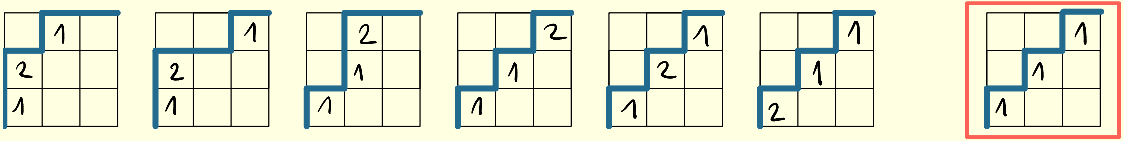

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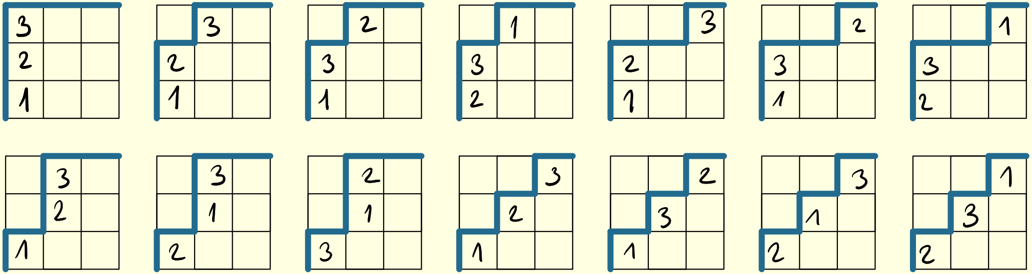

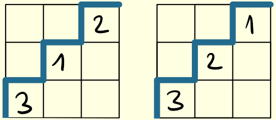

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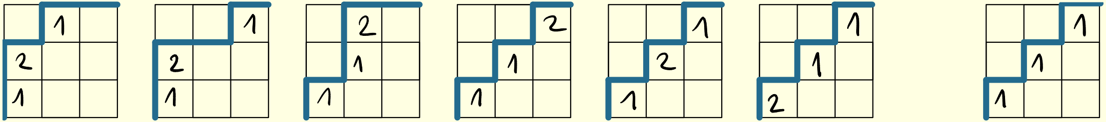

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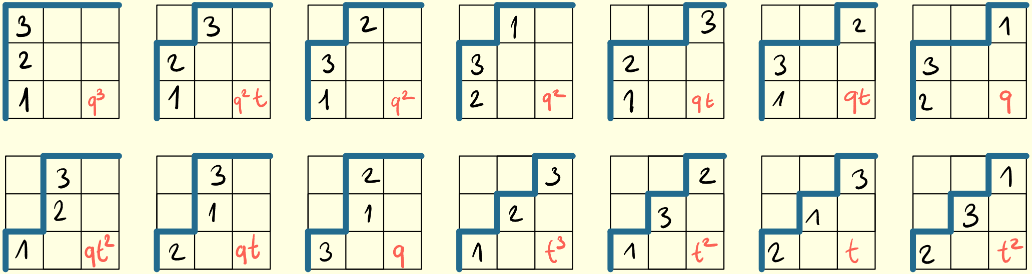

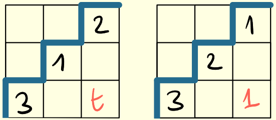

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* Loehr - Remmel 2004

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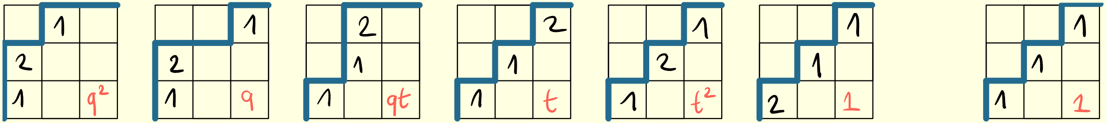

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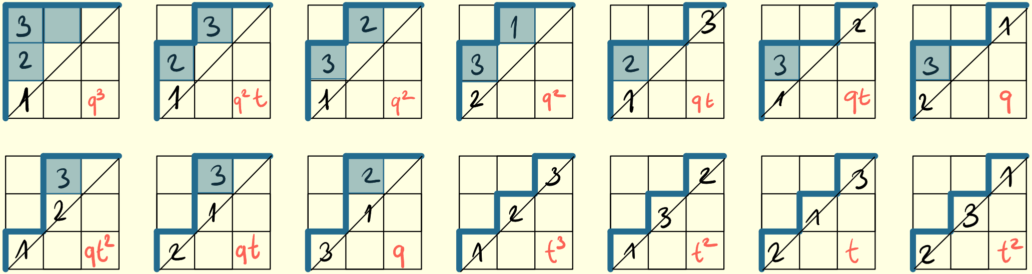

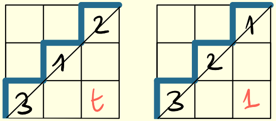

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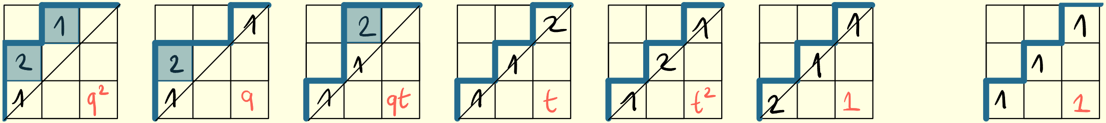
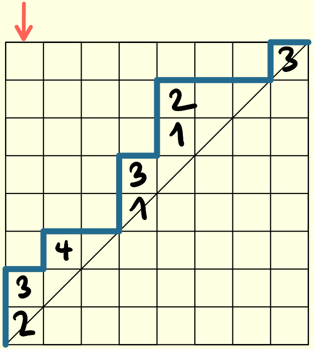


Pmaj



Pool : $\{3, 2\}$

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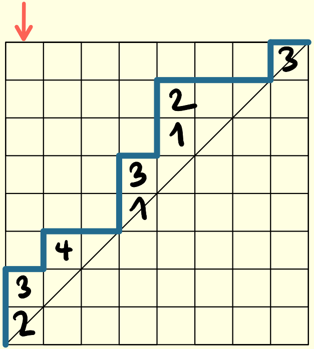

Pool : {3, 2}

Word : 3

Create a word by picking the max. value of the pool that is smaller than the previous letter

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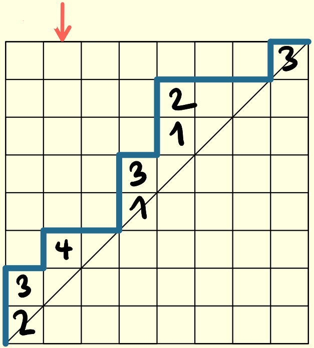

Pool : $\{2, 4\}$

Word : 3

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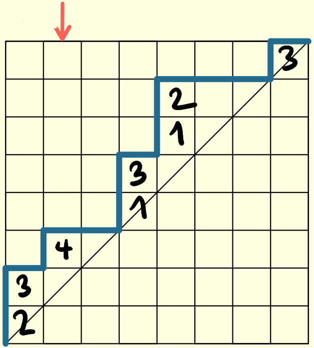

Pool : {2, 4}

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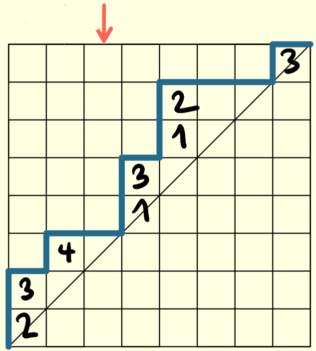

Pool : $\{4\}$

Word : 3 2

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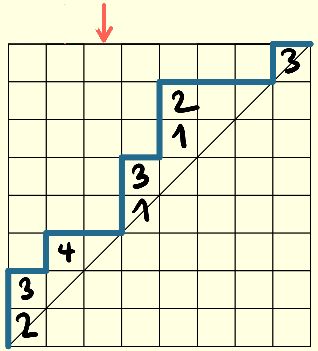

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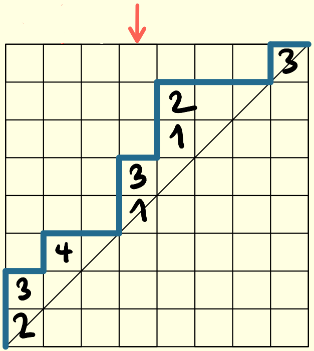

Pool : {3, 1}

Word : 3 2 4 3

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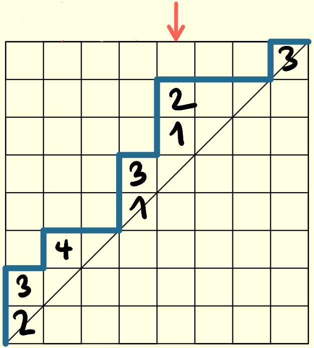

Pool : {1, 2, 1}

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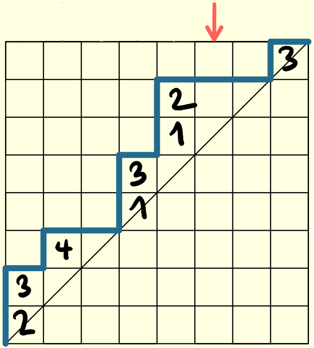

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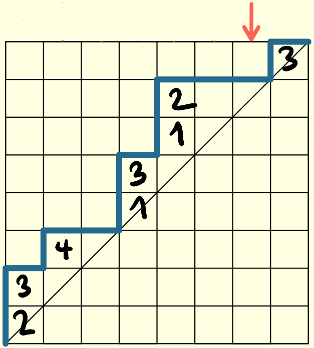

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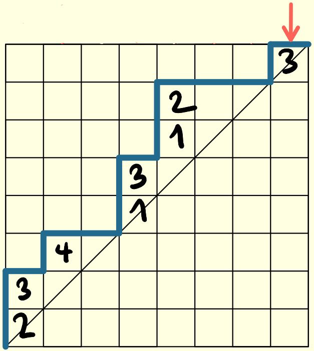

Pool : {3}

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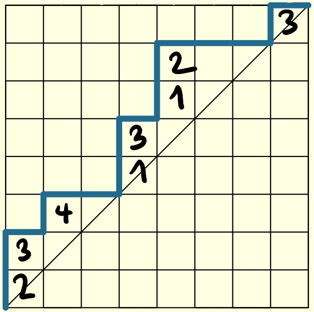

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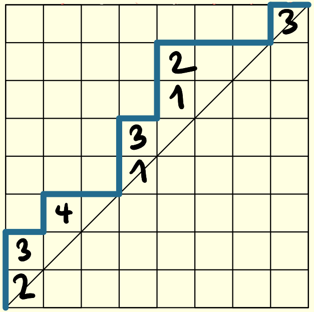

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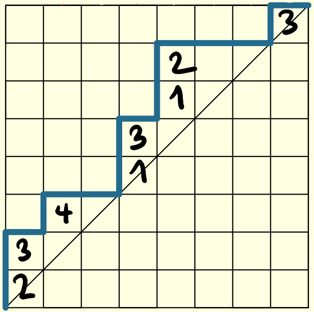

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1 2 3 4 5 6 7 8

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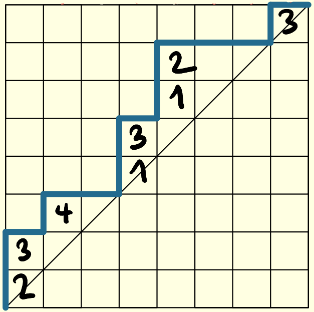

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$$\underline{3}\ 1\ 1\ 2\ 3\ \underline{4}\ 2\ 3 \longrightarrow \text{major index } 1 + 6 = 7$$

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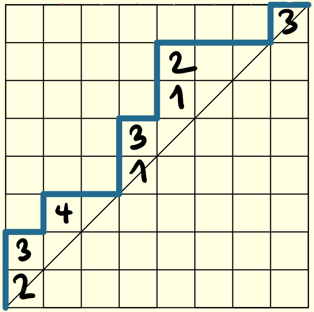
Pmaj



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$$\underset{1}{\underline{3}}\ \underset{2}{1}\ \underset{3}{1}\ \underset{4}{2}\ \underset{5}{3}\ \underset{6}{\underline{4}}\ \underset{7}{2}\ \underset{8}{3} \longrightarrow \text{major index } 1 + 6 = \boxed{7}$$

pmaj of the path

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A new combinatorial formula for ∇e_n, using the abelian sandpile model

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This scalar product, in the shuffle theorem formula, can be interpreted as "selecting" certain labelled paths, whose labelling satisfies a μ,ν-shuffle condition

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↳ statistical mechanics.

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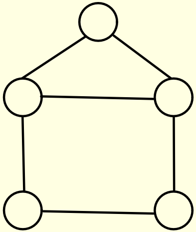
Given a (finite simple) graph



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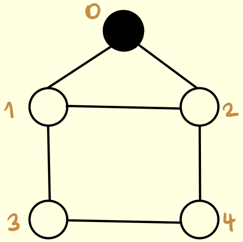

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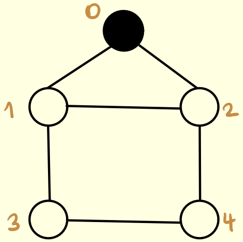

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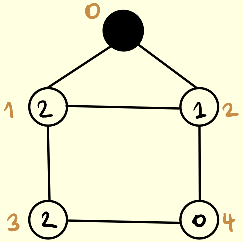

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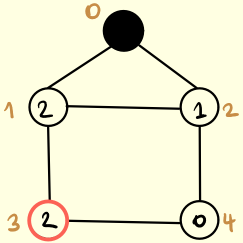

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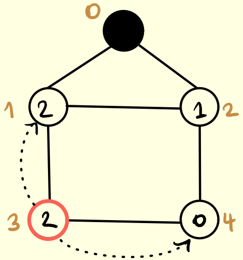

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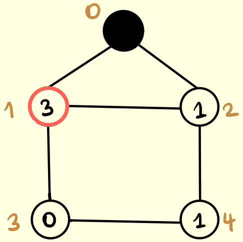

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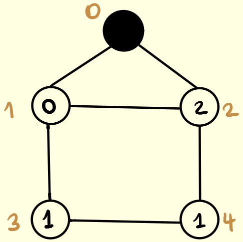

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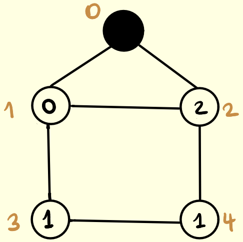

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Abelian : the stabilization is unique

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We need the following

PROP Take c a configuration on a graph G and d the configuration obtained from c by "firing the sink".

Then c is recurrent $\Leftrightarrow$ d stabilizes to c

The plan

For all partitions μ, ν we will define a graph $G_{\mu,\nu}$

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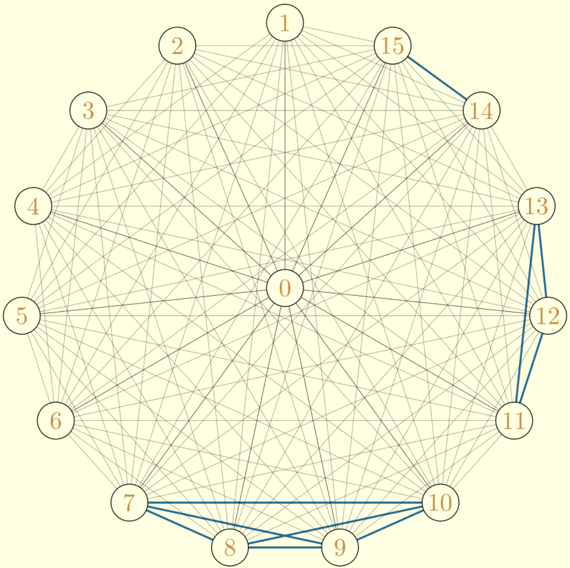
THE PROOF will be an explicit bijection between $\mathrm{Sortrec}(G_{\mu,\nu})$ and labelled Dyck paths, sending (level, delay) to (area, pmaj)


1
15
2
14
3
13
4
0
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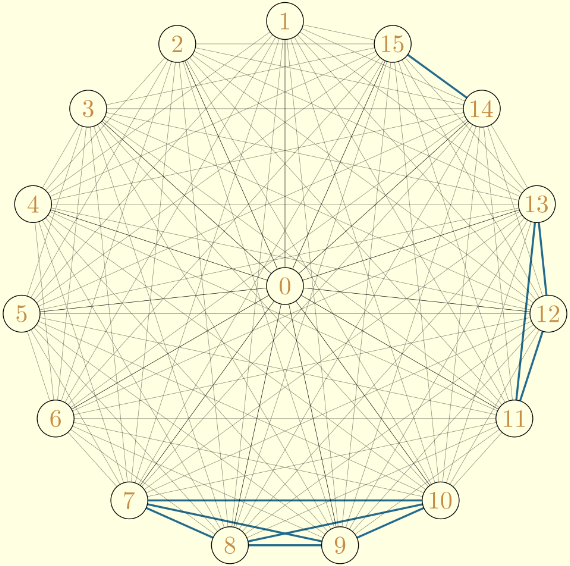
7
9
8

Sorted recurrents

$G_{(3,2,1),(4,3,2)} =$



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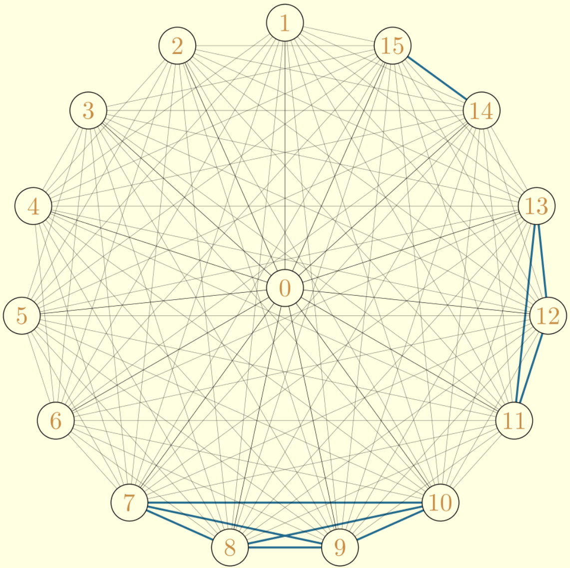

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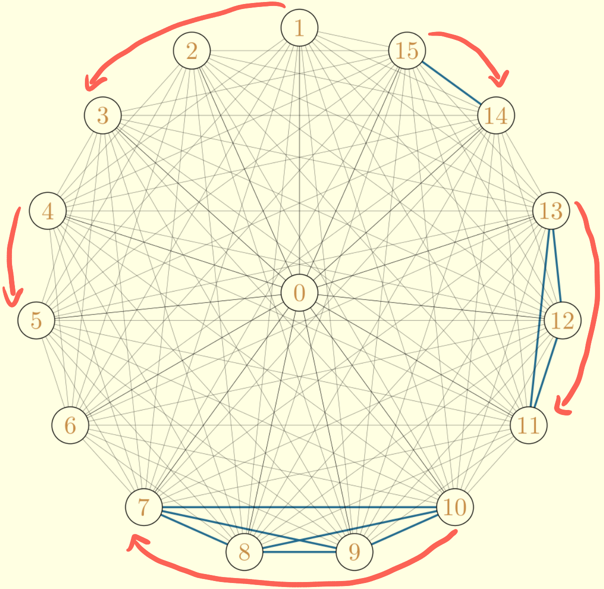

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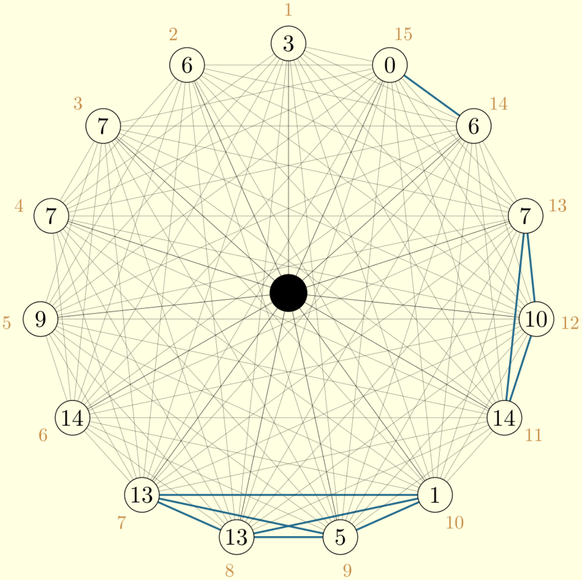
Only consider configs
- decreasing in cliques
- increasing in anti-cliques

The bijection



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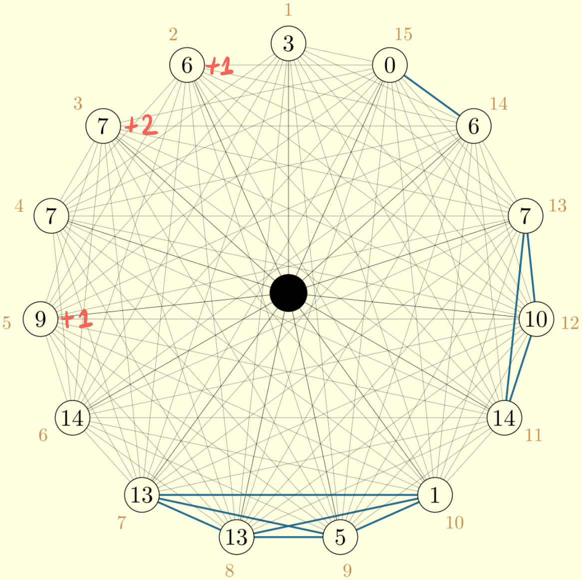

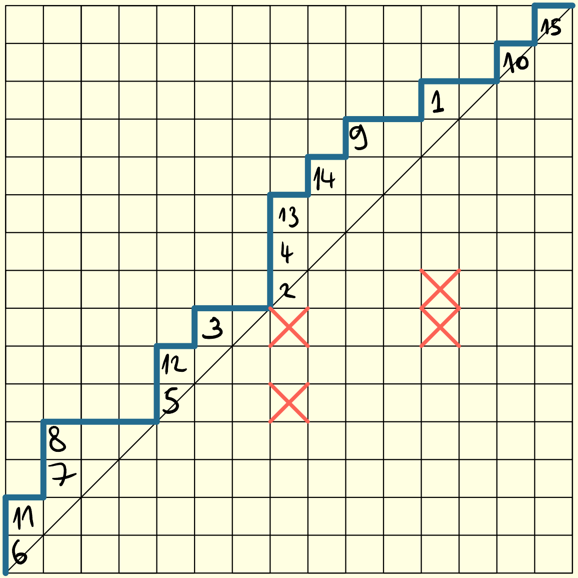

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Will this new combinatorial model shed some light?

Thanks for listening