Operations and Supply Chain Management

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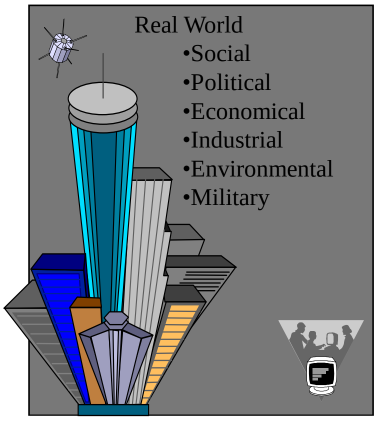
Summary

Decision-Making

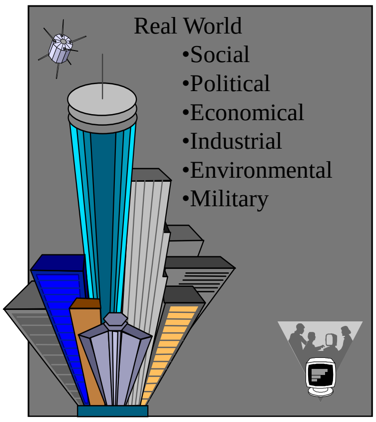


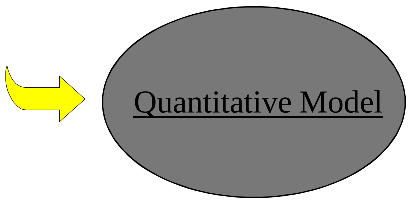
- To Describe,
- To Understand,
- To Manage.

2 Approaches :

- Qualitative Approach,
- <u>Quantitative Approach</u>.

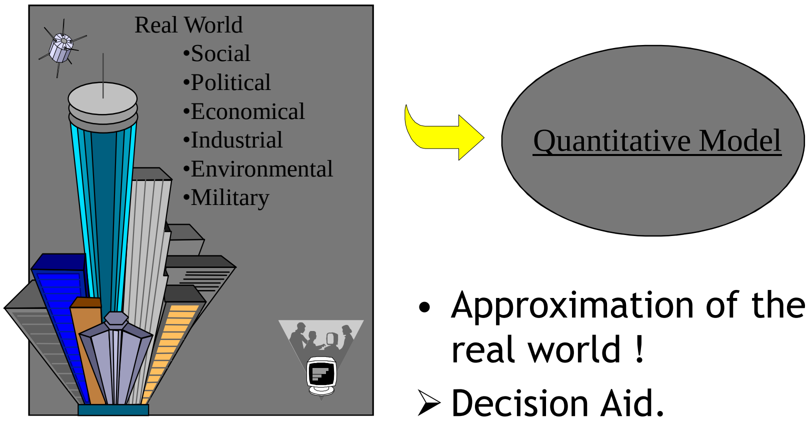
Decision Aid





- Possible decisions ?
- How to compare them ?
- Preferences, Objectives ?

Decision Aid



- Approximation of the real world !
- ➢ Decision Aid.

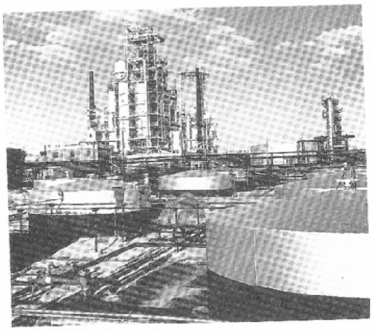
2. Optimization

- Linear programming
- Integer programming
- Model building
- Optimization software:
  - Excel's solver
  - MPL 4

Management Science Application

A Refinery Linear Programming System at Citgo Petroleum

In 1983 Southland Corporation, the parent company of the 7-Eleven convenience store chain, acquired Citgo Petroleum Corporation. Prior to this acquisition, Citgo had lost money for several years; thus, a primary objective of Southland was to improve Citgo's profitability. The Southland Corporation invested in a number of management science applications to achieve this objective, and one of the largest and most important was a refinery linear programming system. The linear programming system allowed for effective management of crude stock acquisition, processing costs, and energy costs, which were almost $4 billion in 1984. The refinery linear programming system is used routinely to make decisions regarding crude selection and acquisition, refinery run levels, feedstock acquisitions, unit turnaround options, and hydrocracker conversion. The linear programming system is now one of the primary corporate operational planning tools. In 1985 Citgo achieved a pretax profit of over $70 million, and the linear programming system was cited as a significant contributor to this turnaround.

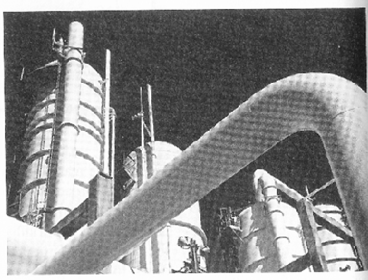


Source: D. Klingman et al., "The Successful Deployment of Management Science Throughout Citgo Petroleum Corporation," *Interfaces* 17, no. 1 (January–February 1987): 4–25.

Management Science Application

Chemical Production at Monsanto

Monsanto produces maleic anhydride (a chemical used in making plastic) at plants in St. Louis and Pensacola. The combined capacity of the two plants is more than 45% of total U.S. output of 359 million pounds per year. Capacity at both plants exceeded demand, resulting in a need to assign production to each plant in an optimal manner. Three linear programming models were developed for this purpose, including a global model to determine the amount produced at each plant and individual plant models to adjust operating plans during a production period. The models, which encompassed over a thousand variables and a dozen or more constraints, minimize cost subject to meeting a production target. Use of the system has resulted in estimated annual savings of between $1 and $3 million (depending on plant operating rules).



Source: R. Boykin, "Optimizing Chemical Production at Monsanto," *Interfaces* 15, no. 1 (January–February 1985): 88–95.

Grape Juice Management at Welch's

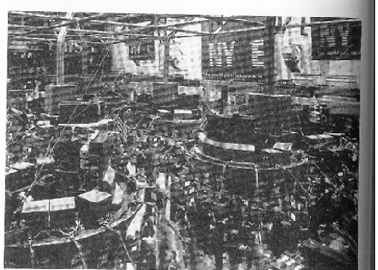
With annual sales over $550 million, Welch's is one of the world's largest grape-processing companies. Founded in 1869 by Dr. Thomas B. Welch, it processes raw grapes (nearly 300,000 tons per year) into juice, as well as jellies and frozen concentrates. Welch's is owned by the National Grape Cooperative Association (NGCA), which has a membership of 1,400 growers. Welch's is NGCA's production, distribution, and marketing organization. Welch's operates its grape processing plants near its growers. Because of the dynamic nature of product demand and customer service, Welch's holds finished goods inventory as a buffer, and maintains a large raw materials inventory stored as grape juice in refrigerated tank farms. Packaging operations at each plant draw juice from the tank farms during the year as needed. The value of the stored grape juice often exceeds $50 million. Harvest yields and grape quality vary between localities. In order to maintain national quality and consistency in its products, Welch's transfers juices between plants and adjusts product recipes. To do this Welch's uses a spreadsheet-based linear programming model. The juice logistics model (JLM) encompasses 324 decision variables and 361 constraints, that minimizes the combined costs of transporting grape juice between plants and the product recipes at each plant, and the carrying cost of storing grape juice. The model decision variables include the grape juice shipped to customers for different product groups, the grape juice transferred between plants, and inventory levels at each plant. Constraints are for recipe requirements, inventories, and grape juice usage and transfers. During the first year the linear programming model was used, it saved Welch's between $130 thousand

and $170 thousand in carrying costs alone by showing Welch's it did not need to purchase extra grapes that were then available. The model has enabled Welch's management to make quick decisions regarding inventories, purchasing grapes, and adjusting product recipes when grape harvests are higher or lower than expected, and when demand changes, resulting in significant cost savings.

Source: E. W. Schuster and S. J. Allen, "Raw Material Management at Welch's, Inc.," *Interfaces* 28, no. 5 (September–October 1998): 13–24.

A Linear Programming Model for Optimal Portfolio Selection at Prudential Securities, Inc.

In the secondary mortgage market, government agencies purchase mortgage loans from the original mortgage lenders and pool them to create mortgage-backed securities (MBSs). These securities are traded in capital markets along with other fixed-income securities such as treasury and corporate bonds. The total size of this market for MBSs is well over $1 trillion. There are a number of types of MBSs. The market for MBSs is maintained by a network of dealers, including Prudential Securities, Inc. These firms and others like it underwrite and issue new market-based securities. This market is somewhat more complex than standard bond investments such as treasury or corporate securities because the principal on mortgages is returned gradually over the life of the security rather than in a lump sum at the end. In addition, cash flows fluctuate because of the homeowner's right to prepay mortgages. In order to deal with these complexities, Prudential Securities has developed and implemented a number of management science models to reduce investment risk and properly value securities for its investors. One such model employs linear programming to design an optimal securities portfolio to meet various investors' criteria under different interest rate environments. The linear programming model determines the amount to invest in different MBSs to meet the client's objectives for portfolio performance. Constraints might include maximum and minimum percentages (of the total portfolio investment) that could be invested in any one or more securities, the duration of the securities, the amount to be invested, and the amount of the different



securities available. This linear programming model and other management science models are used hundreds of times each day at Prudential Securities by traders, salespeople, and clients. The models allow the firm to participate successfully in the mortgage market. Prudential's secondary market trading in MBSs typically exceeds $5 billion per week.

Source: Y Ben-Dow, L. Hayre, and V. Pica, "Mortgage Valuation Models at Prudential Securities," *Interfaces* 22, no. 1 (January–February 1992): 55–71.

Gasoline Blending at Texaco

The petroleum industry first began using linear programming to solve gasoline blending problems in the 1950s. A single grade of gasoline can be a blend of from three to ten different components. A typical refinery might have as many as 20 different components it blends into four or more grades of gasoline. Each grade of gasoline differs according to octane level, volatility, and area marketing requirements.

At Texaco, for example, the typical gasoline blends are Power and unleaded regular, Plus, and Power Premium. These different grades are blended from a variety of available stocks that are intermediate refinery products such as distilled gasoline, reformate gasoline, and catalytically cracked gasoline. Additives include, among other things, lead and other octane enhancers. As many as 15 stocks can be blended to yield up to eight different blends. The properties or attributes of a blend are determined by a combination of properties that exist in the gasoline stocks and those of any additives. Examples of stock properties (that originally emanated from crude oil) include to some extent vapor pressure, sulfur content, aromatic content, and octane value, among other items. A linear programming model determines the volume of each blend subject to constraints for stock availability, demand, and the property (or attribute) specifications for each blend. A single blend may have up to 14 different characteristics. In a typical blend analysis involving 7 input stocks and 4 blends, the problem will include 40 variables and 71 constraints.

Source: C. E. Bodington and T. E. Baker, "A History of Mathematical Programming in the Petroleum Industry," *Interfaces* 20, no. 4 (July–August 1990): 117–27; and, C. W. DeWitt et al., "OMEGA: An Improved Gasoline Blending System for Texaco," *Interfaces* 19, no. 1 (January–February 1989): 85–101.

Analyzing Bank Branch Efficiency with DEA

Data envelopment analysis (DEA) is an application of linear programming used to determine the less productive (i.e., inefficient) service units among a group of similar service units. DEA bases this determination on the inputs (resources) and outputs of the service units.

A major northeastern bank, with 33 branches, used DEA to identify ways to improve its productivity. Bank management did not know how to identify the more efficient branches. The bank identified five resource inputs available to all branches—customer service (tellers), sales service, management, expenses, and office space. Five groups of branch outputs were also identified—deposits, withdrawals, and checking; bank and traveler's checks and bonds; night deposits; loans; and new accounts. The DEA model found that only 10 branches were operating efficiently, whereas 23 branches were using excess resources to provide their volume and mix of services and were, thus, inefficient. The DEA model results provided the basis for reviewing and evaluating branch operations, which revealed operating differences between inefficient branches and best-practice branches. The bank was able to substantially improve its branch productivity and profits, reducing its total branch staff by approximately 20% within one year of the completion of the DEA analysis, and implementing changes in branch operations that resulted in annual savings of over $6 million.

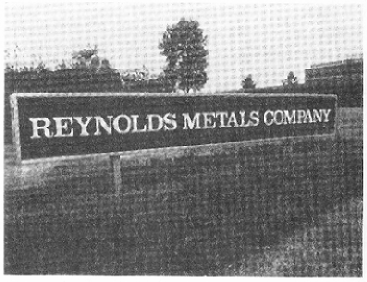
Source: H. D. Sherman and G. Ladino, "Managing Bank Productivity Using Data Envelopment Analysis (DEA)," *Interfaces* 25, no. 2 (March–April 1995): 60–73.

Selecting Freight Carriers at Reynolds Metals Company

In 1991, Reynolds Metals Company spent over $0.25 billion to deliver its products and receive raw materials through a transportation network that included truck, rail, ocean, and air shipments from over 120 shipping locations to over 5,000 receiving destinations. Truck shipments accounted for over half the company's freight costs, and interstate truck shipments alone cost over $80 million.

Reynolds Metals encompasses 12 decentralized operating divisions; and consistent with their decentralized operating philosophy, each division and plant traditionally was responsible for negotiating with and selecting its own carriers and arranging for shipments.

However, because of concerns about variability in service quality and high costs, in 1987 the company developed a new central dispatch system located in Richmond, Virginia, to select all (independent) truck carriers and dispatch them centrally. A vital component of this new central dispatch was a mixed integer programming model that selected the number and type of carrier to be further evaluated for final selection with a simulation model. The objective of the model was to optimize central dispatch freight costs. The model optimally selected a specific number of truck carriers and assigned them to shipping locations. Constraints included a limit on the number of carriers to be selected, carrier limits on the number of trucks they would provide Reynolds, and limits on the number of carriers allowed to service individual locations. A typical model had over 5,000 constraints, 200 integer variables, and 9,000 total variables. Based on the model results, the number of truck carriers used by Reynolds was reduced from over 200 to 14. Savings in freight costs using the entire central dispatch system is over $7 million annually, and on-time delivery service was increased from 80% to 95%.



Source: E. W. Moore Jr., "The Indispensable Role of Management Science in Centralizing Freight Operations at Reynolds Metals Company," *Interfaces* 21, no. 1 (January–February 1991): 107–29.

Minimizing Color Photographic Paper Waste at Kodak

Kodak (Australasia) Pty. Ltd., a division of Eastman Kodak Company, produces rolls of photographic color paper used to produce color photographs. In producing these rolls, the company loses some of the color paper as waste or "trim loss" in the cutting process, which results in a significant cost. The paper is originally purchased in large bulk rolls from 42 to 52.5 inches wide and up to 8,750 feet in length. The paper also comes in three different blends (i.e., different chemical coatings on the paper that react differently to light). Customers, such as one-hour photo-processing shops, order rolls from 3.5 to 11 inches wide and in lengths from 275 to 1,150 feet. To produce the customer rolls from its bulk rolls, the company must cut an entire bulk roll according to a specific design for length and width, which is referred to as a cutting plan. In order to minimize the amount of waste created when these bulk rolls are cut, Kodak developed a system of mixed integer and 0–1 programming models to determine the cutting plans for customer orders. The variables for the mixed integer model included the number of customer order lots and the number of bulk rolls used for a cutting plan, while the 0–1 integer model variables were for the paper blend for the cutting plan used. General benefits of the models included a reduction in trim waste, an increase in productivity, a reduction in the planning effort for diagramming cutting plans, and a reduced planning horizon; in addition, the company was able to better match production to customer orders. The savings in the reduction of waste alone was over $2 million in the first year of operation since waste levels were reduced by 50%.

Source: A. A. Farley, "Planning the Cutting of Photographic Color Paper Rolls for Kodak (Australasia) Pty. Ltd.," *Interfaces* 21, no. 1 (January–February 1991): 92–106.

Optimal Assignment of Gymnasts to Events

An integer programming model was developed to assign members of a women's gymnastics team to the four events conducted at a typical NCAA meet—vault, uneven bars, balance beam, and floor exercises. Each team can enter up to six gymnasts in each event, and the top five scores among these entrants contribute to the team score. At least four of the entrants must participate in all four events. These conditions formed the model constraints; the objective was to maximize the team's overall expected score. The model was tested at Utah State University and allowed officials to analyze the effects of changing conditions, such as improved performance or injuries, on the team score; to indicate to a team member why she was not selected for a particular event; and to eliminate the time the coach spent manually evaluating different team combinations.

Source: P. Ellis and R. Corn. "Using Bivalent Integer Programmin to Select Teams for Intercollegiate Women's Gymnastics Compe tion," *Interfaces* 14, no. 3 (May–June 1984): 41–46.

Fleeting the Schedule at Delta Airlines

Each day, Delta Airlines flies over 2,500 flight legs in the United States, Canada, and Mexico. Delta uses about 450 different aircraft for these flights, arranged into 10 groups or fleets. The pattern of routes that these aircraft fly through the airline's system is the schedule, which is a crucial aspect of an airline's profitability. The schedule must be designed to maximize revenue potential by eliminating empty seats while simultaneously minimizing operating costs, which have historically been high throughout the aircraft industry. A small change in the daily system flight schedule can result in millions of dollars in revenues or losses. As a result, an airline like Delta is continuously refining and attempting to improve its schedule.

Each Delta fleet is made up of different types of aircraft, and the assignment of a particular set of aircraft to specific markets is called *fleeting the schedule*. In 1991 Delta began the Coldstart project to address the problem of fleeting its schedule. The primary goal of the fleeting process is to have the right plane at the right place at the right time; otherwise, it either will not be able to accommodate customer demand or will fly with empty seats. The results of the Coldstart project was a mixed integer linear programming model that assigns fleet types to flight legs to minimize operating and spill costs, where spill is the number of passengers not carried because of insufficient aircraft capacity.

Model operational constraints include the number of aircraf available in each fleet and scheduling requirements. The model is run daily. It normally contains about 40,000 constraints as well as 60,000 variables, about 40,000 of which are integer. The Coldstart model was implemented in September 1992, and by the summer of 1993 it was saving Delta approximately $220,000 per day. The model was expected to save Delta approximately $200 million during its first three years.

Source: R. Subramanian, et al., "Coldstart: Fleet Assignment at Delta Airlines," *Interfaces* 24, no. 1 (January–February 1994), 104–20.

A simple example

- Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains.
- A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by $14.
- A train sells for $21 and uses $9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by $10.

A simple example

- The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing.
- A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor.
- A train requires 1 hour of finishing and 1 hour of carpentry labor.

A simple example

- Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours.
- Demand for trains is unlimited, but at most 40 soldiers are bought each week.
- Giapetto wants to maximize weekly profit (revenues - costs).

Formulation of a mathematical model (1)

- Decision variables:
  - x_1 = number of soldiers produced each week
  - x_2 = number of trains produced each week
- Objective function:
  - Weekly revenues = $27x_1 + 21x_2$
  - Weekly raw material costs = $10x_1 + 9x_2$
  - Other weekly variable costs = $14x_1 + 10x_2$
  - Objective function to maximize =

$$z = 3x_1 + 2x_2$$

Formulation of a mathematical model (2)

- Constraints:
  - Finishing time (100h) : $2x_1 + x_2 \leq 100$
  - Carpentry time (80h): $x_1 + x_2 \leq 80$
  - Soldiers (max 40): $x_1 \leq 40$
- Sign restrictions:

$$x_1 \geq 0 \qquad x_2 \geq 0$$

Linear program (LP)

$$\text{Max } z = 3x_1 + 2x_2$$

$$2x_1 + x_2 \leq 100$$
$$x_1 + x_2 \leq 80$$
$$x_1 \leq 40$$
$$x_1 \geq 0 \qquad x_2 \geq 0$$

Linear programming assumptions

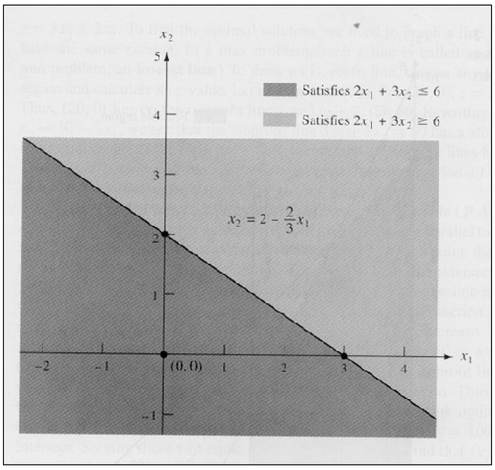
- Proportionality: The contributions of each variable to the objective function and to the constraints are proportional to the value of that variable.
- Additivity: The contributions of each variable to the objective function and to the constraints are independent from the values of the other variables.
- Divisibility: Each decision variable can assume fractional values. (cf. integer programming)
- Certainty: Each parameter is known with certainty.

Some definitions

- Feasible region:
  - Set of all solutions satisfying all the LP's constraints and sign restrictions (feasible points or feasible solutions).
- Optimal solution:
  - Feasible solution that optimizes (max or min) the objective function.
  - Does it exist ?
  - Is it unique ?

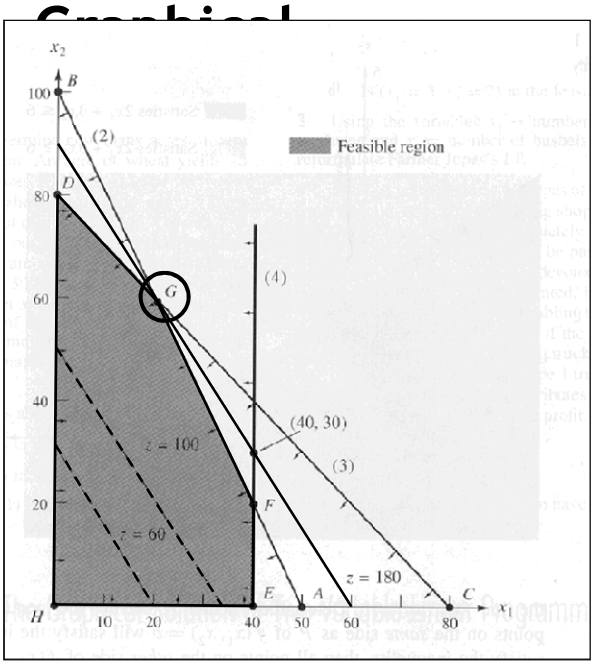
Graphical solution of a 2-variable LP

- Feasible region = region in the plane.
- Constraint: equation = straight line



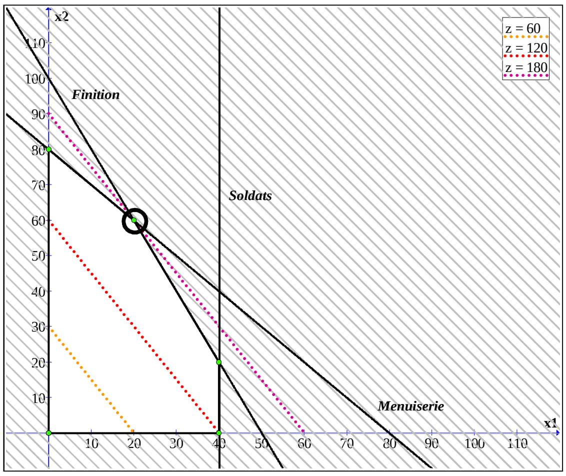
Graphical

- Constraints
- Iso-profit lines
- Optimal solution



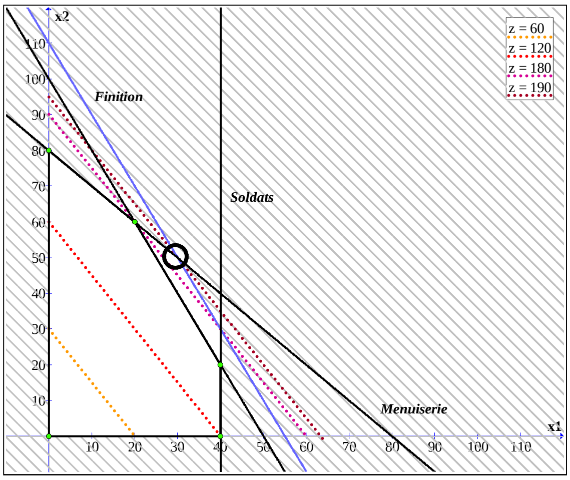
Binding constraint - Finishing

100 hours



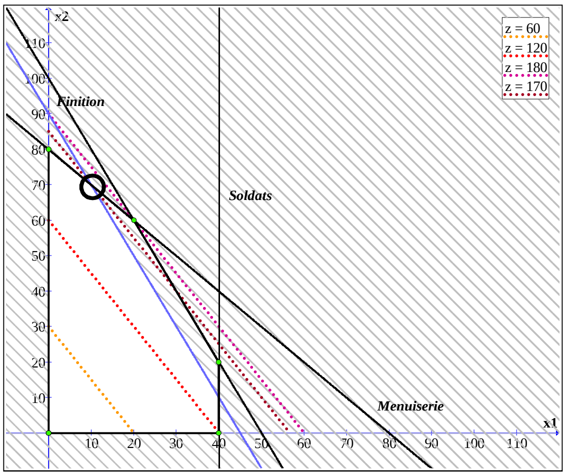
Binding constraint - Finishing

110 hours



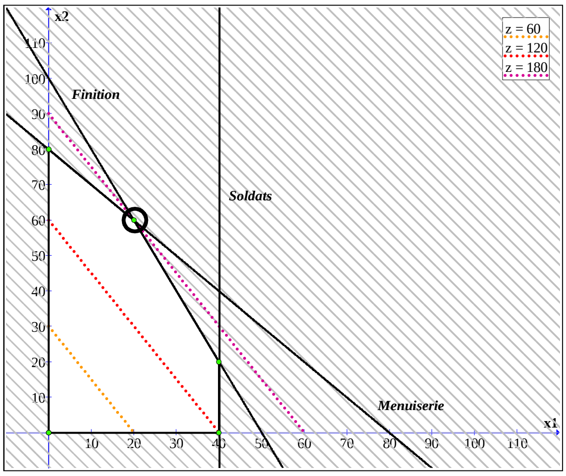
Binding constraint - Finishing

90 hours



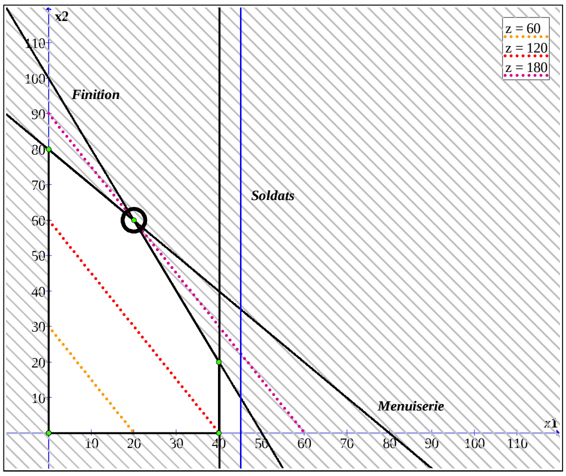
Non-binding constraint - Soldiers

40 soldiers



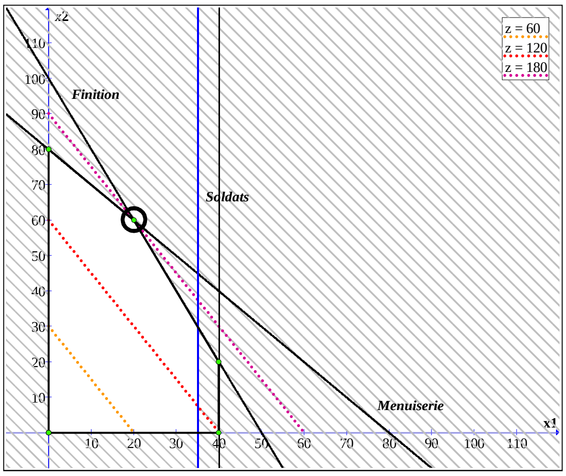
Non-binding constraint - Soldiers

45 soldiers



Non-binding constraint - Soldiers

35 soldiers



Multiple optimal solutions example

An auto company manufactures *cars* and *trucks*. Each vehicle must be processed in the *paint shop* and *body assembly shop*. If the paint shop were only painting trucks, 40 per day could be painted. If the paint shop were only painting cars, 60 per day could be painted. If the body shop were only producing cars, it could process 50 per day. It it were only producing trucks, it could process 50 per day. Each truck produced contributes $300 to *profit*, and each car contributes $200. Determine a *daily production schedule* that will *maximize the company's profit*.

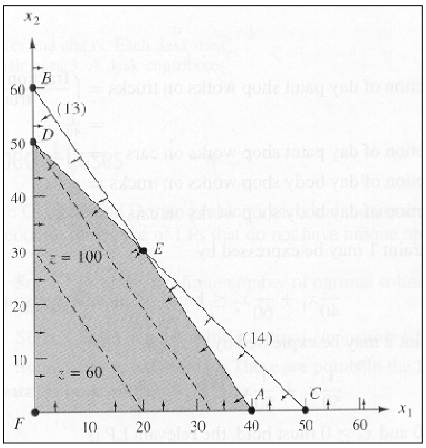
Multiple optimal solutions example

x_1 = number of trucks produced daily
x_2 = number of cars produced daily

$$\max\ z = 300x_1 + 200x_2$$

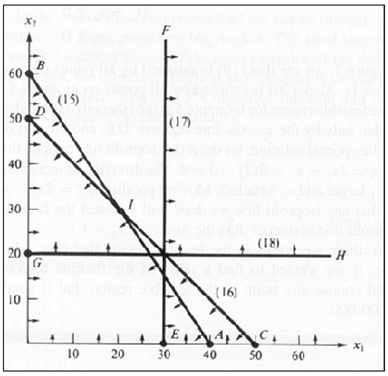
s.t.

$$\frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1$$
$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1$$
$$x_1 \ge 0 \qquad x_2 \ge 0$$



Unfeasible LP example

Suppose now that the auto company is required to product at least 30 trucks and 20 cars per day.
→ 2 additional constraints:
$x_1 \geq 30 \quad x_2 \geq 20$



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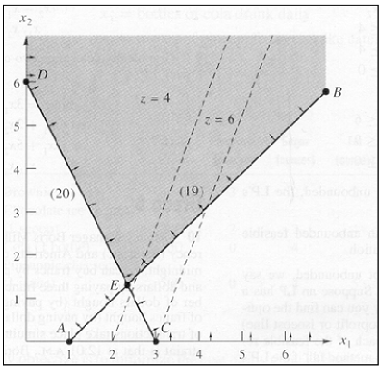
Unbounded LP example

$$\max\ z = 2x_1 - x_2$$

s.t.

$$x_1 - x_2 \leq 1$$
$$2x_1 + x_2 \geq 6$$
$$x_1 \geq 0 \qquad x_2 \geq 0$$



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Optimal solution

- Can be:
  - Unique → extreme point of the feasible region,
  - Multiple → side of the feasible region,
  - Unbounded (missing constraints),
  - Non-existent (conflicting constraints).
- Some contraints are binding:
  LHS = RHS
- Some contraints are nonbinding:
  LHS ≠ RHS (difference = slack)

Typical LP's: Diet problems

Four foods are available for consumption: brownies, chocolate ice cream, cola and pineapple cheesecake. One brownie costs ¢50, one scoop of chocolate ice cream costs ¢20, one bottle of cola costs ¢30, and one piece of cheesecake costs ¢80. Each day, you must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar and 8 oz of fat. Formulate a LP to satisfy these requirements at minimum cost.

Per unit:	Calories	Chocolate	Sugar	Fat
Brownie	400	3	2	2
Chocolate ice cream	200	2	2	4
Cola	150	0	4	1
Cheesecake	500	0	4	5

Typical LP's: Diet problems

x_1	number of brownies
x_2	number of scoops of chocolate ice cream
x_3	bottles of cola
x_4	pieces of pineapple cheesecake

- LP formulation:

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$\begin{array}{rrrrcr} 400x_1 & +200x_2 & +150x_3 & +500x_4 & \geq & 500 \\ 3x_1 & +2x_2 & & & \geq & 6 \\ 2x_1 & +2x_2 & +4x_3 & +4x_4 & \geq & 10 \\ 2x_1 & +4x_2 & +x_3 & +5x_4 & \geq & 8 \end{array}$$

$$x_i \geq 0 \quad (i = 1,2,3,4)$$

- LP solution:

$$x_1 = 0 \quad x_2 = 3 \quad x_3 = 1 \quad x_4 = 0 \quad z = 90$$

- Slacks:

$$t_1 = 250 \quad t_2 = 0 \quad t_3 = 0 \quad t_4 = 5$$

Typical LP's: Work scheduling problems

A fast food restaurant requires different numbers of full-time employees on different days of the week.

Day	Full-time employees
1: Monday	17
2: Tuesday	13
3: Wednesday	15
4: Thursday	19
5: Friday	14
6: Saturday	16
7: Sunday	11

Each full-time employee must work five consecutive days and then receive two days off. The manager wants to use only full-time employees. Formulate a LP to minimize the number of full-time employees that must be hired.

Typical LP's: Work scheduling problems

x_i = number of employees beginning work on day i

• LP formulation:

$$\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

$$\begin{array}{lllllllll}
x_1 & & & +x_4 & +x_5 & +x_6 & +x_7 & \geq & 17 \\
x_1 & +x_2 & & & +x_5 & +x_6 & +x_7 & \geq & 13 \\
x_1 & +x_2 & +x_3 & & & +x_6 & +x_7 & \geq & 15 \\
x_1 & +x_2 & +x_3 & +x_4 & & & +x_7 & \geq & 19 \\
x_1 & +x_2 & +x_3 & +x_4 & +x_5 & & & \geq & 14 \\
 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & & \geq & 16 \\
 & & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & \geq & 11
\end{array}$$

$$x_i \geq 0 \quad (i = 1, 2, \ldots, 7)$$

• LP solution:

$x_1 = \frac{4}{3}$ $x_2 = \frac{10}{3}$ $x_3 = 2$ $x_4 = \frac{22}{3}$ $x_5 = 0$ $x_6 = \frac{10}{3}$ $x_7 = 5$ $z = \frac{67}{3}$

• Rounded up solution:

$x_1 = 2$ $x_2 = 4$ $x_3 = 2$ $x_4 = 8$ $x_5 = 0$ $x_6 = 4$ $x_7 = 5$ $z = 25$

• IP optimal solution:

$x_1 = 4$ $x_2 = 4$ $x_3 = 2$ $x_4 = 6$ $x_5 = 0$ $x_6 = 4$ $x_7 = 3$ $z = 23$

Typical LP's: Capital budgeting problems

Star Oil Company is considering 5 different investment opportunities. The cash outflows and net present values (in M$) are the following:

	Inv.1	Inv.2	Inv.3	Inv.4	Inv.5
Time 0 cash outflow	$11	$53	$5	$5	$29
Time 1 cash outflow	$3	$6	$5	$1	$34
NPV	$13	$16	$16	$14	$39

Star Oil has $40 million available for investment at the present time (time 0); it estimates that one year from now (time 1) $20 million will be available for investment. Star Oil may purchase any fraction of each investment. In this case, the cash outflows and NPV are adjusted accordingly. Star Oil wants to maximize the NPV that can be obtained by investment. Formulate an LP to achieve this goal. Any funds left over at time 0 cannot be used at time 1.

Typical LP's: Capital budgeting problems

x_i = fraction of investment i purchased by Star Oil

- LP formulation:

$$\max z = 13x_1 + 16x_2 + 16x_3 + 14x_4 + 39x_5$$

$$\begin{array}{llllll} 11x_1 & +53x_2 & +5x_3 & +5x_4 & +29x_5 & \leq 40 \\ 3x_1 & +6x_2 & +5x_3 & +x_4 & +34x_5 & \leq 20 \\ x_1 & & & & & \leq 1 \\ & x_2 & & & & \leq 1 \\ & & x_3 & & & \leq 1 \\ & & & x_4 & & \leq 1 \\ & & & & x_5 & \leq 1 \end{array}$$

$$x_i \geq 0 \quad (i = 1, 2, \ldots, 5)$$

- LP solution:

$x_1 = 1 \quad x_2 = 0.201 \quad x_3 = 1 \quad x_4 = 1 \quad x_5 = 0.288 \quad z = 57.449$

Typical LP's: Blending problems

Sunco Oil manufactures 3 types of gasoline (gas 1, gas 2 and gas 3). Each type is produced by blending 3 types of crude oil (crude 1, crude 2, and crude 3). The sales price per barrel of gasoline and the purchase price per barrel of crude oil are as follows:

	Sales price		Purchase price
Gas 1	$70	Crude 1	$45
Gas 2	$60	Crude 2	$35
Gas 3	$50	Crude 3	$25

Sunco can purchase up to 5000 barrels of each type of crude oil daily.

The 3 types of gasoline differ in their octane rating and sulfur content. Gas 1 must have an octane rating of at least 10 and contain at most 1% of sulphur. Gas 2 must have an octane rating of at least 8 and contain at most 2% of sulphur. Gas 3 must have an octane rating of at least 6 and contain at most 1% of sulphur. It costs $4 to transform one barrel of oil into one barrel of gasoline, and Sunco's refinery can produce up to 14,400 barrels of gasoline daily.

	Octane rating	Sulfur content
Crude 1	12	0.5%
Crude 2	6	2.0%
Crude 3	8	3.0%

Sunco's customers require the following amounts of each gasoline: gas 1 – 3000 barrels per day, gas 2 – 2000 barrels per day, gas 3 – 1000 barrels per day. The company wants to meet these demands. Sunco has also the option of advertising to stimulate demand for its products. Each dollar spent daily in advertising a particular type of gas increases the daily demand for that type of gas by 10 barrels.

Formulate an LP to maximize the daily profits (revenues – costs) of Sunco.

Typical LP's: Blending problems

a_i = dollars spent daily on advertising gas i

x_{ij} = barrels of crude oil i used daily to produce gas j

- LP formulation:

x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	a_1	a_2	a_3	
21	11	1	31	21	11	41	31	21	-1	-1	-1	MAX
1	0	0	1	0	0	1	0	0	-10	0	0	= 3000
0	1	0	0	1	0	0	1	0	0	-10	0	= 2000
0	0	1	0	0	1	0	0	1	0	0	-10	= 1000
1	1	1	0	0	0	0	0	0	0	0	0	≤ 5000
0	0	0	1	1	1	0	0	0	0	0	0	≤ 5000
0	0	0	0	0	0	1	1	1	0	0	0	≤ 5000
1	1	1	1	1	1	1	1	1	0	0	0	≤14000
2	0	0	-4	0	0	-2	0	0	0	0	0	≥ 0
0	4	0	0	-2	0	0	0	0	0	0	0	≥ 0
-0.005	0	0	0.01	0	0	0	0	0.02	0	0	0	≤ 0
0	-0.015	0	0	0	0	0	0.01	0	0	0	0	≤ 0
0	0	-0.005	0	0	0.01	0	0	0.02	0	0	0	≤ 0

- LP solution:

$$z = 287{,}500$$

$$x_{11} = 2222.22 \quad x_{12} = 2111.11 \quad x_{13} = 666.67$$

$$x_{21} = 444.44 \quad x_{22} = 4222.22 \quad x_{23} = 333.34$$

$$x_{31} = 333.33 \quad x_{23} = 3166.67 \quad x_{33} = 0$$

$$a_1 = 0 \quad a_2 = 750 \quad a_3 = 0$$

Multi-period decision problems: An inventory model

Sailco Corp. must determine how many sailboats should be produced during each of the next four quarters. The demand during each quarter is as follows: Q1 – 40 sailboats, Q2 – 60, Q3 – 75, Q4 – 25. Sailco must meet demands on time. At the beginning of Q1, Sailco has an inventory of 10 sailboats. At the beginning of each quarter, Sailco must decide how many sailboats should be produced during this quarter. We assume that sailboats manufactured during a quarter can be used to meet demand for that quarter. During each quarter, Sailco can produce up to 40 sailboats with regular-time labor at a total cost of $400 per sailboat. Additional boats can be produced with overtime labor at a total cost of $450 per sailboat.

At the end of each quarter (after demand has been satisfied), a carrying or holding cost of $20 per sailboat is incurred. Use LP to schedule production to minimize the sum of production and inventory costs during the next four quarters.

Multi-period decision problems: An inventory model

x_t = number of sailboats produced (regular-time) during Qt

y_t = number of sailboats produced (overtime) during Qt

i_t = number of sailboats on hand at the end of Qt

- LP formulation:

$$\min 400(x_1+x_2+x_3+x_4)+450(y_1+y_2+y_3+y_4)+20(i_1+i_2+i_3+$$

$$x_1 \le 40 \quad x_2 \le 40 \quad x_3 \le 40 \quad x_4 \le 40$$

$$i_1 = 10 + x_1 + y_1 - 40 \quad i_2 = i_1 + x_2 + y_2 - 60$$

$$i_3 = i_2 + x_3 + y_3 - 75 \quad i_4 = i_3 + x_4 + y_4 - 25$$

$$x_t \ge 0 \quad y_t \ge 0 \quad i_t \ge 0 \quad (t = 1,2,3,4)$$

- LP solution: $z = 78,450$

$$x_1 = x_2 = x_3 = 40 \quad x_4 = 25$$

$$y_1 = 0 \quad y_2 = 10 \quad y_3 = 35 \quad y_4 = 0$$

$$i_1 = 10 \quad i_2 = i_3 = i_4 = 0$$

Multiperiod financial models

Finco Investment Corp. must determine investment strategy for the firm during the next three years. At present time (time 0), $100,000 is available for investment. Investments A, B, C, D and E are available. The cash flow associated with investing $1 in each investment are:

	0	1	2	3
A	-$1	+$0.50	+$1	$0
B	$0	-$1	+$0.50	+$1
C	-$1	+$1.2	$0	$0
D	-$1	$0	$0	+$1.9
E	$0	$0	-$1	+$1.5

To ensure that the company's portfolio is diversified, Finco requires that at most $75,000 be placed in a single investment. In addition to investments A-E, Finco can earn interest at 8% per year by keeping univested cash in money market funds. Returns from investments may be immediately reinvested. Finco cannot borrow funds. Formulate an LP to maximize cash on hand at time 3.

Multiperiod financial models

A = dollars invested in A
B = dollars invested in B
C = dollars invested in C
D = dollars invested in D
E = dollars invested in E
S_t = dollars invested in MM funds at time t

- LP formulation:

$$\max z = B + 1.9D + 1.5E + 1.08S_2$$
$$A + C + D + S_0 = 100{,}000$$
$$0.5A + 1.2C + 1.08S_0 = B + S_1$$
$$A + 0.5B + 1.08S_1 = E + S_2$$
$$A \le 75{,}000 \quad B \le 75{,}000 \quad C \le 75{,}000 \quad D \le 75{,}000 \quad E \le 75{,}0$$
$$A, B, C, D, E, S_1, S_2 \ge 0$$

Some definitions

- Linear program:

$$\text{Max } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad (1)$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \end{cases} \quad (2)$$

$$x_j \ge 0 \quad j = 1, 2, \ldots, n \quad (3)$$

Definitions

- Objective function (max or min) :

  Min z = – Max (– z)

- Constraints : ≤ or ≥ or =

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i$$
$$\Updownarrow$$
$$-a_{i1}x_1 - a_{i2}x_2 - \dots - a_{in}x_n \leq -b_i$$

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$$
$$\Updownarrow$$
$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \end{cases}$$

Notations

- Standard LP form:

$$\text{Min } z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad (1)$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \end{cases} \quad (2)$$

$$x_j \geq 0 \quad j = 1, 2, \dots, n \quad (3)$$

$$\text{Min } z = \sum_{j=1}^{n} c_j x_j$$
$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \leq b_i , \quad i = 1, 2, \dots, m \\ x_j \geq 0 , \quad j = 1, 2, \dots, n \end{cases}$$

où $c_1, c_2, \dots, c_n, a_{11}, a_{12}, \dots, a_{mn}, b_1, b_2, \dots, b_m \in R$

Notations

- Vector notation:

$$\text{Min } z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{cases} \sum_{j=1}^{n} x_j P_j \le P_0 \\ x_j \ge 0 , \quad j = 1, 2, \ldots, n \end{cases} \qquad P_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} , \quad \cdots , \quad P_0 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- Matrix notation: $\text{Min } \{ CX \mid AX \le b , X \ge 0 \}$

$$C' = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Slack variables

- To transform unequalities into equations.

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i$$
$$\Updownarrow$$
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + t_i = b_i$$
avec $t_i \ge 0$

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ge b_i$$
$$\Updownarrow$$
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - t_i = b_i$$
avec $t_i \ge 0$

- All constraints can be changed to equations, with additional slack variables $\rightarrow$ 2nd LP standard form.

Example 2

$$\text{Max } z = 40x_1 + 30x_2$$

$$\begin{aligned} x_1 \quad\quad &\leq 8 \\ x_2 &\leq 6 \\ x_1 + 2x_2 &\leq 15 \\ 2x_1 + x_2 &\leq 18 \end{aligned}$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

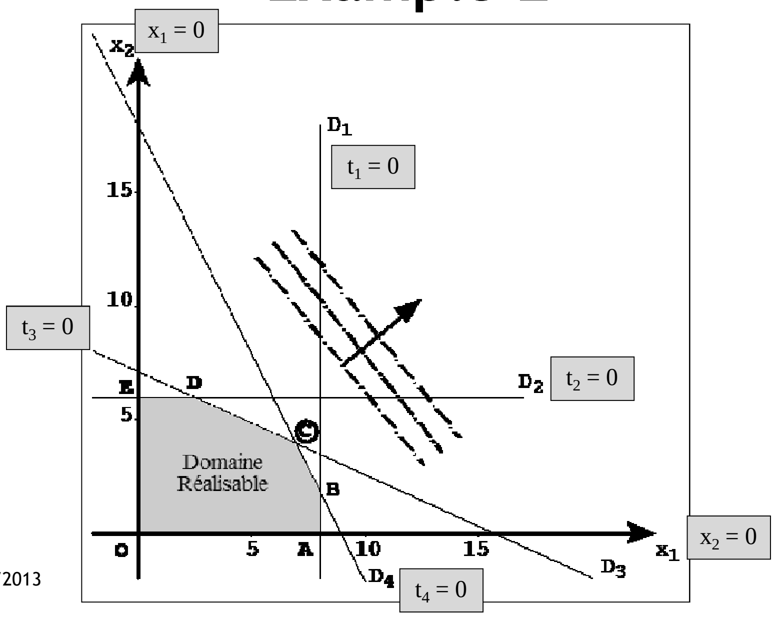
⇓⇑

$$\text{Max } z = 40x_1 + 30x_2$$

$$\begin{aligned} x_1 + t_1 &= 8 \\ x_2 + t_2 &= 6 \\ x_1 + 2x_2 + t_3 &= 15 \\ 2x_1 + x_2 + t_4 &= 18 \end{aligned}$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad t_1 \geq 0 \quad t_2 \geq 0 \quad t_3 \geq 0 \quad t_4 \geq 0$$

Example 2



Example 2 - Remarks

- The feasible region is a convex polygon.
- Constraint = side of the polygon = one variable is equal to 0.
- Vertex = intersection of two sides = two variables equal to 0.
- $n = 6,\ m = 4$

Basis

- Non singular submatrix B, $m \times m$, of A.
- Columns of B :
  - $B = \{ P_{j1}, P_{j2}, \dots, P_{jm} \}$.
  - $I(B) = \{ j_1, j_2, \dots, j_m \}$ = basic indices.
  - $x_{j1}, x_{j2}, \dots, x_{jm}$: **basic variables.**
  - $J(B)$ = non basic indices $\rightarrow$ **non basic variables.**

Basic solution

- Solution obtained from a basis B, by setting all $n-m$ non basic variables equal to 0 and solving the resulting basic system:

$$AX = b \;\Rightarrow\; BX_B = b \;\Rightarrow\; X_B = B^{-1} b$$

- Special cases:
  - Feasible basic solution (f.b.s.): $X_B \geq 0$
  - Explicit basic solution: if $B = I$, $\Rightarrow\; X_B = b$

Example 2 : bases

$$\text{Max } z = 40x_1 + 30x_2$$

$$\begin{array}{rcrcl} x_1 & & & \leq & 8 \\ & & x_2 & \leq & 6 \\ x_1 & + & 2x_2 & \leq & 15 \\ 2x_1 & + & x_2 & \leq & 18 \end{array}$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

⇓⇑

$$\text{Max } z = 40x_1 + 30x_2$$

$$\begin{array}{rcrcccccccl} x_1 & & & + & t_1 & & & & & = & 8 \\ & & x_2 & & & + & t_2 & & & = & 6 \\ x_1 & + & 2x_2 & & & & & + & t_3 & = & 15 \\ 2x_1 & + & x_2 & & & & & & + t_4 & = & 18 \end{array}$$

$$x_1 \geq 0 \; x_2 \geq 0 \quad t_1 \geq 0 \; t_2 \geq 0 \; t_3 \geq 0 \; t_4 \geq 0$$

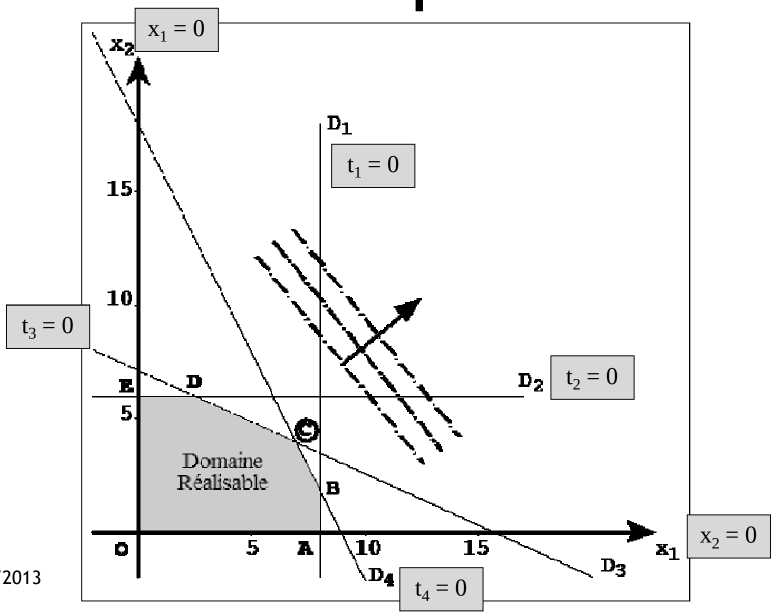
Example 2 - Remarks

- The feasible region is a convex polygon.
- Constraint = side of the polygon = one variable is equal to 0.
- Vertex = intersection of two sides = two variables equal to 0.
- $n = 6,\ m = 4$
- Basic solution: $n–m = 2$ non-basic variables (equal to 0) !

f.b.s. for example 2

Enumération des Solutions de Base								
Sommet	x_1	x_2	t_1	t_2	t_3	t_4	z	Solution
O	0	0	8	6	15	18	0	s.b.r.
	0		0				-	-
E	0	6	8	0	3	12	180	s.b.r.
	0	7,5	8	-1,5	0	10,5	-	s.b.
	0	18	8	-12	-21	0	-	s.b.
A	8	0	0	6	7	2	320	s.b.r.
		0		0			-	-
	15	0	-7	6	0	-12	-	s.b.
	9	0	-1	6	6	0	-	s.b.
	8	6	0	0	-5	-4	-	s.b.
	8	3,5	0	2,5	0	-1,5	-	s.b.
B	8	2	0	4	3	0	380	s.b.r.
D	3	6	5	0	0	6	300	s.b.r.
	6	6	2	0	-3	0	-	s.b.
C	7	4	1	2	0	0	400	s.b.r.≡s.o.

Example 2

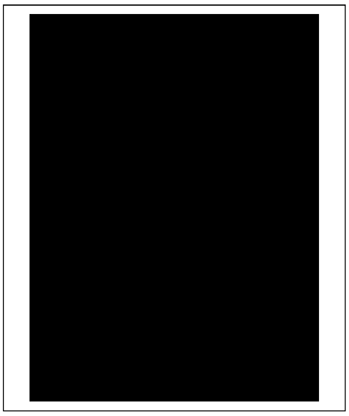


Theoretical results

- If the feasible region is bounded and non-empty, there is at least one vertex of the feasible region that is an optimal solution of the LP.
- There is a 1-1 relation between the vertices of the feasible region and the feasible basic solutions.

Simplex algorithm

- Principle :
  To go from vertex (f.b.s.) to vertex improving each time the value of the objective function z, until either an optimal solution is obtained or it appears that the problem is not bounded.



Simplex algorithm

(included in Excel Solver a.o.)

- Iterative algorithm (1948).
- Finds optimal basic solution (extreme point of feasible region).
- Detects unbounded solutions and unfeasibility.
- Provides additional results:
  Post-optimal analysis (sensitivity analysis)
  - **Reduced costs**:
    impact of a unit increase of a non-basic variable (0 → 1) on the optimal value of the objective function.
  - **Shadow prices**:
    impact of a unit change of the RHS of a constraint on the optimal value of the objective function.

Optimization algorithms

- Continuous variables (LP) :
  - Simplex,
  - «Interior point» algorithms.
- Integer variables (IP) :
  - «Branch and bound» algorithms,
  - «Branch and cut» algorithms
- Mixed programs (continuous-integer)

Giapetto and Excel's solver

Microsoft Excel 11.0 Rapport des réponses
Feuille: [LPexemples.xls]Giapetto
Date du rapport: 16/10/2005 18:50:49

Cellule cible (Max)

Cellule	Nom	Valeur initiale	Valeur finale
$D$4	Fonction écon. MAX	180	180

I.

Cellules variables

Cellule	Nom	Valeur initiale	Valeur finale
$B$3	x1	20	20
$C$3	x2	60	60

Contraintes

II.

Cellule	Nom	Valeur	Formule	État	Marge
$D$6	Finition MAX	100	$D$6<=$F$6	Lié	0
$D$7	Menuiserie MAX	80	$D$7<=$F$7	Lié	0
$D$8	Soldats MAX	20	$D$8<=$F$8	Non lié	20
$B$3	x1	20	$B$3>=0	Non lié	20
$C$3	x2	60	$C$3>=0	Non lié	60

Results

I. Optimal solution found by the solver.
II. Binding and nonbinding constraints, slack variables.

Giapetto and Excel's solver

Microsoft Excel 11.0 Rapport de la sensibilité
Feuille: [LPexemples.xls]Giapetto
Date du rapport: 16/10/2005 18:50:49

III. IV.

Cellules variables

Cellule	Nom	Finale Valeur	Réduit Coût	Objectif Coefficient	Admissible Augmentation	Admissible Réduction
$B$3	x1	20	0	3	1	1
$C$3	x2	60	0	2	1	0,5

Contraintes

Cellule	Nom	Finale Valeur	Ombre Coût	Contrainte à droite	Admissible Augmentation	Admissible Réduction
$D$6	Finition MAX	100	1	100	20	20
$D$7	Menuiserie MAX	80	1	80	20	20
$D$8	Soldats MAX	20	0	40	1E+30	20

V. VI.

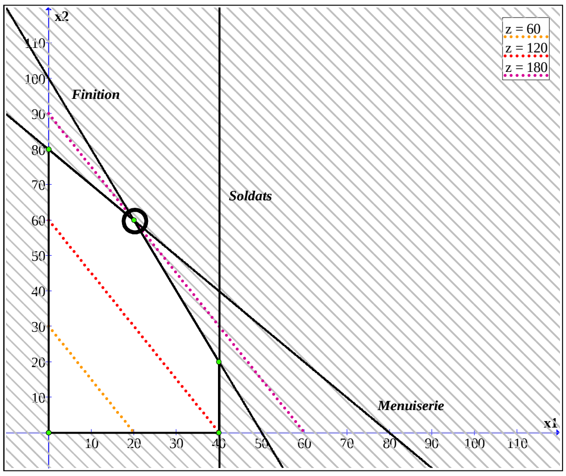
Results

III. Reduced costs for non-basic variables.
IV. Stability intervals for the optimal basic solution (objective function coefficients).
V. Shadow prices of the constraints.
VI. Validity intervals for the shadow prices.

Objective function - Trains

2€/train

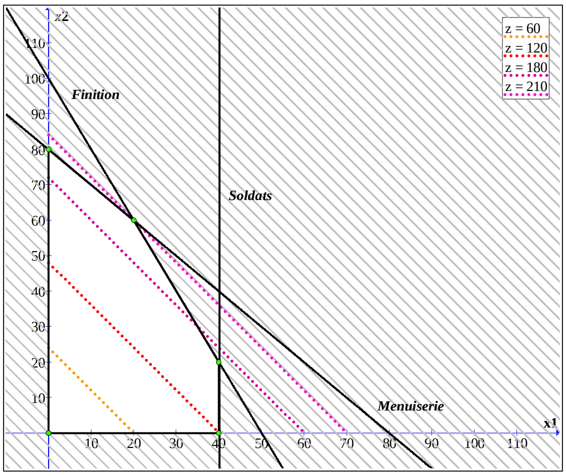
z = 180



Objective function - Trains

2,5€/train

z = 210

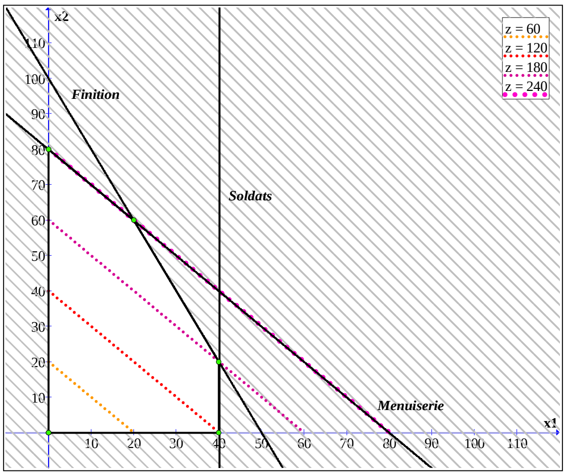


Objective function - Trains

3€/train

z = 240

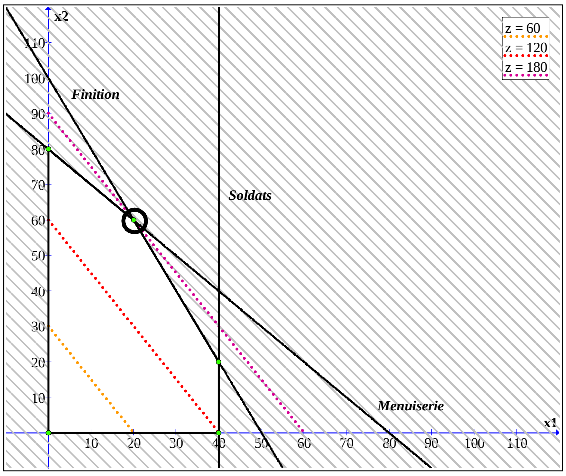
multiple o.s.



Objective function - Trains

2€/train

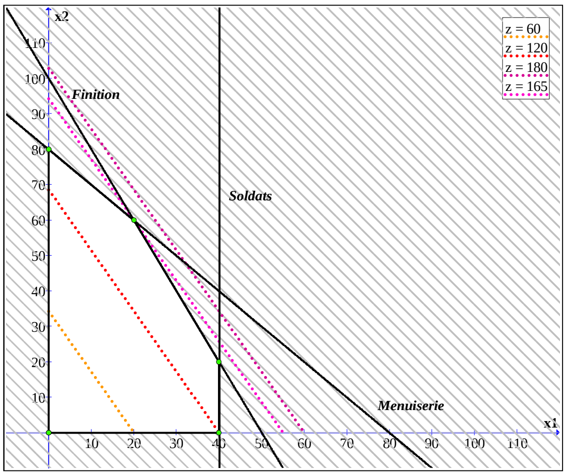
z = 180



Objective function - Trains

1,75€/train

z = 165

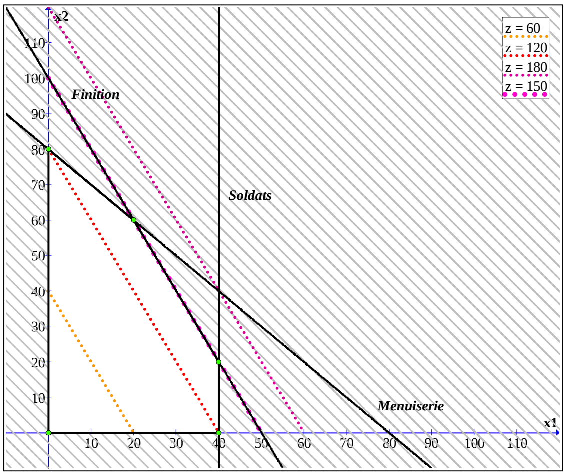


Objective function - Trains

1,5€/train

z = 150

multiple o.s.

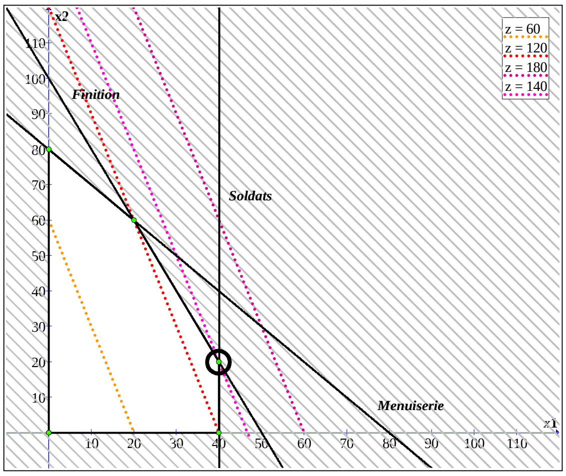


Objective function - Trains

1€/train

z = 140

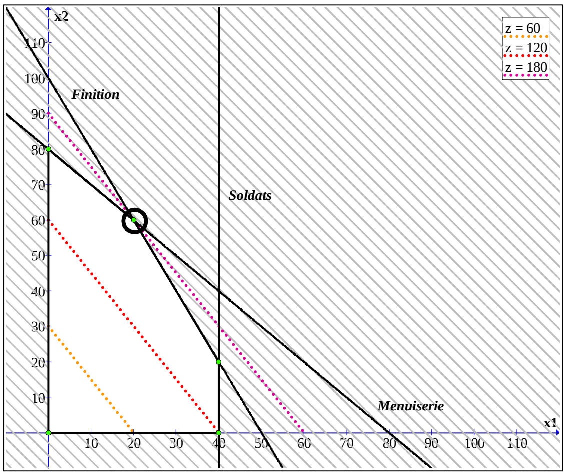
New o.s.



Shadow price - Finishing

100 hours

PM = 1



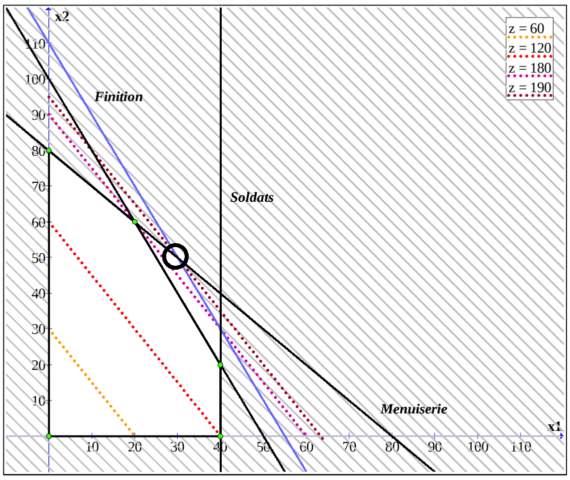
2012/2013

Shadow price - Finishing

110 hours

PM = 1

z = 190



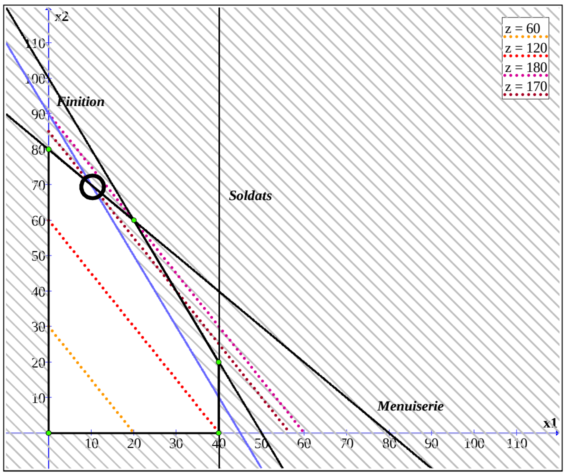
2012/2013

Shadow price - Finishing

90 hours

PM = 1

z = 170



Giapetto and Excel's solver

Microsoft Excel 11.0 Rapport de la sensibilité
Feuille: [LPexemples.xls]Giapetto
Date du rapport: 16/10/2005 18:50:49

III. IV.

Cellules variables

Cellule	Nom	Finale Valeur	Réduit Coût	Objectif Coefficient	Admissible Augmentation	Admissible Réduction
$B$3	x1	20	0	3	1	1
$C$3	x2	60	0	2	1	0,5

Contraintes

Cellule	Nom	Finale Valeur	Ombre Coût	Contrainte à droite	Admissible Augmentation	Admissible Réduction
$D$6	Finition MAX	100	1	100	20	20
$D$7	Menuiserie MAX	80	1	80	20	20
$D$8	Soldats MAX	20	0	40	1E+30	20

V. VI.

…

- Big Mac Example

- MPL4 solver:

  http://www.maximal-usa.com

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Typical LP's

- Allocation models.
- Covering models.
- Blending models.
- Network models:
  - Transportation model,
  - Assignment model,
  - Transshipment model.

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Covering model - example

Herrick Foods Company

Herrick Foods Company wishes to introduce packaged trail mix as a new product. The ingredients for the trail mix are seeds, raisins, flakes, and two kinds of nuts. Each ingredient contains certain amounts of vitamins, minerals, protein, and calories; the marketing department has specified the product be designed so that a certain minimum nutritional profile is met. The decision problem is to minimize the product cost and determine the product composition—that is, by choosing the amount of each ingredient in the mix. The data shown below summarize the parameters of the problem.

	Grams/Pound					Nutritional Requirement
	SEEDS	RAISINS	FLAKES	PECANS	WALNUTS	
Vitamins	10	20	10	30	20	16
Minerals	5	7	4	9	2	10
Protein	1	4	10	2	1	15
Calories	500	450	160	300	500	600
Cost/Pound ($)	4	5	3	7	6	

Blending model - example

Keogh Coffee Roasters

Keogh Coffee Roasters blends three types of coffee beans (Brazilian, Colombian, and Peruvian) into ground coffee that is sold at retail. Each kind of bean has a distinctive aroma and taste, and the company has a chief taster who can rate the fragrance of the aroma and the strength of the taste on a scale of 1 to 100. The features of the beans are tabulated below:

Bean	Aroma Rating	Strength Rating	Cost/Pound ($)
Brazilian	75	15	.50
Colombian	60	20	.60
Peruvian	85	18	.70

Keogh would like to create a blend that has an aroma rating of at least 78 and a strength rating of at least 16. Its supplies of the various beans are limited, however. The available quantities are 1500 pounds of Brazilian, 1200 pounds of Colombian, and 2000 pounds of Peruvian beans, all delivered under a previously arranged purchase agreement. Keogh wants to make 4000 pounds of the blend at the lowest possible cost. ■

Transportation model - example

Goodwin Manufacturing Company

Goodwin Manufacturing Company is planning next week's shipments from its three manufacturing plants to its four distribution warehouses and seeking a minimum-cost shipping schedule. Each plant has a potential capacity, expressed in cartons of product, and each warehouse has a demand requirement for the week that must be met. There are 12 possible shipment routes, and for every plant–warehouse combination, the unit shipping cost is known. The following table provides the given information:

	To: Warehouse				
From: Plant	ATLANTA	BOSTON	CHICAGO	DENVER	Capacity
Minneapolis	$0.60	$0.56	$0.22	$0.40	10,000
Pittsburgh	$0.36	$0.30	$0.28	$0.58	15,000
Tucson	$0.65	$0.68	$0.55	$0.42	15,000
Requirement	8,000	10,000	12,000	9000	

Assignment model - example

Europa Auto Company

Europa Auto Company is an automaker with six manufacturing plants and six vehicle types to produce this year. The firm has learned that it makes sense to produce each vehicle at a unique plant, even though some of the plants are older and less efficient than others. For each possible assignment of a vehicle to a plant, the firm has estimated the annual cost (in millions of dollars) of implementing the assignment. The cost data take the form shown in the following table, which lists plant locations and identifies the products by number. The automaker's objective is to minimize the total cost of the assignment.

	Compact (1)	Coupe (2)	Sedan (3)	SUV (4)	Truck (5)	Van (6)
Akron	80	56	43	62	46	58
Buffalo	94	50	88	64	63	52
Columbus	94	46	50	40	55	73
Detroit	98	79	71	65	91	59
Evansville	61	59	89	98	45	52
Flint	77	49	65	95	72	91

Transshipment model - example

DeMont Chemical Company

DeMont Chemical Company manufactures its fertilizer in three plants, referred to as P1, P2, and P3. The company ships its products from plants to two central DCs, designated D1 and D2, and then from the DCs to five regional warehouses, W1 through W5. No demand occurs at the DCs, and there are no capacity limits at the DCs.

Demand is associated with the warehouses, and capacities exist at the plants. Data describing the system are shown in the following two tables, one for each stage of the system. The units for capacity and demand are pounds of fertilizer, and the unit costs are given per pound.

	To DC		
From Plant	D1	D2	Capacity (lb)
P1	$1.36	$1.22	2400
P2	$1.28	$1.35	2750
P3	$1.68	$1.55	2500

	To Warehouse				
From DC	W1	W2	W3	W4	W5
D1	$0.60	$0.36	$0.22	$0.44	$0.72
D2	$0.80	$0.56	$0.42	$0.40	$0.55
Requirement (lb)	1250	1000	1600	1750	1500

Integer programming (IP)

- Integer programming model:
  - Some variables are integer variables.
  - Some variables are binary variables (yes/no).
  - Expression of logical or qualitative constraints by binary variables.
- Branch and bound procedure

Integer variables

Callum Communications

Callum Communications runs a small call center that operates 7 days a week. Callum requires a specified minimum number of employees to be at work each day, to provide the necessary level of customer service. Under union regulations, employees at the call center must all work full-time schedules, which means 5 consecutive workdays and 2 days off per week. Furthermore, employees whose regular schedules include a weekend day receive a pay premium. Specifically, employees who work 5 weekdays are paid $400 per week. Employees who work one of the weekend days are paid $440, and employees who work both of the weekend days are paid $470. The minimum daily staffing requirements for workers are described in the following table:

Day	Sun.	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.
Requirement	16	18	18	17	13	8	5

Callum's management wishes to minimize the cost of salaries paid to the workforce while meeting the staffing requirements. ■

Binary variables

Newton Corporation

Division A of Newton Corporation has been allocated $40 million for capital projects this year. Managers in division A have examined various possibilities and have proposed five projects for the capital-budgeting committee to consider. The projects cover a variety of activities and functional areas, and there is just one of each type. The projects are the following:

P1: Renovate the production facility for greater efficiency.

P2: License a new technology for use in production.

P3: Expand advertising by naming a stadium.

P4: Purchase land and construct a new headquarters building.

P5: Introduce a new product to complement the current line.

Each project has an estimated net present value (NPV), and each requires a capital expenditure, which must come out of the budget for capital projects. The following table summarizes the possibilities, as they have been provided to the committee, with all figures in millions of dollars.

Project	P1	P2	P3	P4	P5
NPV	2.0	3.6	3.2	1.6	2.8
Expenditure	12	24	20	8	16

The committee would like to maximize the total NPV from projects selected, subject to a $40-million limit on capital expenditures. ■

Qualitative constraints

- Additional constraints:
  1. Select at least one project from the international area (P2 or P5).
  2. P2 and P5 are mutually exclusive.
  3. P5 requires that P3 be selected.

The matching problem

Oxbridge College

Oxbridge College faces the problem of devising an exam schedule at the end of every term. By tradition, the exam period lasts 4 days, and exams are scheduled in the morning of each day. In other words, there are four available exam periods. To create an exam schedule, the college registrar assigns courses to exam days according to the course meeting time. Thus, all courses that meet Monday at 9 a.m. are assigned the same exam day. There are eight distinct meeting times in the college calendar, and the registrar wants to assign two to each of the four exam days. However, when two times are assigned to the same exam day, there may be some students who have an exam conflict because they are taking courses that meet at those two times. Special arrangements have to be made for such students. The registrar's office has an information system that can determine, for any pair of class times, how many students are taking courses at both times. With this information, the registrar would like to devise an exam schedule that makes the number of exam conflicts as small as possible because that minimizes the number of cases in which special arrangements have to be made. ∎

ModelSheets

Set-covering problem

DixieNet Company

DixieNet is an Internet service provider for residential consumers in a southern state. The company is small now but plans to expand. Its first major goal is to establish a set of hubs throughout the state so that all residents of the state can access a hub via a local phone call. Local phone service is available between all pairs of adjacent counties in the state. Thus, if there is a hub in a given county or in one of its adjacent counties, then residents of that county have the desired access.

County adjacencies, which can be obtained from a map of the state, are described in the data array shown in Figure 6.14. Values of 1 in the array indicate counties that are adjacent to each other, and values of 0 indicate counties that are not adjacent to each other. (Each county is considered to be ac
manager would like to minimize the number of
the state.

Set Covering: Locating Hubs

Data — *Adjacency*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	0	0	1	0	0	0	0	0	0	0	0	0
2	1	1	1	0	1	0	0	0	0	0	0	0	0	0
3	0	1	1	1	1	1	0	0	0	0	0	0	0	0
4	0	0	1	1	0	1	0	1	0	0	0	0	0	0
5	1	1	1	0	1	1	0	0	1	1	0	0	0	0
6	0	0	1	1	1	1	1	0	0	1	1	0	0	0
7	0	0	0	0	0	1	1	1	0	0	1	1	0	0
8	0	0	0	1	0	0	1	1	0	0	0	1	0	0
9	0	0	0	0	1	0	0	0	1	1	0	0	1	0
10	0	0	0	0	1	1	0	0	1	1	1	0	1	0
11	0	0	0	0	0	1	1	0	0	1	1	1	1	0
12	0	0	0	0	0	0	1	1	0	0	1	1	1	1
13	0	0	0	0	0	0	0	0	1	1	1	1	1	1
14	0	0	0	0	0	0	0	0	0	0	0	1	1	1

Match | Cover

Qualitative constraints

- Linking constraints
  - Force two variables to behave consistently.
- Disjunctive constraints
  - Choice of one option or its opposite.
- Tour constraints
  - Requirement that a travel path must stop at every location.

Linking constraints

Moore Office Products

Moore Office Products has been producing and selling three product families (F1, F2, and F3) and planning for those products using a product mix type of linear programming model. Each product family requires production hours in each of three departments. In addition, each family requires its own sales force, which must be supported no matter how large or small the sales volume happens to be. The parameters describing the situation are summarized in the following table. Moore's management is wondering whether it should continue to market the three product families.

	Hours Required/1000 Units			
	F1	F2	F3	Hours Available
Department A	3	4	8	2000
Department B	3	5	6	2000
Department C	2	3	9	2000
Profit/Unit ($)	1.20	1.80	2.20	
Sales Cost ($1000s)	60	200	100	
Demand (1000s)	300	200	50	

ModelSheets

Disjunctive constraints Machine-sequencing problem

Miles Manufacturing

Miles Manufacturing is a regionally focused production shop that fabricates metal components for automobile companies. Its scheduling efforts are centered around a large piece of equipment that handles a variety of operations such as drilling, shaping, polishing, and mechanical testing. Work arrives at the machine in batches—each batch corresponding to a customer order—and the information system provides data on the size of the order, how long it will take to process, and when it is due (the due dates having been previously negotiated with customers). These due dates, which apply to the shop, are adjusted for the delivery time that will be needed to put the order in the customer's hands. When several orders are waiting to be processed, the supervisor looks for guidance on how the orders should be sequenced. The minimization of job tardiness is an accepted criterion for a schedule.

This morning's workload consists of six jobs, as described by the following table. The problem is to sequence the six jobs so that the supervisor can start work.

Job Number	1	2	3	4	5	6
Processing Time (h)	5	7	9	11	13	15
Due Date (hours from now)	28	35	24	32	30	40

With 60 total hours of work to schedule and a latest due date of 40, it is obvious that the jobs cannot all be finished on time, and there will be some unavoidable tardiness even in the best schedule. ■

Tour constraints
Traveling salesperson problem

Douglas Electric Cart Company

Douglas Electric Cart Company assembles small electric vehicles, which are sold for use on golf courses, at university campuses, and in sports stadiums. In these markets, customers like to buy in a variety of colors, so Douglas offers several choices. As a result, its manufacturing operations include a sophisticated painting operation, which is separately scheduled.

In today's schedule, there are six colors (C1–C6) with cleaning times as shown in the table below.

	C1	C2	C3	C4	C5	C6
C1	—	16	63	21	20	66
C2	57	—	40	46	69	42
C3	23	11	—	55	53	47
C4	71	53	58	—	47	5
C5	27	79	53	35	—	30
C6	57	47	51	17	24	—

The entry in row *i* and column *j* of the table gives the cleaning time required between product lots of color *Ci* and color *Cj*. Each production run consists of a cycle through the full set of colors, and the operations manager wishes to sequence the colors so that the total cleaning time in a cycle is minimized. ■