Delta conjectures and Theta refinements

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19 november 2020

ULB

Guide to this presentation

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Original work is joint with Michele D'Adderio and/or Alessandro Iraci.

My research field in one sentence

Studying combinatorial formulas for interesting symmetric functions.

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Set of symmetric functions $\Lambda_{\mathbb{K}}$

Ingredients:

- a (countably) infinite number of variables: $x_1, x_2, x_3, \ldots$

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Examples

$$2x_1 + 2x_2 + 2x_3 + \cdots$$

$$\frac{1}{2}x_1x_2 + \frac{1}{2}x_1x_3 + \frac{1}{2}x_2x_3 + \cdots$$

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Non-examples

$$x_1 + 2x_2 + 3x_3 + \cdots$$
$$\frac{1}{2}x_1x_2 + \frac{1}{2}x_1x_3 + \frac{1}{2}x_2x_3$$

Symmetric functions

The "numbers" in our symmetric function space will be $\mathbb{K} = \mathbb{Q}(q, t)$: anything we can obtain from natural numbers, q and t, using $+, \times, -, \div$

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Example of an element in $\mathbb{Q}(q, t)$

$$\frac{\frac{2}{3}q^3t^2 - q^2t + \frac{1}{3}t}{q^4 - q^2t^2 - t + 4}$$

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Example of an element in $\Lambda_{\mathbb{Q}(q,t)}$

$$\frac{q}{t}x_1 + \frac{q}{t}x_2 + \frac{q}{t}x_3 + \cdots$$

Basis of the symmetric function ring

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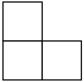
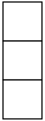
Example The partitions of 3 are (3), $(2,1)$ and $(1,1,1)$

(3)

$(2,1)$



$(1,1,1)$



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- elementary symmetric functions

$$e_{(2,1)} = e_2 \cdot e_1 = (x_1x_2 + x_1x_3 + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$$

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- Schur symmetric functions

$$s_{(2,1)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + \cdots$$

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & 3 \\ \hline\end{array}$$

Interesting symmetric functions

In one sentence

Symmetric functions made up of q, t-counting are interesting.

Counting

Let's play "stupid scrabble" with an incomplete set (only 4 first letters of the alphabet)

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A_1

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- "Words" with 1 letter:

A_1 B_3

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- "Words" with 1 letter:

A_1 B_3 C_3

Counting

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- "Words" with 1 letter:

A_1 B_3 C_3 D_2

$\rightarrow$ 4 possible words

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$A_1 A_1$ $A_1 B_3$ $A_1 C_3$ $A_1 D_2$

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- "Words" with 2 letters:

$A_1 A_1$ $A_1 B_3$ $A_1 C_3$ $A_1 D_2$

$B_3 A_1$ $B_3 B_3$ $B_3 C_3$ $B_3 D_2$

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$A_1 A_1$	$A_1 B_3$	$A_1 C_3$	$A_1 D_2$
$B_3 A_1$	$B_3 B_3$	$B_3 C_3$	$B_3 D_2$
$C_3 A_1$	$C_3 B_3$	$C_3 C_3$	$C_3 D_2$

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Let's play "stupid scrabble" with an incomplete set (only 4 first letters of the alphabet)

- "Words" with 2 letters:

$A_1\,A_1$	$A_1\,B_3$	$A_1\,C_3$	$A_1\,D_2$
$B_3\,A_1$	$B_3\,B_3$	$B_3\,C_3$	$B_3\,D_2$
$C_3\,A_1$	$C_3\,B_3$	$C_3\,C_3$	$C_3\,D_2$
$D_2\,A_1$	$D_2\,B_3$	$D_2\,C_3$	$D_2\,D_2$

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Let's play "stupid scrabble" with an incomplete set (only 4 first letters of the alphabet)

- "Words" with 2 letters:

A_1 A_1	A_1 B_3	A_1 C_3	A_1 D_2
B_3 A_1	B_3 B_3	B_3 C_3	B_3 D_2
C_3 A_1	C_3 B_3	C_3 C_3	C_3 D_2
D_2 A_1	D_2 B_3	D_2 C_3	D_2 D_2

$\rightarrow 4 \times 4 = 4^2$ possible words

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$B_3\,A_1$	$B_3\,B_3$	$B_3\,C_3$	$B_3\,D_2$
$C_3\,A_1$	$C_3\,B_3$	$C_3\,C_3$	$C_3\,D_2$
$D_2\,A_1$	$D_2\,B_3$	$D_2\,C_3$	$D_2\,D_2$

  $\rightarrow 4 \times 4 = 4^2$ possible words

- "Words" with n letters: $\text{Scrab}(n) = 4^n$.

q-counting: how?

Refining the count to take into account the score of the word

$A_1\,A_1$	$A_1\,B_3$	$A_1\,C_3$	$A_1\,D_2$
$B_3\,A_1$	$B_3\,B_3$	$B_3\,C_3$	$B_3\,D_2$
$C_3\,A_1$	$C_3\,B_3$	$C_3\,C_3$	$C_3\,D_2$
$D_2\,A_1$	$D_2\,B_3$	$D_2\,C_3$	$D_2\,D_2$

q-counting: how?

Refining the count to take into account the score of the word

A_1 A_1	A_1 B_3	A_1 C_3	A_1 D_2
B_3 A_1	B_3 B_3	B_3 C_3	B_3 D_2
C_3 A_1	C_3 B_3	C_3 C_3	C_3 D_2
D_2 A_1	D_2 B_3	D_2 C_3	D_2 D_2

Score	0	1	2	3	4	5	6
Words	0	0	1	2	5	4	4

q-counting: how?

Refining the count to take into account the score of the word

A_1 A_1	A_1 B_3	A_1 C_3	A_1 D_2
B_3 A_1	B_3 B_3	B_3 C_3	B_3 D_2
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D_2 A_1	D_2 B_3	D_2 C_3	D_2 D_2

Score	0	1	2	3	4	5	6
Words	0	0	1	2	5	4	4

$\rightarrow \mathrm{Scrab}(2, q) :=$

q-counting: how?

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$A_1 A_1$	$A_1 B_3$	$A_1 C_3$	$A_1 D_2$
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$C_3 A_1$	$C_3 B_3$	$C_3 C_3$	$C_3 D_2$
$D_2 A_1$	$D_2 B_3$	$D_2 C_3$	$D_2 D_2$

Score	0	1	2	3	4	5	**6**
Words	0	0	1	2	5	4	**4**

$\rightarrow \mathrm{Scrab}(2, q) := 4q^6$

q-counting: how?

Refining the count to take into account the score of the word

$A_1 A_1$	$A_1 B_3$	$A_1 C_3$	$A_1 D_2$
$B_3 A_1$	$B_3 B_3$	$B_3 C_3$	$B_3 D_2$
$C_3 A_1$	$C_3 B_3$	$C_3 C_3$	$C_3 D_2$
$D_2 A_1$	$D_2 B_3$	$D_2 C_3$	$D_2 D_2$

Score	0	1	2	3	4	**5**	6
Words	0	0	1	2	5	**4**	4

$\rightarrow \text{Scrab}(2, q) := 4q^6 + 4q^5$

q-counting: how?

Refining the count to take into account the score of the word

$A_1 A_1$	$A_1 B_3$	$A_1 C_3$	$A_1 D_2$
$B_3 A_1$	$B_3 B_3$	$B_3 C_3$	$B_3 D_2$
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Score	0	1	2	3	**4**	5	6
Words	0	0	1	2	**5**	4	4

$\to \mathrm{Scrab}(2, q) := 4q^6 + 4q^5 + 5q^4$

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Refining the count to take into account the score of the word

$A_1 A_1$	$A_1 B_3$	$A_1 C_3$	$A_1 D_2$
$B_3 A_1$	$B_3 B_3$	$B_3 C_3$	$B_3 D_2$
$C_3 A_1$	$C_3 B_3$	$C_3 C_3$	$C_3 D_2$
$D_2 A_1$	$D_2 B_3$	$D_2 C_3$	$D_2 D_2$

Score	0	1	2	**3**	4	5	6
Words	0	0	1	**2**	5	4	4

$\rightarrow \mathrm{Scrab}(2, q) := 4q^6 + 4q^5 + 5q^4 + 2q^3$

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Score	0	1	**2**	3	4	5	6
Words	0	0	**1**	2	5	4	4

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Score	0	1	2	3	4	5	6
Words	0	0	1	2	5	4	4

$\rightarrow \mathrm{Scrab}(2, q) := 4q^6 + 4q^5 + 5q^4 + 2q^3 + q^2$

$\rightarrow \mathrm{Scrab}(3, q) = 8q^9 + 12q^8 + 18q^7 + 13q^6 + 9q^5 + 3q^4 + q^3$

q-counting: how?

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$A_1 A_1$	$A_1 B_3$	$A_1 C_3$	$A_1 D_2$
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Score	0	1	2	3	4	5	6
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$$\boxed{\mathrm{Scrab}(n,1) = \mathrm{Scrab}(n)}$$

q-counting: why?

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$$
\begin{aligned}
&\mathrm{Scrab}(2,q)\times\mathrm{Scrab}(3,q)\\
=&(4q^6+4q^5+5q^4+2q^3+q^2)\\
\times&(8q^9+12q^8+18q^7+13q^6+9q^5+3q^4+q^3)\\
=&32q^{15}+80q^{14}+160q^{13}+200q^{12}+210q^{11}+161q^{10}\\
&+105q^9+50q^8+20q^7+5q^6+q^5
\end{aligned}
$$

q-counting: why?

Arithmetic! How can we q-count the ways to play a 2-letter word followed by a 3-letter word?

$$\begin{aligned} &\mathrm{Scrab}(2,q)\times\mathrm{Scrab}(3,q)\\ =&(4q^6+4q^5+5q^4+2q^3+q^2)\\ \times&(8q^9+12q^8+18q^7+13q^6+9q^5+3q^4+q^3)\\ =&32q^{15}+80q^{14}+160q^{13}+200q^{12}+210q^{11}+161q^{10}\\ &+105q^9+50q^8+20q^7+5q^6+q^5 \end{aligned}$$

We could further refine our counting to account for another aspect of words (e.g. number of vowels) $\to q,t$-counting.

Interesting symmetric functions

Any symmetric function $f \in \Lambda^{(n)}_{\mathbb{Q}(q,t)}$ can be written as

$$f = \sum_{\lambda \vdash n} c_\lambda s_\lambda \qquad\qquad c_\lambda \in \mathbb{Q}(q,t).$$

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Such a function is **Schur positive** if $c_\lambda \in \mathbb{N}[q,t]$ for all λ.

The set $\mathbb{N}[q,t]$ is anything that be obtained from natural numbers, q and t, using $+, \times$.

Example of element in $\mathbb{N}[q,t]$

$$9q^7t^2 + 4q^3t^4 + 2q^2 + 7t$$

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$$9q^7t^2 + 4q^3t^4 + 2q^2 + 7t$$

$\rightarrow$ q,t-countings always lie in $\mathbb{N}[q,t]$.

Why do we care about Schur positivity?

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Any finite-dimensional representation of $\mathfrak{S}_n$ is entirely determined by its character and semi-simple

Why do we care about Schur positivity?

Let $\mathfrak{S}_n$ be the n-th symmetric group. The Frobenius characteristic map gives a correspondence

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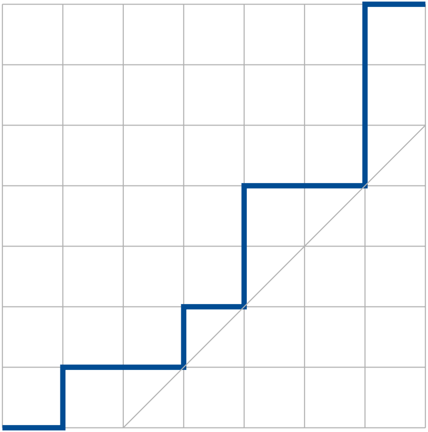
$$V \leftrightarrow \textstyle\sum_{i,j \in \mathbb{N}} q^i t^j \mathcal{F}(V^{(i,j)}) = \text{Schur positive element of } \Lambda_{\mathbb{Q}(q,t)}^{(n)}$$

Combinatorial formulas

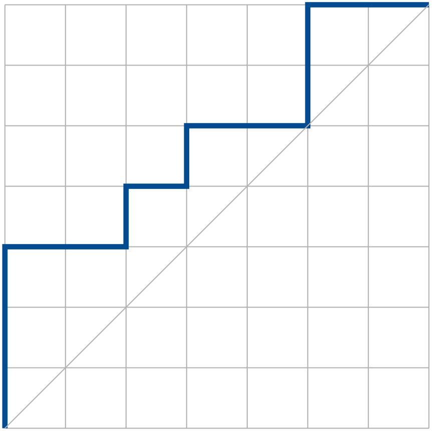
Given an interesting symmetric function f, we look for "combinatorial objects" (e.g. scrabble words), with which we can build f.

Objects and statistics



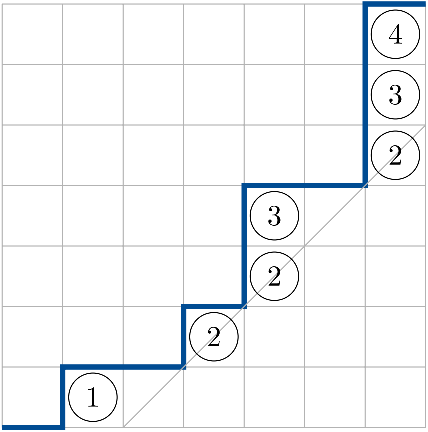
Square paths $\mathsf{SQ}(7)$.

Objects and statistics



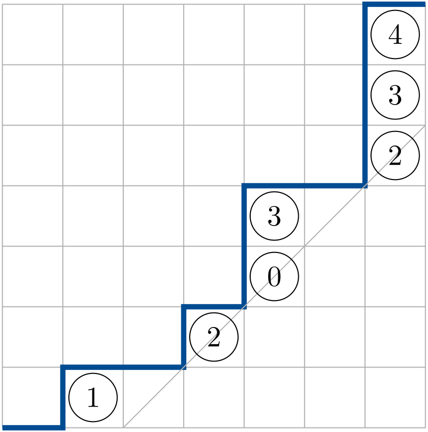
above *main diagonal* → Dyck paths $\mathsf{D}(7)$

Objects and statistics



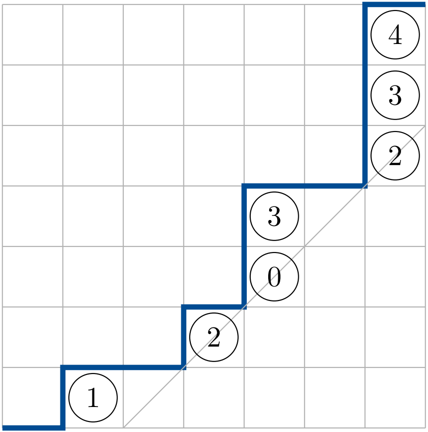
Labelled square paths $\mathsf{LSQ}(7)$

Objects and statistics



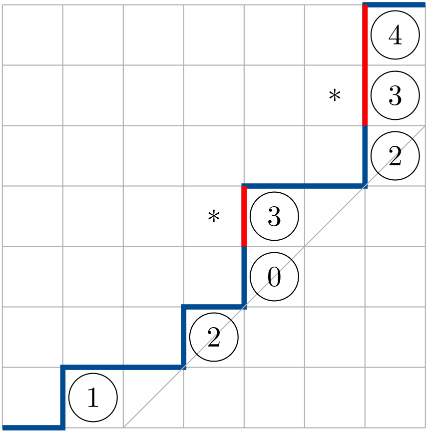
Partially labelled square paths $\mathsf{LSQ}(1, 6)$

Objects and statistics



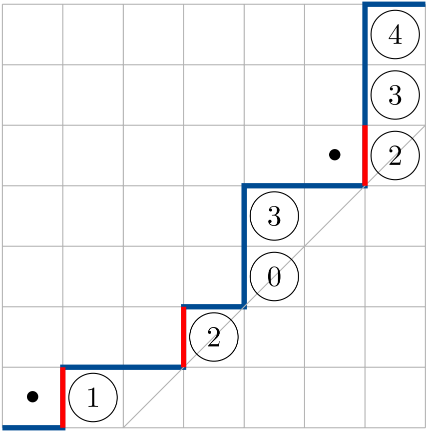
$$x^P = x_1 x_2^2 x_3^2 x_4$$
$$x_0 \mapsto 1$$

Objects and statistics



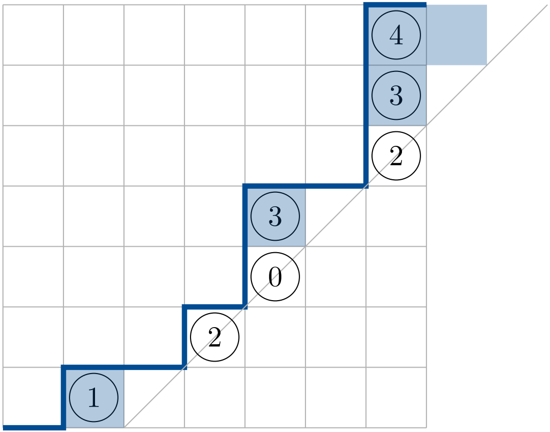
Rise-decorated partially labelled square paths $\mathsf{LSQ}(1,6)^{*2}$.

Objects and statistics



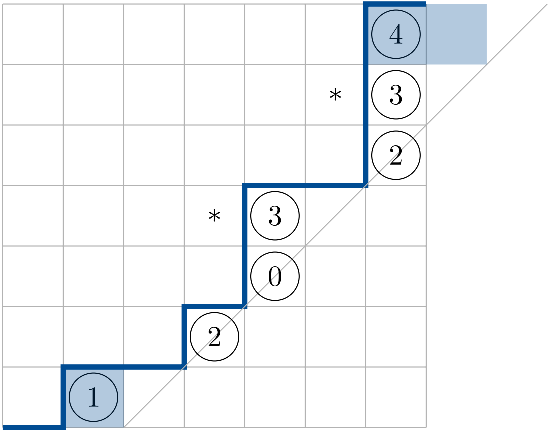
Valley-decorated partially labelled square paths $\mathsf{LSQ}(1,6)^{\bullet 2}$.

Objects and statistics



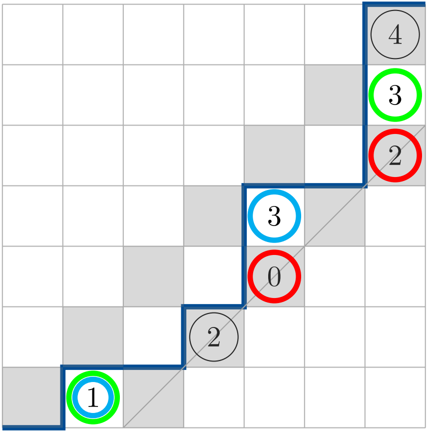
$\mathsf{area}(P) = 5$

Objects and statistics



$\mathsf{area}(P) = 3$

Objects and statistics



Second statistic $\mathsf{dinv}(P) = 10$

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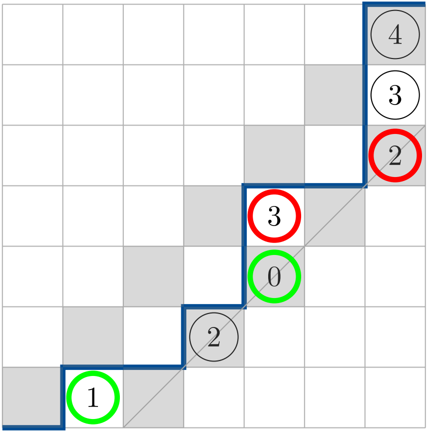

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A combinatorial formula: the shuffle Theorem

∇e_n is Schur positive.
∇ is a simple operator on $\Lambda_{\mathbb{Q}(q,t)}$ related to Macdonald polynomials.

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Shuffle theorem For $n \in \mathbb{N}$

$$\nabla e_n = \sum_{P \in \mathsf{LD}(n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Conjectured by Haglund, Haiman, Loehr, Remmel and Ulyanov (2002) and proved by Carlsson and Mellit (2018).

Example $n = 3$

$$\begin{aligned}\nabla e_3 = & x_1^3 + \left(q^2 + qt + t^2 + q + t + 1\right) x_1^2 x_2 + \\ & \left(q^2 + qt + t^2 + q + t + 1\right) x_1 x_2^2 + x_2^3 + \cdots\end{aligned}$$

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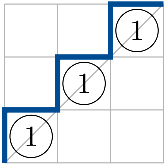

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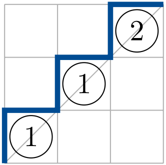

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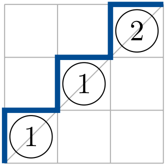

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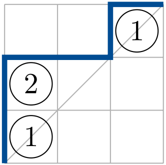

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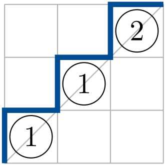

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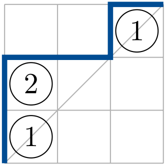

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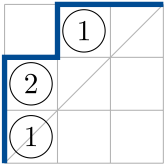

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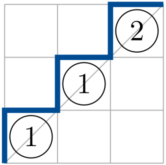

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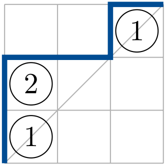

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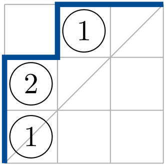

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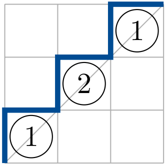

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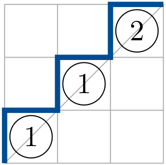

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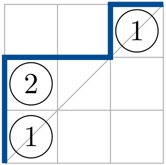

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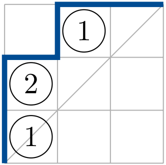

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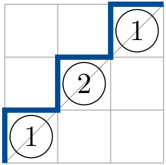

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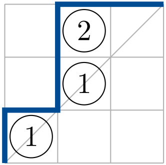

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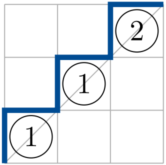

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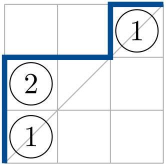

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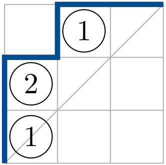

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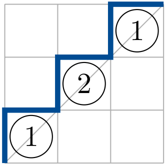

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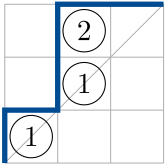

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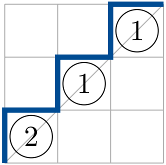

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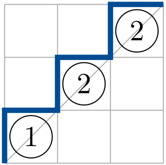

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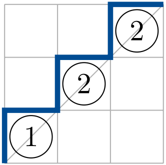

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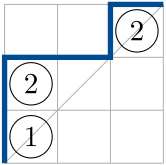

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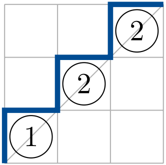

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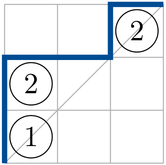

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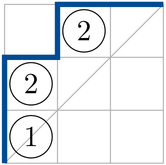

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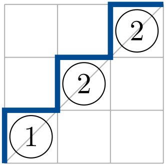

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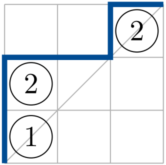

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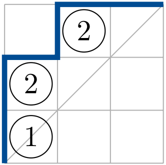

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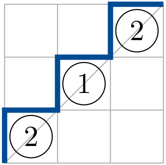

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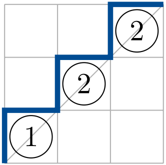

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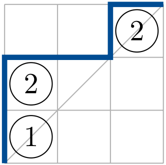

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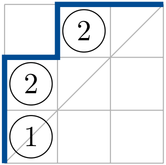

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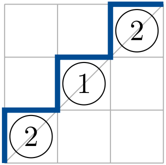

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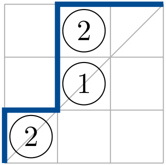

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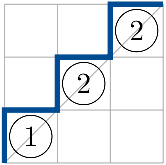

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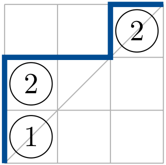

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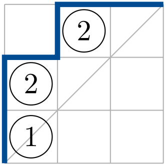
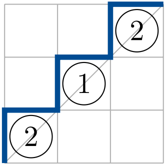
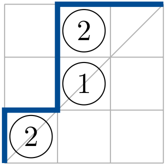


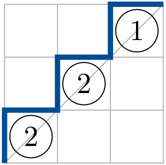
dinv 0 area 2



dinv 1 area 0



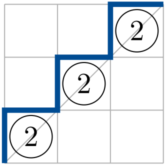
dinv 0 area 1



dinv 0 area 0

Example $n = 3$

$$\nabla e_3 = x_1^3 + \left(q^2 + qt + t^2 + q + t + 1\right) x_1^2 x_2 + \left(q^2 + qt + t^2 + q + t + 1\right) x_1 x_2^2 + x_2^3 + \cdots$$



dinv 0 area 0

The square theorem

Square theorem For $n \in \mathbb{N}$

$$(-1)^n \nabla p_n = \sum_{P \in \mathsf{LSQ}(n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Conjectured by Loehr and Warrington (2007) and proved by Sergel (2017), using the shuffle theorem.

Example $n = 3$

$$
\begin{aligned}
-\nabla p_3 = & x_1^3 + \left(q^2t^2 + q^2t + qt^2 + q^2 + qt + t^2 + q + t + 1\right) x_1^2 x_2 \\
& + \left(q^2t^2 + q^2t + qt^2 + q^2 + qt + t^2 + q + t + 1\right) x_1 x_2^2 + x_2^3 + \cdots
\end{aligned}
$$

Example $n = 3$

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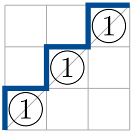
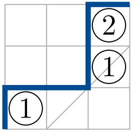


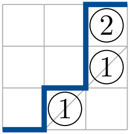
dinv 0 area 0

Example $n = 3$

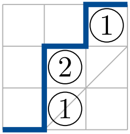
$$-\nabla p_3 = x_1^3 + \left(q^2t^2 + q^2t + qt^2 + q^2 + qt + t^2 + q + t + 1\right) \textcolor{red}{x_1^2 x_2}$$
$$+ \left(q^2t^2 + q^2t + qt^2 + q^2 + qt + t^2 + q + t + 1\right) x_1 x_2^2 + x_2^3 + \cdots$$



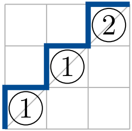
dinv 2 area 2



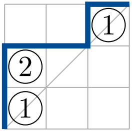
dinv 2 area 1



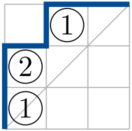
dinv 1 area 2



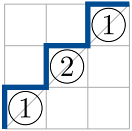
dinv 2 area 0



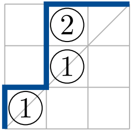
dinv 1 area 1



dinv 0 area 2



dinv 1 area 0



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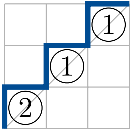

dinv 0 area 0

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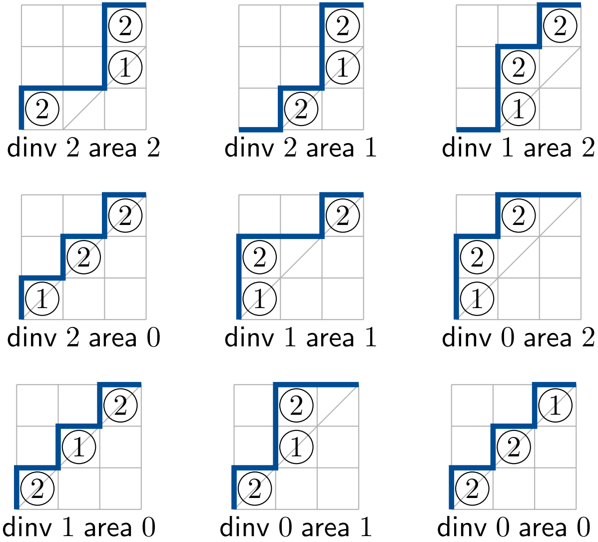

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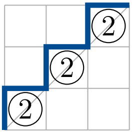

dinv 0 area 0

Generalisations of shuffle theorem

Two more symmetric function operators: Θ_k and Δ_{h_m}.

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Generalised Delta conjecture (Theta reformulation)

$$\Delta_{h_m}\Theta_k \nabla e_{n-k} = \sum_{P \in \mathsf{LD}(m,n)^{*k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P$$

$$= \sum_{P \in \mathsf{LD}(m,n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P$$

Proposed by Haglund, Remmel and Wilson (2015). Open problem.

Example $n = 2$, $k = 1$, $m = 1$

$$\Delta_{h_1}\Theta_1\nabla e_1 = x_1^2 + (q + t + 2)\, x_1x_2 + x_2^2 + \cdots$$

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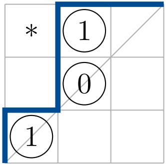
→ **Rise decorations**



dinv 0 area 0

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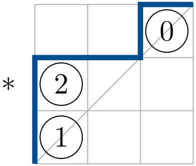
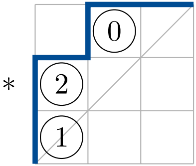


dinv 1 area 0



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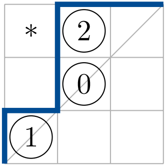
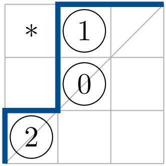


dinv 0 area 0



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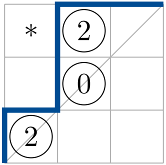

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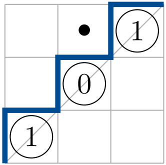

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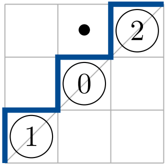
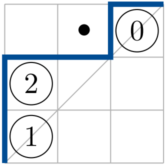


dinv 1 area 0



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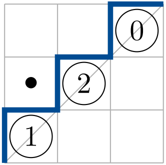

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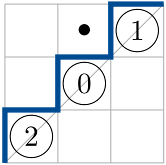

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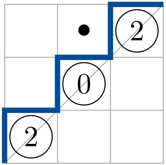

dinv 0 area 0

Refinements of the shuffle theorem

A **composition** $\alpha \vDash n$ is a vector of positive integers whose sum is n. Its length is denoted $\ell(\alpha)$.

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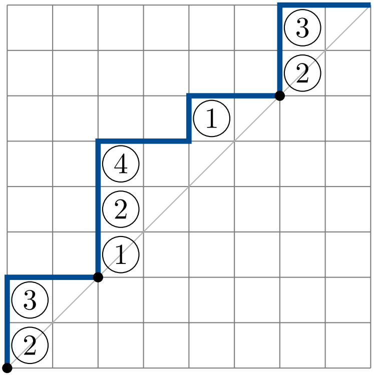

$\mathsf{touch}(P) = 3$

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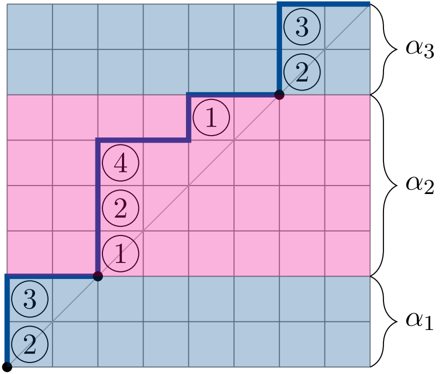

Composition $\alpha = (2,4,2) \vDash n = 8$

Refinements of the shuffle theorem

C_α and $E_{n,r}$ are symmetric functions refining e_n:

$$e_n = \sum_{r=0}^{n} E_{n,r} \qquad\qquad E_{n,r} = \sum_{\alpha \vDash n \,:\, \ell(\alpha)=r} C_\alpha$$

Touching shuffle theorem For $n, r \in \mathbb{N}$

$$\nabla E_{n,r} = \sum_{\substack{P \in \mathsf{LD}(n) \\ \mathsf{touch}(P)=r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

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Compositional Shuffle theorem For α a composition

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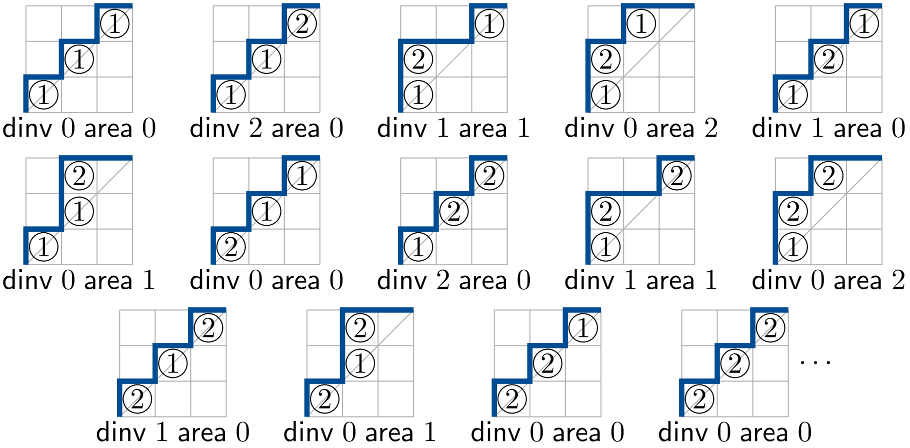
This is what by Carlsson and Mellit (2018) proved, implying the shuffle conjecture. The refinement played a key role.

Back to our example ∇e_3



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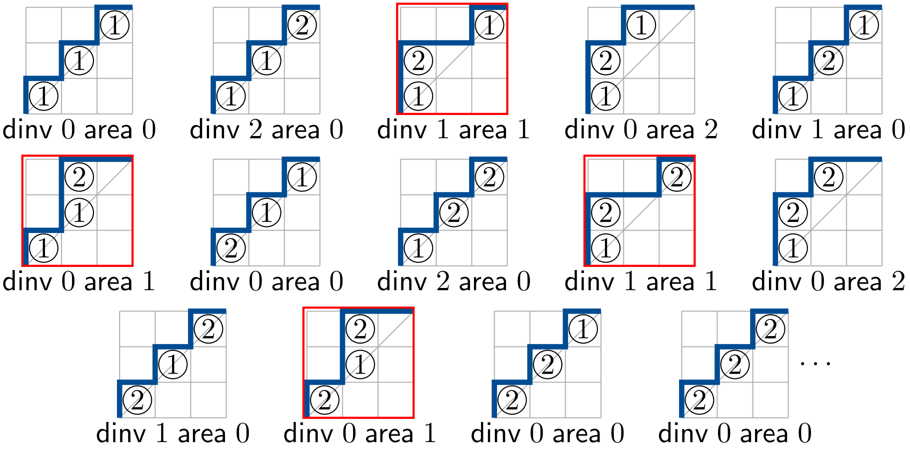
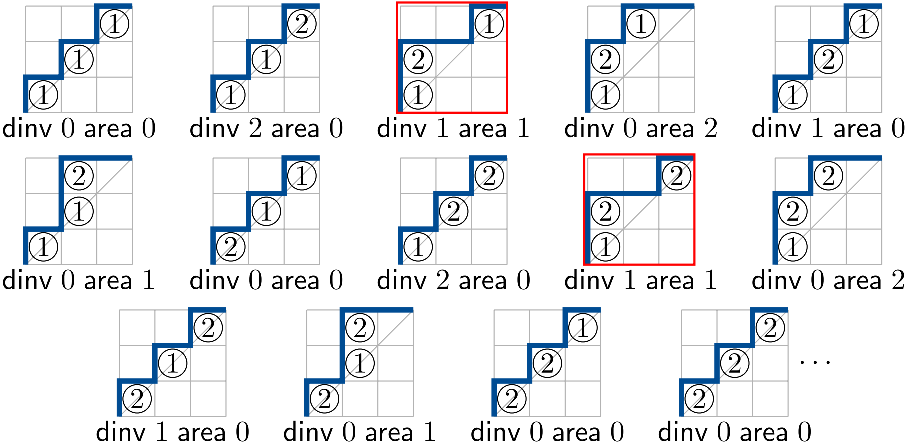


$$\nabla E_{3,2} = (qt + t)\, x_1^2 x_2 + (qt + t)\, x_1 x_2^2 + \cdots$$

Back to our example ∇e_3



$$\nabla C_{(2,1)} = qtx_1^2x_2 + qtx_1x_2^2 + \cdots$$

Content of the thesis

In my thesis I

The definition of the Θ_k operator seems to give the following refinement of the rise Delta conjecture.

Conjecture For α a composition

$$\begin{aligned}\Theta_k \nabla C_\alpha &= \sum_{P \in \mathsf{LD}(\alpha)^{*k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P \\ &= \sum_{P \in \mathsf{LD}(\alpha)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P\end{aligned}$$

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We generalise Carlsson and Mellit's combinatorial recursion to the rise decorated context
$\to$ operator Delta conjecture.

In my thesis II

We found a way to add the 0's to the shuffle theorem.

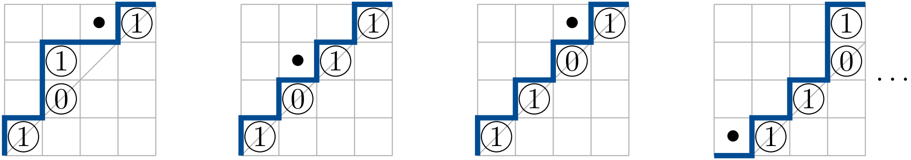
Touching generalised shuffle theorem

$$\Delta_{h_m} \nabla E_{n,r} = \sum_{\substack{P \in \mathsf{LD}(m,n) \\ \mathsf{touch}(P) = r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

In my thesis III

Generalised Delta square conjecture (valley version)

$$(-1)^{n-k}\Delta_{h_m}\Theta_k \nabla p_{n-k} = \sum_{P \in \mathsf{LSQ}'(m,n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$



dinv 0 area 1 — dinv 1 area 0 — dinv 0 area 0 — dinv 1 area 1

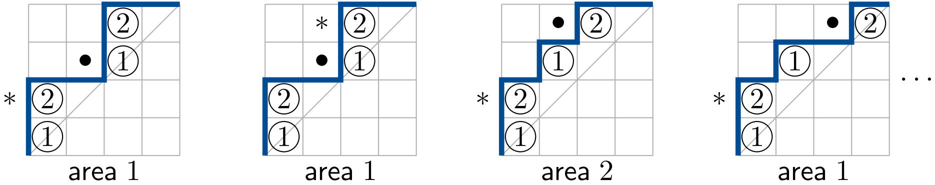
Sergel showed that (touching shuffle ⇒ square).

Theorem Touching (generalised) Delta conjecture
⇒ (generalised) Delta square conjecture

Next up: a Theta conjecture?

Computer evidence suggests that

$$\Theta_l \Theta_k \nabla e_{n-k-l}\Big|_{q=1} = \sum_{P \in \mathsf{LD}(n)^{*k,\bullet l}} t^{\mathsf{area}(P)} x^P$$



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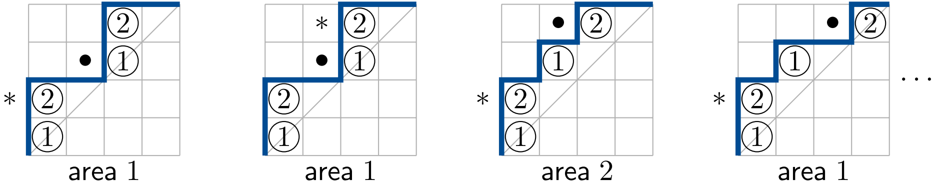

→ A unified Delta conjecture?

Thank you for your attention

The module of the shuffle theorem

Consider the ring $\mathcal{R}_n \coloneqq \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and the **diagonal action** of $\mathfrak{S}_n$ on $\mathcal{R}_n$

$$\sigma \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}).$$

The **diagonal coinvariants** are $\mathcal{D}_n \coloneqq \mathcal{R}_n / \mathcal{I}_n$ where $\mathcal{I}_n$ is the ideal of constant-free invariants of the diagonal action.
This module is naturally bi-graded.

> Theorem (Haiman) $\mathcal{F}_{q,t}(\mathcal{D}_n) = \nabla e_n$