Tiered trees, polyominoes and Theta operators

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Summary

In this talk we will discuss our new conjecture

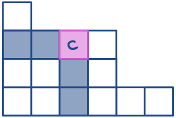
$$\Theta_{e_\lambda} e_1 \Big|_{t=1} = \sum_T q^{\mathrm{inv}(T)} x^T$$

where the sum ranges over rooted λ-tiered trees

- Why do we care about $\Theta_{e_\lambda} e_1$? $*$
- What are tiered trees?
- Proof of cases $\lambda = 1^{n-1}$ and $\langle \ , e_1^{|\lambda|+1} \rangle$
- For $\ell(\lambda) = 2$, connection with labelled parallelogram polyominoes and the abelian sandpile model on inversion graphs

Theta operators

For any cell c in a partition μ let



$a'_\mu(c) = \#$ cells strictly to the left of c

$l'_\mu(c) = \#$ cells strictly below c

$$\Pi_\mu = \prod_{c \in \mu/(1)} \left(1 - q^{a'_\mu(c)} t^{l'_\mu(c)}\right) \in \mathbb{Q}[q,t]$$

Denote by $\{\tilde{H}_\mu\}$ the basis of Macdonald Polynomials of $\Lambda_{\mathbb{Q}(q,t)}$

Define the Macdonald eigenoperator on $\Lambda_{\mathbb{Q}(q,t)}$ via $\Pi \tilde{H}_\mu = \Pi_\mu \tilde{H}_\mu$

For any $f \in \Lambda^{(m)}_{\mathbb{Q}(q,t)}$ and $g \in \Lambda^{(n)}_{\mathbb{Q}(q,t)}$ define

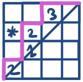
$$\Theta_f\, g = \begin{cases} 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ f \cdot g & \text{if } m = n = 0 \\ \Pi\left(f\left[\frac{X}{(1-q)(1-t)}\right] \Pi^{-1}(g)\right) & \text{otherwise} \end{cases}$$

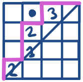
And extend linearly for any $f, g \in \Lambda_{\mathbb{Q}(q,t)}$

D'Adderio, Iraci, VW

Theta operators

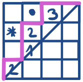
- $\Theta_{e_k} \nabla e_{n-k} = \boxed{\Delta'_{e_{n-k-1}} e_n} \longrightarrow$ the symmetric function of the Delta conjectures (rise and valley)





Essential in the proof of the (compositional) rise version

- $\Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell}$ might give a unified Delta conjecture
$\rightsquigarrow$ decorations on both rises and valleys



- $h_j^\perp \Theta_{e_{(m,n)}} e_1 = \Theta_{e_{m-j}} \Theta_{e_{n-j}} \nabla e_{j+1} + \Theta_{e_{m-j+1}} \Theta_{e_{n-j}} \nabla e_j$
$+ \Theta_{e_{m-j}} \Theta_{e_{n-j+1}} \nabla e_j + \Theta_{e_{m-j+1}} \Theta_{e_{n-j+1}} \nabla e_{j-1}$

$\Theta_{e_\lambda} e_1$ for $\ell(\lambda) = 2$

Essentially the unified Delta conjecture

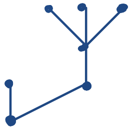
- $\langle \Theta_{e_{1^n}} e_1 , e_{n+1} \rangle = \langle \nabla e_n , e_1^n \rangle$

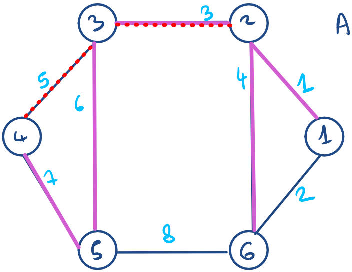
$\lambda = 1^n$ Catalan case of our conjecture (proved at $t=1$)

Hilbert series of the shuffle theorem

Graphs, trees and Tutte polynomials

A simple, connected graph $G = (V, E)$





A tree is a connected graph without cycles

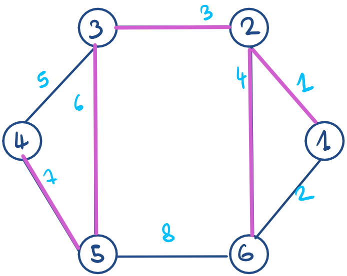
A spanning tree T of G is a subtree of G containing all vertices of G

Fix $<_E$ an order on the edges of G eg lexicographic

An edge $e \in G \setminus T$ is exterior active if it is minimal for $<_E$ in the unique cycle of $T \cup \{e\}$

An edge $e \in T$ is interior active if it is minimal for $<_E$ in the set of edges of G joining the 2 components of $T \setminus \{e\}$

Graphs, trees and Tutte polynomials



$ext(T) = \#$ exterior active edges
$= 1$

$int(T) = \#$ interior active edges
$= 2$

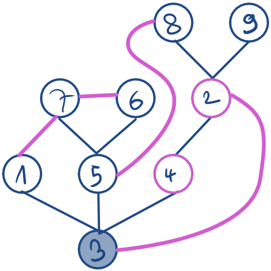
Tutte polynomial of G

$$T_G(q,t) := \sum_{\text{spanning trees } T \text{ of } G} q^{ext(T)}\, t^{int(T)}$$

Does not depend on the choice of $<_E$!

Rooted trees and κ-inversions

A rooted graph is a graph with a distinguished vertex r



For $v \in T$ a rooted tree, its height
$ht(v) =$ distance between v and r

Its parent $p(v)$ is its unique neighbour at strictly lower height
$\rightsquigarrow$ child, descendent, ancestor

A labelling of $G=(V,E)$ is any assignment $w : V \longrightarrow \mathbb{N}$
It is called standard if $\mathrm{Im}(w) = \{1, 2, \dots, |V|\}$

An inversion of a standardly labelled rooted tree is a pair of vertices $(v, \tilde{v})$ such that $\tilde{v}$ is a descendant of v and $w(v) > w(\tilde{v})$

If T is a spanning tree of a graph G then a κ-inversion is an inversion $(v, \tilde{v})$ such that $\{p(v), \tilde{v}\}$ is an edge of G $(v \neq r)$

Rooted trees and k-inversions

THM (Gessel '95) For any standardly labelled graph G

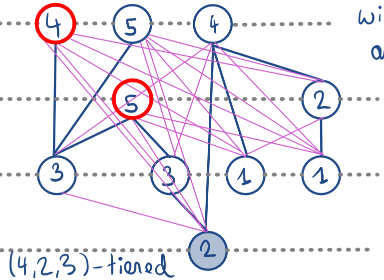
$$\sum_{T \text{ spanning tree of } G} q^{\mathrm{inv}_k(T)} = T_G(q,1)$$

where $\mathrm{inv}_k(T) = \#$ k-inversions of T

In other words, inv_k is distributed like ext on the spanning trees of G

Rooted tiered trees

$x^T = x_1^2 x_2^3 x_3 x_4^2 x_5^2$



(4,2,3)-tiered

A **rooted tiered tree** is a rooted tree $T=(V,E)$ with a label function $w: V \to \mathbb{N}_0$ and a **level function** $lv: V \to \mathbb{N}$ such that

1) $\{v,\tilde{v}\} \in E \Rightarrow lv(v) \neq lv(\tilde{v})$
2) $\{v,\tilde{v}\} \in E$ and $lv(v) < lv(\tilde{v}) \Rightarrow w(v) < w(\tilde{v})$
3) $p(v) = p(\tilde{v})$ and $lv(v) = lv(\tilde{v}) \Rightarrow w(v) \neq w(\tilde{v})$
4) $lv^{-1}(0) = \{x\}$

Such a tree is called **α-tiered**, for some composition α, if $lv(i) = \alpha_i$ $RTT(\alpha)$

The **compatibility graph** of T is $T \cup \{\{v,\tilde{v}\} \mid \underbrace{lv(v) < lv(\tilde{v}) \text{ and } w(v) < w(\tilde{v})}_{\text{compatible}}\}$

An **inversion** of T is a pair of vertices $(v,\tilde{v})$ such that

1) $\tilde{v}$ descendant of v
2) $\tilde{v}$ and $p(v)$ are compatible
3) $w(\tilde{v}) < w(v)$ or $(w(v) = w(\tilde{v})$ and $lv(\tilde{v}) > lv(v))$

$inv(T) = \#$ inversions

The main conjecture

$$\boxed{\Theta_{e_\lambda} e_1 \Big|_{t=1} = \sum_{T \in RTT(\lambda)} q^{\mathrm{inv}(T)} x^T}$$

Obvious question: find tstat $: RTT(\lambda) \to \mathbb{N}$ to complete the conjecture

Besides computational evidence, we proved the cases

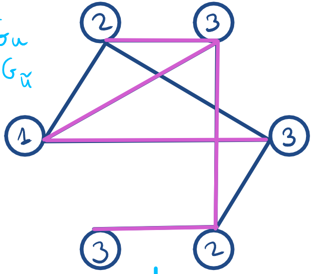
- $\lambda = 1^{n-1}$ fully tiered trees
- $\langle \, , e_1^n \rangle$ standardly labelled tiered trees

which are closely related.

Tiered trees and inversion graphs

For any word $u \in \mathbb{N}^n$, its inversion graph $G_u = (\{1,\dots,n\}, E)$ the labelled graph such that $w(i) = u_i$ and $\{i,j\} \in E \iff i<j$ and $u_i > u_j$

$G_u \simeq G_{\tilde{u}}$



eg $u = 3\ 3\ 2\ 1\ 3\ 2$ STANDARDISATION

$\tilde{u} = 4\ 5\ 2\ 1\ 6\ 3$

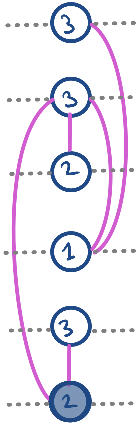
= word with standard letters and the same inversion graph

$\in S_{(1,2,3)}$

S_α is the set of α-shuffles

⇝ Standardisations of words with α_i occurences of i

$$\boxed{x^u = x^\alpha}$$



Tiering with compatibility graph G_u

Spanning trees of inversion graphs

⟷ fully tiered trees

Tiered trees and Tutte polynomials

$$\sum_{\sigma\in S_\alpha} \boxed{\sum_{T\in ST(G_\sigma)} q^{k-\mathrm{inv}(T)}} = \sum_{\substack{T\in RTT(1^{|\alpha|-1}) \\ x^T = x^\alpha}} q^{\mathrm{inv}(T)}$$

$\|$ Gessel

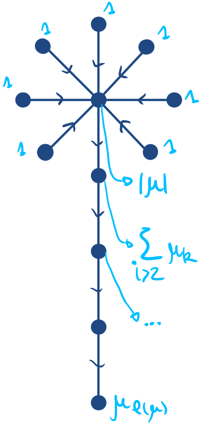
$$T_{G_\sigma}(q,1)$$

$\|$?

$$\langle 4_{e_\alpha} e_1 , h_\alpha \rangle$$

Kac Polynomial of Dandelion quivers

Take Γ to be the Dandelion graph.



Define $v_\mu = (\underbrace{1, 1, \ldots, 1}_{\#\text{ short legs}}, |\mu|, |\mu| - \mu_1, \ldots, \mu_{\ell(\mu)})$

for some partition μ st $\ell(\mu)$ = length of long leg.

Let $A_{\Gamma, v_\mu}(q)$ be the Kac polynomial of Γ of dimension vector μ

Gunnels-Letellier-Villegas-2018

$$\boxed{A_{\Gamma, v_{\mu^-}}(q) = \sum_{\sigma \in S_\mu} T_{G_\sigma}(q, 1)}$$

$\mu^- = (\mu_2, \mu_3, \ldots)$

And, for $\mathcal{A}_n$ the family of SF's defined by

$$\widetilde{H}_{(n+1)} = \sum_{k=0}^{n} \binom{n}{k} (q-1)^{n-k} \widetilde{H}_{(k)} \mathcal{A}_{n-k+1}(X; q)$$

$$\boxed{A_{\Gamma, v_{\mu^-}}(q) = \langle \mathcal{A}_{|\mu|}, h_\mu \rangle}$$

Proof of the case $\lambda = 1^{n-1}$

$$\boxed{A_{\Gamma, v_{\mu^-}}(q) = \sum_{\sigma \in S_\mu} T_{G_\sigma}(q,1)}$$

$$\boxed{A_{\Gamma, v_{\mu^-}}(q) = \langle \mathcal{A}_{|\mu|}, h_\mu \rangle}$$

THM $\mathcal{A}_n(X;q) = \Theta_{e_{1^{n-1}}} e_1 \big|_{t=1}$

Proof Show that $\Theta_{e_{1^{n-1}}} e_1$ satisfies the same defining relation as $\mathcal{A}_n$

$$\widetilde{H}_{(n+1)} = \sum_{k=0}^{n} \binom{n}{k} (q-1)^{n-k} \widetilde{H}_{(k)} \left(\Theta_{e_{1^{n-k}}} e_1\right)\Big|_{t=1}$$

$\square$

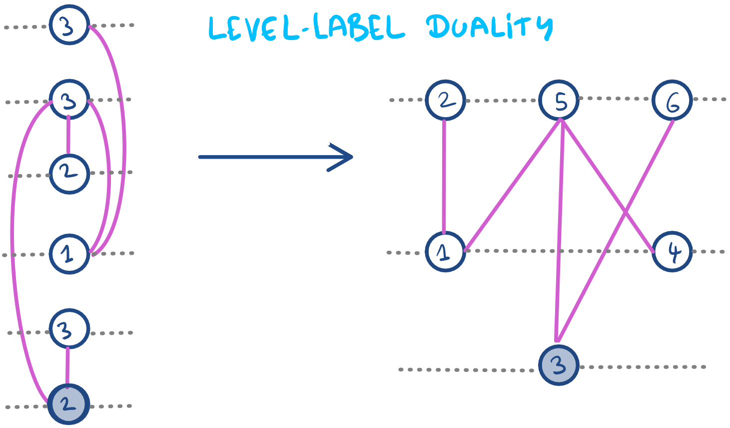
Thus

$$\left\langle \Theta_{e_{1^{n-1}}} e_1 \big|_{t=1}, h_\mu \right\rangle = \sum_{\sigma \in S_\mu} T_{G_\sigma}(q,1) \quad \forall \mu$$

$$\sum_{\sigma \in S_\mu} T_{G_\sigma}(q,1) = \sum_{\substack{T \in RT(1^{n-1}) \\ x^T = x^\mu}} q^{\mathrm{inv}(T)}$$

$$\Rightarrow \boxed{\Theta_{e_{1^{n-1}}} e_1 \big|_{t=1} = \sum_{T \in RT(1^{n-1})} q^{\mathrm{inv}(T)} x^T}$$

The case $\langle \, , e_1^{|\lambda|+1} \rangle$



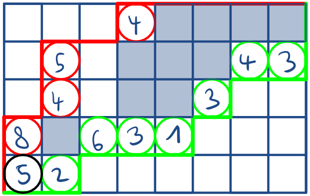
$$\langle \Theta_{e_1^{n-1}} e_1 , \underbrace{h_{\mu,1}} \rangle = \langle \Theta_{e_\mu} e_1 , e_1^n \rangle$$

1 label = 1 ⟶ on label at level 0 = root

→ We may deduce case $\langle \, , e_1^n \rangle$ from case $\lambda = 1^{n-1}$

Labelled polyominoes

When $\ell(\lambda) = 2$, we have another combinatorial model



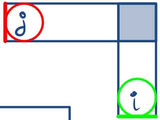
AREA = 8

red labels  green labels 

black label 

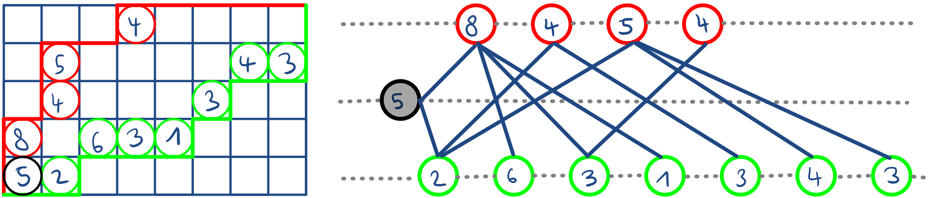
Strictly increasing $\uparrow$ $\leftarrow$

The area is the # squares inside the polyomino, not containing a label and such that $i < j$



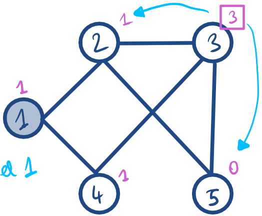
Conjecture (4) $e_{m-1,n-1} \, e_1 \big|_{t=1} = \sum_{P \in LPP(m,n)} q^{area(P)} x^P$

Labelled Polyominoes and tiered trees



We proved the case $\langle \, , e_1^n \rangle$, i.e. standard labels of our polyomino conjecture using the abelian sandpile model on the compatibility graph of the associated tiered tree

Abelian Sandpile model



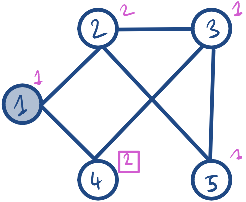
level 1

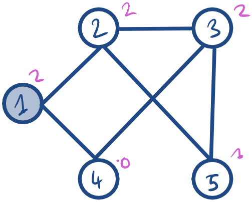
Let $G=(V,E)$ graph with a distinguished vertex s sink

configuration $c : V \longrightarrow \mathbb{N}_0$

c is unstable if $\exists\, v \neq s$ such that $\deg(v) \leqslant c(v)$

We topple unstable vertex i $\quad c \xrightarrow{i} \tilde{c}$





$\longrightarrow$ Stable

Stablization does not depend on toppling order

A recurrent configuration c is such that $c(s)=\deg(s)$ and $c \xrightarrow{s} \tilde{c} \xrightarrow{\text{stab}} c$

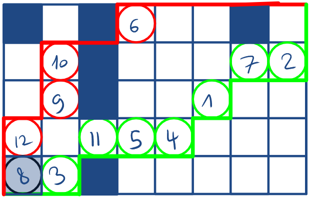
The level of a configuration is $\sum_{v \in V} c(v) - |E|$

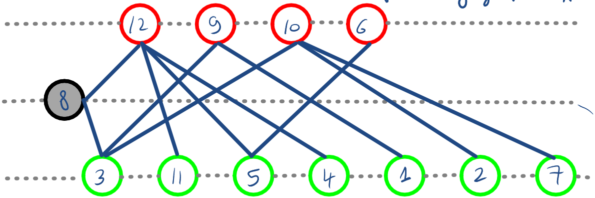
$$\boxed{\text{THM} \quad \sum_{c \in Rec(G)} q^{\mathrm{level}(c)} = T_G(q,1)}$$

Labelled polyominoes and ASM

Fix a choice $\pi = (\{1,2,3,4,5,7,11\},\{8\},\{6,9,10,12\})$ of green/red/black labels

Compatibility graph G_π





$LPP(\pi) \longleftrightarrow$ Spanning trees of G_π

white squares $\longleftrightarrow$ edges of G_π

We define $\alpha : LPP(\pi) \longrightarrow Rec(G_\pi)$ by setting

c(green vertex) = # white squares on top of squares containing it

eg $c(11)=0$, $c(5)=c(4)=3$

c(red vertex) = # white squares to the right of squares containing it

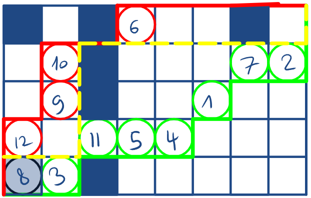
eg $c(9)=5$, $c(6)=3$

c(black label) = # white squares on top + to the right of square containing it

$c(8)=9$

Dukes - Le Borgne

Labelled polyominoes and ASM



THM α is
- well defined
- bijective
- such that $area(P) = level(\alpha(P))$

The proof uses the canonical toppling order induced by the bounce path

8 3 12 9 10 11 5 4 1 7 2 6

$$\Rightarrow \sum_{P \in LPP(\pi)} q^{area(P)} = \sum_{c \in Rec(G_\pi)} q^{level(c)} = T_{G_\pi}(q,1)$$

$\sum_\pi$

$$\sum_{P \in LPP(m,n)} q^{area(\pi)} = \sum_\pi T_{G_\pi}(q,1) = \sum_{\sigma \in S_{(m-1,1,n-2)}} T_{G_\sigma}(q,1)$$

$$= \langle 4_{e_{(m-1,n-1)}} e_1 , e_1 \rangle$$

Thank You very much for listening