COMPACTNESS OF SIGN-CHANGING SOLUTIONS TO SCALAR CURVATURE-TYPE EQUATIONS WITH BOUNDED NEGATIVE PART

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Abstract. We consider the equation $\Delta_g u + hu = |u|^{2^*-2}u$ in a closed Riemannian manifold (M, g), where $h \in C^{0,\theta}(M)$, $\theta \in (0,1)$ and $2^* = \frac{2n}{n-2}$, $n := \dim(M) \geq 3$. We obtain a sharp compactness result on the sets of sign-changing solutions whose negative part is *a priori* bounded. We obtain this result under the conditions that $n \geq 7$ and $h < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ in M, where Scal_g is the Scalar curvature of the manifold. We show that these conditions are optimal by constructing examples of blowing-up solutions, with arbitrarily large energy, in the case of the round sphere with a constant potential function h.

1. Introduction and main results

Let (M, g) be a closed (i.e. compact, without boundary) Riemannian manifold of dimension $n \geq 3$. We are interested in this paper in the asymptotic behavior of sequences of *sign-changing* solutions $(u_k)_k$ to the scalar curvature-type equation

$$\Delta_g u + hu = |u|^{2^*-2}u \quad \text{in } M, \tag{1.1}$$

where $\Delta_g := -\operatorname{div}_g \nabla$ is the Laplace–Beltrami operator, $h \in C^{0,\theta}(M)$, $\theta \in (0,1)$ and $2^* = \frac{2n}{n-2}$ is the critical exponent for the embeddings of the Sobolev space $H^1(M)$ into the Lebesgue spaces $L^q(M)$.

The case of *positive* solutions of (1.1) has originated a vast amount of work in the last decades and is now well understood. In particular, assuming that the operator $\Delta_g + h$ is coercive (which is a necessary condition to the existence of positive solutions for (1.1)), Druet [14] showed that if

$$h < \frac{n-2}{4(n-1)} \operatorname{Scal}_g \quad \text{in } M, \tag{1.2}$$

where Scal_g is the Scalar curvature of the manifold, then there exists a constant $C > 1$ such that every solution u of (1.1) satisfies

$$\frac{1}{C} \leq u \leq C \quad \text{in } M.$$

We are concerned in this article with sign-changing solutions. Our first result establishes the boundedness of the set of solutions of (1.1) whose negative part is *a priori* bounded:

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Theorem 1.1. *Let (M,g) be a closed manifold of dimension $n \geq 7$, and let $h \in C^{0,\theta}(M)$ and $\theta \in (0,1)$. Assume that (1.2) holds true. Then for every $A > 0$, there exists $C_A > 0$ such that for every solution u of (1.1), if $u \geq -A$ in M, then*

$$u \leq C_A \quad \text{in } M.$$

We prove in this paper a slightly more general result, Theorem 2.1 below, which also addresses the case of subcritical powers and allows to take into account perturbations of the potential h and the exponent 2^*. By standard elliptic regularity results, Theorem 1.1 establishes in particular the compactness in $C^{2,\theta}(M)$ of the set of solutions to (1.1) which are uniformly bounded from below. Note that in the statement of Theorem 1.1, the operator $\Delta_g + h$ is not assumed to be coercive, unlike in the positive case, and in particular (M,g) is not assumed to be of positive Yamabe type.

The next result shows that the assumptions of Theorem 1.1 are sharp in the case of the round sphere $(\mathbb{S}^n, g_0)$ with a constant potential function h (note that in this case $\mathrm{Scal}_{g_0} \equiv n(n-1)$):

Theorem 1.2. *Let $(\mathbb{S}^n, g_0)$ be the n–dimensional round sphere. Assume that h is a constant and $h > 0$ in case $n \in \{3,4,5\}$, $h > 2$ in case $n = 6$, $h > n(n-2)/4$ in case $n \geq 7$. Assume moreover that $h \neq j(j+n-1)(n-2)/4$ for all $j \geq 1$. Then there exists a sequence of solutions $(u_k)_{k\in\mathbb{N}}$ to the equation*

$$\Delta_{g_0} u_k + h u_k = |u_k|^{2^*-2} u_k \quad \text{in } \mathbb{S}^n \tag{1.3}$$

such that

$$-\infty < \lim_{k\to\infty} \min_M u_k < 0 \quad \text{and} \quad \lim_{k\to\infty} \max_M u_k = \infty. \tag{1.4}$$

The sequence $(u_k)_{k\in\mathbb{N}}$ that we construct in Theorem 1.2 also satisfies

$$\lim_{k\to\infty} \|u_k\|_{H^1(\mathbb{S}^n)} = +\infty. \tag{1.5}$$

We point out that according to Druet's result [14], it is not possible to construct positive blowing-up solutions of (1.3) when $h < n(n-2)/4$; when $h > n(n-2)/4$, however, such solutions were constructed by Chen, Wei and Yan [7] (see also Vétois and Wang [44]). Theorem 1.2 highlights the specificity of the six-dimensional case: when $n = 6$, $n(n-2)/4 = 6$, but we manage to obtain non-compactness under the weaker assumption $h > 2$. This is due to a subtle interaction between the bubbling profiles, the negative part of the solution and the potential h, see for instance (3.57).

The result of Theorem 1.2 is also true when $h \equiv n(n-2)/4$, namely in the case of the Yamabe equation. Solutions of this type have been constructed in this case by del Pino, Musso, Pacard and Pistoia [10, 11].

Few results are known concerning the question of compactness of the set of solutions of (1.1) in the context of sign-changing solutions. On the one hand, the non-compactness of the whole set of solutions in the case of the Yamabe equation on the sphere was established by Ding [13]. More recently, several examples of solutions were constructed in the case of the Yamabe equation on the sphere. In this case, del Pino, Musso, Pacard and Pistoia [10, 11] obtained examples of solutions satisfying (1.4) and concentrating along some special submanifolds. Musso and Wei [28] constructed another type of solutions satisfying a non-degeneracy property. In a different direction, Clapp [8] and Clapp and Fernández [9] used topological methods to obtain examples of solutions satisfying equivariance properties and, very recently, by using ODE methods, Fernández and Petean [18] discovered the existence of a new type of solutions vanishing on an arbitrary number of hypersurfaces.

On the other hand, a compactness result for energy-bounded, sign-changing solutions was established by Vétois [43]. In this work, it is the compactness of sets of solutions whose energy is *a priori* bounded which was obtained, also in the case where $n \geq 7$ and (1.2) holds true in M, with the additional assumption that (M, g) is locally conformally flat. The proof of [43] uses in a crucial way the H^1–bubble tree decomposition result obtained by Struwe [41]. This type of result (see also the C^0–blow-up theory developed by Druet, Hebey and Robert [16] in the context of positive solutions) applies to energy-bounded sequences of solutions. Theorem 1.1 and the results in [43] are therefore very different in nature, and so are their proofs. It is in particular worth noting that Theorem 1.1 is the first compactness result for sign-changing solutions of (1.1) which does not require an *a priori* bound on the energy. It is also worth noting that, in view of Theorem 1.3 in Vétois [43], the boundedness assumption on the negative part of the solutions in Theorem 1.1 is optimal, since – at least on locally conformally flat manifolds – infinite-energy blow-up occurs otherwise.

Different types of existence results of sign-changing blowing-up solutions were also established by Deng, Musso and Wei [12], Pistoia and Vétois [31] and Robert and Vétois [35,37]. More precisely, these papers are concerned with the existence of families of sign-changing blowing-up solutions $(u_\varepsilon)_{\varepsilon>0}$ to the asymptotically critical equations

$$\Delta_g u_\varepsilon + h u_\varepsilon = |u_\varepsilon|^{2^*-2-\varepsilon} u_\varepsilon \quad \text{in } M$$

for small $\varepsilon > 0$. In particular, we point out that the solutions constructed by Robert and Vétois [35,37] satisfy (1.4), however, in contrast with (1.5), their energy is bounded from above. The existence of such solutions was obtained in [35] for manifolds with positive Yamabe invariant, under the conditions that there exists a non-degenerate solution u_0 to (1.1) with either [$n \in \{3, 4, 5\}$ and h is arbitrary], [$n = 6$ and $\frac{n-2}{4(n-1)} \operatorname{Scal}_g - h < 2u_0$], [$n \in \{7, 8, 9\}$ and $h \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g$] or [$n \geq 10$, $h \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ and (M, g) is locally conformally flat]. The non-degeneracy condition was then relaxed in [37] in the case where u_0 is a strict local minimizer of an energy functional.

More generally, these compactness questions originated with the investigation of the set of positive solutions of the Yamabe equation in manifolds of positive Yamabe type. In the case of the sphere, the positive solutions of the Yamabe equation were classified by Obata [29]. For more general manifolds, references in the context of positive solutions include Druet [14], Khuri, Marques, and Schoen [23], Li and Zhang [24, 25], Li and Zhu [26], Marques [27] and Schoen [39, 40] for compactness results and Brendle [4] and Brendle and Marques [5] for non-compactness results.

We prove Theorem 1.1 in Section 2. Its proof is based on an *a priori* asymptotic analysis of sequences of blowing-up solutions $(u_k)_{k\in\mathbb{N}}$ of (1.1). We identify, for each k, a suitable set of points in M where u_k is likely to blow-up (the number of such points is *not* known to be *a priori* bounded in k). Around each one of these points x_k a local analysis is carried on, and we prove that u_k blows-up at first order, and on a controlled scale around x_k, as a canonical bubbling profile. The conclusion is then obtained by analyzing the pointwise interactions between all these defects of compactness. The new difficulty here is of course that the sequence $(u_k)_{k\in\mathbb{N}}$

that we investigate changes sign. We overcome this issue by adapting the approach introduced in Druet–Premoselli [17] (see also Premoselli [32]).

We prove Theorem 1.2 in Section 3. The proof of this result relies on a Lyapunov–Schmidt-type method, which was invented, developed and successfully used in a series of works by Wang, Wei and Yan [45, 46] and Wei and Yan [47–50]; see also del Pino, Musso, Pacard and Pistoia [10, 11] and Guo, Li, Pistoia and Yan [20] for more recent works inspired from this method. The solutions that we construct are of the form

$$u_k = \sum_{i=1}^{k} B_{i,k} - \lambda_0 + \Phi_k,$$

where $k \in \mathbb{N}$, $B_{1,k}, \dots, B_{k,k}$ are standard bubbles concentrating at k equidistant points of the equator, $\lambda_0 := h^{1/(2^*-2)}$ is the constant solution of (1.1) and $\Phi_k \to 0$ as $k \to \infty$ in $H^1(M)$. This ansatz is similar to the one used by del Pino, Musso, Pacard and Pistoia [10, 11] in the case of the Yamabe equation. Note, however, that in contrast with [10, 11], our proof does not rely on weighted L^∞–norms. Instead, we use the Sobolev norm induced by the operator $\Delta_g + h$, an approach which is closer to the one used for instance by Chen, Wei and Yan [7] in the context of positive solutions.

2. Proof of Theorem 1.1

In this section, we will prove the following result, which is more general than Theorem 1.1:

Theorem 2.1. *Let (M, g) be a closed manifold of dimension $n \geq 3$, $q \in (2, 2^*]$ and $h_0 \in C^{0,\theta}(M)$, $\theta \in (0, 1)$. In case $q = 2^*$, assume that $n \geq 7$ and $h_0 < \frac{n-2}{4(n-1)} Scal_g$ in M. Let $A \in \mathbb{R}$. Then for every sequences $(h_k)_k$ in $C^{0,\theta}(M)$ such that $\|h_k - h_0\|_{C^{0,\theta}(M)} \to 0$ as $k \to +\infty$, $(q_k)_{k\in\mathbb{N}} \subset (2, 2^*]$ such that $q_k \to q$ and $(u_k)_k \subset C^{2,\theta}(M)$ such that*

$$\triangle_g u_k + h_k u_k = |u_k|^{q_k-2} u_k, \tag{2.1}$$

if $(u_k)_k$ satisfies

$$u_k^- \leq A \quad \text{for any } k \in \mathbb{N}, \tag{2.2}$$

then up to a subsequence, $u_k \to u_0$ in $C^{2,\theta}(M)$, where u_0 solves

$$\Delta_g u_0 + h_0 u_0 = |u_0|^{q-2} u_0 \quad \text{in } M.$$

In (2.2) we have let $u_k^- := -\min(u_k, 0)$ which is always non-negative. Assumption (2.2) ensures that the sequence (u_k) does not develop sign-changing bubbles.

Remark 2.2. *If $(u_k)_k$ is a sequence satisfying the assumptions of Theorem 2.1,* (2.2) *is in particular true if one of the following two conditions is satisfied:*

- *Either $\|u_k\|_{L^p(M)}$ is uniformly bounded in k for some $p > 2^*$, or*
- *$\|u_k\|_{L^{2^*}(M)}$ is smaller than some (small) constant $\varepsilon_0(n, g, h)$ for all k.*

Proof of the Remark: Following Ouyang [30], u_k^- satisfies, weakly in $H^1(M)$:

$$\triangle_g u_k^- + h_k u_k^- \leq (u_k^-)^{q_k-1}.$$

Now, if u_k^- is uniformly bounded in some $L^p(M)$ with $p > 2^*$ then a bootstrap argument shows that u_k^- is uniformly bounded in $L^\infty(M)$. On the other hand, if we assume that $\|u_k\|_{L^{2^*}(M)}$ is small enough then an adaptation of Trudinger's classical argument (see for instance the proof of Theorem 2.15 in Hebey [22]) similarly yields that u_k^- is uniformly bounded in some $L^s(M)$ with $s > 2^*$, and we again conclude with a bootstrap argument. □

The proof of Theorem 2.1 goes through an *a priori* asymptotic analysis. In what follows we let $(h_k)_k$, $(q_k)_k$ and $(u_k)_k$ be sequences as in the statement of Theorem 2.1. In case $q = 2^*$, we assume that $n \geq 7$ and $h_0 < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ in M. Note that we do not assume that $\Delta_g + h_0$ is coercive anymore. We assume that $(u_k)_k$ satisfies assumption (2.2) and, up to a subsequence, we assume that $u_k \not\equiv 0$ for all k. If the sequence $(\|u_k\|_{L^\infty(M)})_k$ is uniformly bounded Theorem 2.1 easily follows by standard elliptic theory. We therefore proceed by contradiction and assume that

$$\|u_k\|_{L^\infty(M)} \longrightarrow +\infty \tag{2.3}$$

as $k \to +\infty$. We first prove Theorem 1.1 in the subcritical case $q < 2^*$:

Proof of Theorem 1.1 when $q < 2^*$. Assume that $\lim_{k\to+\infty} q_k = q < 2^*$. Let $y_k \in M$ be such that

$$|u_k(y_k)| = \max_M |u_k| \longrightarrow +\infty$$

by (2.3). We then obtain $u_k(y_k) = |u_k(y_k)|$ by (2.2), and we can let $\nu_k := u_k(y_k)^{-(q_k-2)/2}$. For any $x \in B\big(0, i_g(M)/2\nu_k\big)$, where $i_g(M)$ is the injectivity radius of M, define

$$v_k(x) := \nu_k^{\frac{2}{q_k-2}} u_k\big(\exp_{y_k}(\nu_k x)\big).$$

It satisfies $\|v_k\|_\infty \leq 1$ and solves

$$\triangle_{g_k} v_k + \nu_k^2 h_k\left(\exp_{y_k}(\nu_k \cdot)\right) v_k = |v_k|^{q_k-2} v_k \quad \text{in } B\left(0, i_g(M)/2\nu_k\right),$$

where $g_k := \exp_{y_k}^* g(\nu_k \cdot)$. By standard elliptic theory we then get that $v_k \to v_0$ in $C^{2,\eta}_{loc}(\mathbb{R}^n)$, for any $0 < \eta < 1$, where

$$\triangle v_0 = |v_0|^{q-2} v_0 \quad \text{in } \mathbb{R}^n.$$

Here $\triangle := -\sum_{i=1}^n \partial_i^2$ stands for the non-negative Euclidean Laplacian. By assumption (2.2) we have $0 \leq v_k^- \leq A\nu_k^{2/(q_k-2)}$ pointwise for any k, so that $v_0 \geq 0$. Since $q < 2^*$ the classification result of Gidas and Spruck [19] shows that $v_0 \equiv 0$, but this is impossible since $v_0(0) = 1$. This ends the proof of Theorem 1.1 when $q < 2^*$. □

The next two subsections are devoted to the proof of Theorem 1.1 in the asymptotically critical case. We will assume from now on that $\lim_{k\to+\infty} q_k = 2^*$ and therefore that $n \geq 7$ and $h_0 < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$.

2.1. **Local analysis.** In this section we consider sequences of critical points $(x_k)_k$ of u_k and a sequence of positive numbers $(\rho_k)_k$ with $16\rho_k < i_g(M)$ such that $|u_k(x_k)| > 0$,

$$d_g(x_k, x)^{\frac{2}{q_k-2}} |u_k(x)| \leq C \quad \text{for any } x \in B_{x_k}(8\rho_k) \tag{2.4}$$

and

$$\rho_k^{\frac{2}{q_k-2}} \max_{B_{x_k}(8\rho_k)} |u_k| \longrightarrow +\infty \tag{2.5}$$

as $k \to +\infty$. Relevant examples of such sequences $(x_k)_k$ and $(\rho_k)_k$ will be constructed in the next subsection. We first prove that u_k develops a concentration point at x_k. Let

$$\mu_k := |u_k(x_k)|^{-\frac{q_k-2}{2}}. \tag{2.6}$$

Lemma 2.3. *Assume* (2.4) *and* (2.5). *Then as $k \to +\infty$ one has $\mu_k \to 0$ and*

$$\mu_k^{\frac{2}{q_k-2}} u_k\big(\exp_{x_k}(\mu_k \cdot)\big) \longrightarrow \left(1 + \frac{|\cdot|^2}{n(n-2)}\right)^{1-\frac{n}{2}}$$

in $C^2_{loc}(\mathbb{R}^n)$.

Proof. Let $y_k \in B_g(x_k, 8\rho_k)$ be such that

$$|u_k(y_k)| = \max_{B_g(x_k,8\rho_k)} |u_k| \longrightarrow +\infty$$

as $k \to +\infty$ by (2.5). We have in particular $u_k(y_k) = |u_k(y_k)|$ by (2.2), and we let $\nu_k := u_k(y_k)^{-(q_k-2)/2}$. By (2.5) one has $\frac{\rho_k}{\nu_k} \to +\infty$ as $k \to +\infty$. For any $x \in B\left(0, \rho_k/\nu_k\right)$ define

$$v_k(x) := \nu_k^{\frac{2}{q_k-2}} u_k\big(\exp_{x_k}(\nu_k x)\big).$$

It satisfies

$$\triangle_{g_k} v_k + \nu_k^2 h_k\left(\exp_{x_k}(\nu_k \cdot)\right) v_k = |v_k|^{q_k-2} v_k \quad \text{in } B\left(0, \rho_k/\nu_k\right),$$

where $g_k := \exp_{y_k}^* g(\nu_k \cdot)$. Also, $\|v_k\|_\infty \le 1$ by definition of y_k. By standard elliptic theory we get that $v_k \to v_0$ as $k \to \infty$ in $C^{2,\eta}_{loc}(\mathbb{R}^n)$ for any $0 < \eta < 1$, where

$$\triangle v_0 = |v_0|^{2^*-2} v_0 \quad \text{in } \mathbb{R}^n.$$

By assumption (2.2) we have $0 \le v_k^- \le A\nu_k^{2/(q_k-2)}$ pointwise for any k, so that $v_0 \ge 0$. Also v_0 is non-trivial, since $v_0(y_0) = 1$, where $y_0 := \lim_{k\to+\infty} \frac{1}{\nu_k} \exp_{x_k}^{-1}(y_k)$, and the latter limit is finite since $d_g(x_k, y_k) = \mathrm{O}(\nu_k)$ by (2.4). By the classification result in Caffarelli, Gidas and Spruck [6] we get

$$v_0(x) = \left(1 + \frac{|x-y_0|^2}{n(n-2)}\right)^{1-\frac{n}{2}} \quad \text{for all } x \in \mathbb{R}^n.$$

Since $\nabla u_k(x_k) = 0$, 0 is also a critical point of v_0 and therefore $y_0 = 0$. In particular, $u_k(x_k) > 0$ and $\frac{\nu_k}{\mu_k} = v_k(0)^{(q_k-2)/2} \to 1$ as $k \to \infty$, where μ_k is as in (2.6), which concludes the proof of Lemma 2.3. □

Define, for any $k \ge 1$ and for any $x \in M$,

$$B_k(x) := \mu_k^{n-2-\frac{2}{q_k-2}} \left(\mu_k^2 + \frac{d_g(x_k,x)^2}{n(n-2)}\right)^{-\frac{n-2}{2}}. \tag{2.7}$$

Let $\varepsilon \in (0,1)$ be fixed. Following the approach of Druet and Premoselli [17] (see also Premoselli [32]) we define, for any $k \ge 1$,

$$\begin{aligned} r_k := \sup \big\{ \mu_k \le r \le \rho_k \quad &\text{such that} \quad |u_k(x) - B_k(x)| \le \varepsilon B_k(x) \\ &\text{and} \quad |\nabla\left(u_k - B_k\right)(x)| \le \varepsilon |\nabla B_k| \quad \text{for all } x \in B_g(x_k, r) \big\}. \end{aligned} \tag{2.8}$$

Here $B_g(x_k, r)$ denotes the Riemannian ball of center x_k and radius r. The radius r_k measures the distance from x_k at which u_k deviates from the bubbling profile

B_k due to the pointwise influence of other concentration points. By Lemma 2.3 there holds $u_k(x_k) > 0$ hence $\mu_k = u_k(x_k)^{-(q_k-2)/2}$ and

$$\frac{r_k}{\mu_k} \longrightarrow +\infty \quad \text{as} \quad k \longrightarrow +\infty. \tag{2.9}$$

The definition of r_k shows in particular that $u_k > 0$ on $B_g(x_k, r_k)$. We first obtain a control on r_k in terms of μ_k:

Lemma 2.4. *Assume that ε is chosen small enough (independently of k). We have*

$$r_k = \mathrm{O}\left(\mu_k^{\frac{n-4}{n-2}}\right).$$

Proof. Let X_k be the 1-form defined in $B_g(x_k, r_k)$ by

$$X_k(x) := \left(1 - \frac{1}{6(n-1)} \operatorname{Ric}_g(\nabla f_k(x), \nabla f_k(x))\right) \nabla f_k(x),$$

where $f_k(x) := \frac{1}{2} d_g(x_k, x)^2$ and Ric_g is the Ricci curvature of the manifold. We let $\Omega_k := B_g(x_k, r_k)$ and write a Pohozaev identity for u_k in Ω_k. Following Proposition 6.2 in Hebey [22] (and identifying X_k with the associated vector field through g), it can be written as follows

$$\begin{aligned}
&\int_{\Omega_k} h_k u_k \langle X_k, \nabla u_k \rangle \, dv_g + \left(\frac{1}{q_k} - \frac{1}{2^*}\right) \int_{\Omega_k} \operatorname{div}_g X_k |u_k|^{q_k} dv_g \\
&\qquad + \int_{\Omega_k} \left(\nabla X_k - \frac{1}{n} \operatorname{div}_g X_k \cdot g\right) (\nabla u_k, \nabla u_k) dv_g \\
&\qquad + \frac{n-2}{4n} \int_{\Omega_k} \triangle_g \left(\operatorname{div}_g X_k\right) u_k^2 dv_g + \frac{n-2}{2n} \int_{\Omega_k} \operatorname{div}_g X_k h_k u_k^2 dv_g \\
&= \int_{\partial \Omega_k} \left(\frac{1}{q_k} \langle X_k, \nu \rangle |u_k|^{q_k} + \langle X_k, \nabla u_k \rangle \partial_\nu u_k - \frac{1}{2} \langle X_k, \nu \rangle |\nabla u_k|^2 \right. \\
&\qquad \left. - \frac{n-2}{4n} \partial_\nu \left(\operatorname{div}_g X_k\right) u_k^2 + \frac{1}{2^*} \operatorname{div}_g X_k \partial_\nu u_k u_k \right) d\sigma_g.
\end{aligned} \tag{2.10}$$

By definition of X_k it is easily checked that

$$\begin{aligned}
&|X_k(x)| = \mathrm{O}\left(d_g(x_k, x)\right), \\
&\triangle_g \left(\operatorname{div}_g X_k\right)(x) = \frac{n}{n-1} \operatorname{Scal}_g(x_k) + \mathrm{O}\left(d_g(x_k, x)\right), \\
&\operatorname{div}_g X_k(x) = n + \mathrm{O}\left(d_g(x_k, x)^2\right), \\
&\partial_\nu \left(\operatorname{div}_g X_k\right)(x) = \mathrm{O}\left(d_g(x_k, x)\right).
\end{aligned} \tag{2.11}$$

Straightforward computations using Lemma 2.3, (2.8), (2.9), (2.11) and the $C^{0,\theta}$ convergence of the h_k yield

$$\begin{aligned}
&\int_{\Omega_k} h_k u_k \langle X_k, \nabla u_k \rangle \, dv_g + \frac{n-2}{4n} \int_{\Omega_k} \triangle_g \left(\operatorname{div}_g X_k\right) u_k^2 dv_g \\
&+ \frac{n-2}{2n} \int_{\Omega_k} \operatorname{div}_g X_k h_k u_k^2 dv_g = C(n) \left(\frac{n-2}{4(n-1)} \operatorname{Scal}_g(x_k) - h_0(x_k)\right) \mu_k^{n - \frac{4}{q_k - 2}} \\
&\qquad + \mathrm{o}\left(\mu_k^{n - \frac{4}{q_k-2}}\right)
\end{aligned} \tag{2.12}$$

as $k \to \infty$, where

$$C(n) := \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{2-n} dx$$

Similarly one obtains, with (2.8), that

$$\int_{\partial\Omega_k} \left(\frac{1}{q_k} \langle X_k, \nu\rangle |u_k|^{q_k} + \langle X_k, \nabla u_k\rangle \partial_\nu u_k - \frac{1}{2} \langle X_k, \nu\rangle |\nabla u_k|^2 \right. \\ \left. - \frac{n-2}{4n} \partial_\nu \left(\operatorname{div}_g X_k\right) u_k^2 + \frac{1}{2^*} \operatorname{div}_g X_k \partial_\nu u_k u_k \right) d\sigma_g = \mathrm{O}\left(\mu_k^{2(n-2) - \frac{4}{q_k - 2}} r_k^{2-n}\right). \tag{2.13}$$

Now, since B_k defined in (2.7) is radial, one gets by definition of X_k that

$$\left|\left(\nabla X_k - \frac{1}{n} \operatorname{div}_g X_k \cdot g\right) (\nabla B_k, \nabla B_k)\right| = \mathrm{O}\left(d_g(x_k, \cdot)^3 |\nabla B_k|^2\right)$$

(see for instance Lemma 8.10 in Hebey [22]), so that with (2.8) we get

$$\int_{\Omega_k} \left(\nabla X_k - \frac{1}{n} \operatorname{div}_g X_k \cdot g\right) (\nabla u_k, \nabla u_k) dv_g = \mathrm{O}\left(\varepsilon \mu_k^{n - \frac{4}{q_k - 2}}\right). \tag{2.14}$$

Finally, up to reducing ρ_k if necessary and since $q_k \leq 2^*$, it is easily seen with (2.8) and (2.11) that

$$\left(\frac{1}{q_k} - \frac{1}{2^*}\right) \int_{\Omega_k} \operatorname{div}_g X_k |u_k|^{q_k} dv_g \geq 0. \tag{2.15}$$

For ε small enough (but independent of k), plugging (2.12)–(2.15) into (2.10) and using that $h_0 < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ proves Lemma 2.4. ☐

Since $n \geq 7$, Lemma 2.4 shows in particular that

$$r_k = \mathrm{o}\left(\sqrt{\mu_k}\right) \quad \text{as} \quad k \longrightarrow +\infty. \tag{2.16}$$

Coming back to the definition of B_k in (2.7) this implies that for any $R > 0$ and for any sequence $y_k \in B_g(x_k, R r_k)$,

$$B_k(y_k) \geq \left(\frac{n(n-2)}{1 + R^2}\right)^{\frac{n-2}{2}} \mu_k^{\frac{n-2}{2} - \frac{2}{q_k - 2}} \mu_k^{\frac{n-2}{2}} r_k^{2-n},$$

so that with (2.16) we get

$$B_k(y_k) \longrightarrow +\infty \quad \text{as} \quad k \longrightarrow +\infty. \tag{2.17}$$

To prove (2.17) we also used that $\mu_k \leq 1$ for k large enough and that $\frac{n-2}{2} - \frac{2}{q_k - 2} \leq 0$. The following lemma shows in particular that u_k remains positive on balls of radii comparable to r_k:

Lemma 2.5. *Let $R > 0$ be fixed. There exists a sequence $(\eta_k)_k$ of positive numbers with $\eta_k \to 0$ as $k \to +\infty$ such that for any $y \in B_g(x_k, R r_k)$,*

$$u_k(y) \geq (1 - \eta_k) B_k(y).$$

Proof. Let $\delta > 0$ be fixed. Let $\Lambda > \|h_0\|_{L^\infty(M)}$ be a positive constant and for any $x \in B_g(x_k, \delta)$ denote by G_k the Green's function of $\triangle_g + \Lambda$ on $B_g(x_k, \delta)$ with Dirichlet boundary condition. Let $(y_k)_k$ be a sequence of points in $B_g(x_k, \delta/2)$. Since $G_k(y_k, y) > 0$ for $y \in B_g(x_k, \delta) \backslash \{y_k\}$ and $\partial_\nu G_k(y_k, y) < 0$ for $y \in \partial B_g(x_k, \delta)$

(see for instance Robert [34]) and since for k large enough we have $\Lambda - h_k \geq 0$ in M, a representation formula with (2.1) and (2.2) gives

$$\begin{aligned} u_k(y_k) &= \int_{B_g(x_k,\delta)} G_k(y_k,\cdot)|u_k|^{q_k-2}u_k dv_g \\ &\quad + \int_{B_g(x_k,\delta)} G_k(y_k,\cdot)\left(\Lambda - h_k\right) u_k dv_g - \int_{\partial B_g(x_k,\delta)} \partial_\nu G_k(y_k,\cdot) u_k d\sigma_g \\ &\geq \int_{B_g(x_k,r_k)} G_k(y_k,\cdot) u_k^{q_k-1} dv_g - C(n,\delta) A^{q_k-1}, \end{aligned} \tag{2.18}$$

where A is the constant appearing in (2.2) and $C(n,\delta)$ is a numerical constant. Assume now that $y_k \in B_g(x_k, Rr_k)$, so that in particular $d_g(x_k,y_k) = \mathrm{o}(1)$ as $k \to \infty$ by Lemma 2.4. Fatou's Lemma using Lemma 2.3, (2.8) and standard properties of the Green's function show that

$$\int_{B_g(x_k,r_k)} G_k(y_k,\cdot) u_k^{q_k-1} dv_g \geq (1+\mathrm{o}(1))\, B_k(y_k)$$

as $k \to \infty$ (see for instance Hebey [22], Proposition 6.1). With (2.17) and (2.18) this concludes the proof of Lemma 2.5. □

Note that, unlike in the case of *positive* solutions, the lower bound on u_k given by Lemma 2.5 is really a consequence of the estimate on r_k given by Lemma 2.4 and of the assumption that $n \geq 7$. Lemma 2.5 shows in particular that u_k is positive in $B_g(x_k, 7r_k)$. Standard Harnack inequalities for positive solutions of (2.1) then apply (see for instance Han and Lin [21], Theorem 4.17) and using (2.4) we in particular get that

$$\frac{1}{C} B_k \leq u_k \leq C B_k \tag{2.19}$$

on $B_g(x_k, 6r_k)$, for some positive C independent of k. Define now, for $x \in B(0,5)$,

$$\tilde{u}_k(x) := \mu_k^{\frac{2}{q_k-2}-(n-2)} r_k^{n-2} u_k\left(\exp_{x_k}(r_k x)\right). \tag{2.20}$$

Lemma 2.6. *As $k \to \infty$, there holds*

$$\tilde{u}_k \longrightarrow \frac{(n(n-2))^{\frac{n-2}{2}}}{|\cdot|^{n-2}} \tag{2.21}$$

in $C^2_{loc}(B(0,5)\backslash\{0\})$. As a consequence, for k large enough,

$$r_k = \rho_k \tag{2.22}$$

holds.

Proof. By (2.1) $\tilde{u}_k$ satisfies

$$\triangle_{\tilde{g}_k} \tilde{u}_k + r_k^2 h_k\left(\exp_{x_k}(r_k\cdot)\right)\tilde{u}_k = \left(\frac{\mu_k}{r_k}\right)^{(n-2)(q_k-2)-2} \tilde{u}_k^{q_k-1} \tag{2.23}$$

in $B(0,5)$, where $\tilde{g}_k := \exp_{x_k}^* g(r_k\cdot)$, so that by (2.16) and (2.19) $\tilde{u}_k$ converges in $C^{2,\eta}_{loc}(B(0,5)\backslash\{0\})$, for any $0 < \eta < 1$, towards a harmonic function $\tilde{u}_\infty$ in $B(0,5)\backslash\{0\}$. By (2.19) and Bôcher's theorem, $\tilde{u}_\infty$ can be written as

$$\tilde{u}_\infty(x) = \frac{\lambda}{|x|^{n-2}} + H(x) \quad \text{for all } x \in B(0,5),$$

where H is harmonic in $B(0,5)$. By integrating (2.23) in $B(0,1)$, using (2.19) and Lemma 2.3, and since H is harmonic one gets, since $r_k = \mathrm{o}(1)$, that $\lambda = (n(n-2))^{(n-2)/2}$ and hence that

$$\tilde{u}_\infty(x) = \frac{(n(n-2))^{\frac{n-2}{2}}}{|x|^{n-2}} + H(x) \quad \text{for all } x \in B(0,5).$$

We now claim that

$$H \equiv 0 \quad \text{in } B(0,5).$$

First, as a consequence of Lemma 2.5, we have $H \geq 0$ everywhere in $B(0,5)$. We now come back to the Pohozaev identity (2.10). The boundary term in the right-hand side now can be written as

$$\begin{aligned}\int_{\partial B_g(x_k,r_k)} \Bigg(\frac{1}{q_k} \langle X_k, \nu \rangle |u_k|^{q_k} + \langle X_k, \nabla u_k \rangle \partial_\nu u_k - \frac{1}{2} \langle X_k, \nu \rangle |\nabla u_k|^2 \\ - \frac{n-2}{4n} \partial_\nu \left(\mathrm{div}_g X_k \right) u_k^2 + \frac{1}{2^*} \mathrm{div}_g X_k \partial_\nu u_k u_k \Bigg) d\sigma_g \\ = \left(- \frac{(n-2)^2}{2} \left(n(n-2) \right)^{\frac{n-2}{2}} \omega_{n-1} H(0) + \mathrm{o}(1) \right) \mu_k^{2(n-2) - \frac{4}{q_k-2}} r_k^{2-n},\end{aligned}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. With (2.12), (2.14) and (2.15), equality (2.10) gives $H(0) \leq 0$, and hence $H(0) = 0$ and $H \equiv 0$ in $B(0,5)$ since H is harmonic.

It remains to prove (2.22). Assume, up to a subsequence, that $r_k < \rho_k$ for k large enough. By the definition of r_k in (2.8) and by Lemma 2.5 this means that there exists $y_k \in \partial B_g(x_k, r_k)$ such that either $u_k(y_k) = (1+\varepsilon) B_k(y_k)$, or $|\nabla (u_k - B_k)(y_k)| = \varepsilon |\nabla B_k(y_k)|$. But this is impossible by (2.21). This ends the proof of Lemma 2.6. ∎

Lemma 2.6 shows that the assumption $h_0 < \frac{n-2}{4(n-1)} \mathrm{Scal}_g$ forces u_k to be close to the bubble B_k at first order on the whole of $B_g(x_k, r_k)$. With (2.16) and Lemma 2.6 we obtain in particular that

$$\rho_k = \mathrm{o}\left(\sqrt{\mu_k}\right) \tag{2.24}$$

as $k \to +\infty$ and that

$$\left\| \frac{u_k}{B_k} - 1 \right\|_{L^\infty(B_g(x_k,\rho_k))} \leq \varepsilon.$$

Since ε can be chosen as small as needed, up to a subsequence we obtain

$$\left\| \frac{u_k}{B_k} - 1 \right\|_{L^\infty(B_g(x_k,\rho_k))} = \mathrm{o}(1) \tag{2.25}$$

as $k \to +\infty$, where B_k is defined in (2.7).

2.2. **Proof of Theorem 1.1.** Recall that $(u_k)_k$ is the sequence introduced in (2.1) which satisfies (2.2) and (2.3). We first identify a set of points of M where we expect the blow-up for $(u_k)_k$ to occur:

Lemma 2.7. *There exist $N_k \geq 1$ points $(x_{1,k}, \dots, x_{N_k,k})$ of M satisfying, up to a subsequence,*

(1) $\nabla u_k(x_{i,k}) = 0$ for $1 \leq i \leq N_k$,

(2) $d_g(x_{i,k}, x_{j,k})^{\frac{2}{q_k-2}} |u_k(x_{i,k})| \geq 1$ for all $1 \leq i, j \leq N_k$, $i \neq j$, and

(3) there exists a positive constant C independent of k such that

$$\left(\min_{1\le i\le N_k} d_g\left(x_{i,k},x\right)\right)^{\frac{2}{q_k-2}}|u_k(x)|\le C \tag{2.26}$$

for any $x\in M$.

By construction the $x_{i,k}$ satisfy in particular $|u_k(x_{i,k})|>0$.

Proof. The proof of this Lemma follows closely the proof of Lemma 2.3. First, an adaptation of Lemma 1.1 in Druet–Hebey [15] shows that for any k there exist $N_k\ge 1$ critical points $x_{1,k},\dots,x_{N_k,k}$ of u_k such that for any $1\le i,j\le N_k, i\ne j$, one has

$$d_g(x_{i,k},x_{j,k})^{\frac{2}{q_k-2}}|u_k(x_{i,k})|\ge 1$$

and that

$$\left(\min_{1\le i\le N_k} d_g\left(x_{i,k},x\right)\right)^{\frac{2}{q_k-2}}|u_k(x)|\le 1 \tag{2.27}$$

for any critical point x of u_k. We prove Lemma 2.7 by contradiction and we let, up to a subsequence, $y_k\in M$ be such that

$$\left(\min_{1\le i\le N_k} d_g\left(x_{i,k},y_k\right)\right)^{\frac{2}{q_k-2}}|u_k(y_k)| = \max_{y\in M}\left(\min_{1\le i\le N_k} d_g\left(x_{i,k},y\right)\right)^{\frac{2}{q_k-2}}|u_k(y)|\longrightarrow +\infty \tag{2.28}$$

as $k\to+\infty$. By (2.28) we have $u_k(y_k)\ne 0$ and, since M is compact and by (2.2), $u_k(y_k)=|u_k(y_k)|\to+\infty$ as $k\to+\infty$. Letting $\nu_k:=u_k(y_k)^{-(q_k-2)/2}$, (2.28) shows that

$$\frac{1}{\nu_k}\left(\min_{1\le i\le N_k} d_g\left(x_{i,k},y_k\right)\right)\to+\infty \text{ as } k\to+\infty. \tag{2.29}$$

For $0<\delta<\frac{1}{2}i_g(M)$ and $x\in B\left(0,\delta/\nu_k\right)$ we define

$$\hat u_k(x):=\nu_k^{\frac{2}{q_k-2}}u_k\left(\exp_{y_k}(\nu_k x)\right).$$

Using (2.28) and (2.29) we have $v_k(0)=1$ and, for $R>0$,

$$|\hat u_k(x)|\le 1+\mathrm{o}(1)\quad \text{for any } x\in B(0,R).$$

By standard elliptic theory and (2.2), $\hat u_k$ converges in $C^{2,\eta}_{loc}(\mathbb{R}^n)$, for any $0<\eta<1$, towards a non-negative function $\hat u_0$ which solves

$$\triangle \hat u_0=\hat u_0^{2^*-1}\quad\text{in } \mathbb{R}^n.$$

By the classification result in [6], we again have

$$\hat u_0(x)=\left(1+\frac{|x|^2}{n(n-2)}\right)^{1-\frac{n}{2}}.$$

Since 0 is a non-degenerate critical point of $\hat u_0$, this implies in particular that for k large enough u_k possesses a critical point $z_k\in M$, with $d_g(y_k,z_k)=\mathrm{o}(\nu_k)$ and $\nu_k^{2/(q_k-2)}u_k(z_k)=1+\mathrm{o}(1)$ as $k\to\infty$. But then

$$\left(\min_{1\le i\le N_k} d_g\left(x_{i,k},z_k\right)\right)^{\frac{2}{q_k-2}}|u_k(z_k)|\longrightarrow+\infty$$

as $k\to\infty$ by (2.29), which is in contradiction with (2.27). This ends the proof of Lemma 2.7. □

For any k, consider the points $\{x_{1,k}, \ldots, x_{N_k,k}\}$ constructed in Lemma 2.7. It is a first, simple remark that the analysis of subsection 2.1 shows that $N_k \geq 2$ – so there are at least two concentration points. Indeed, if up to a subsequence we had $N_k \equiv 1$, conditions (2.4) and (2.5) would be satisfied for the sequences $(x_{1,k})_k$ and $\rho_k = \frac{1}{32} i_g(M)$. More precisely, (2.4) would follow from (2.26), while (2.5) would follow from (2.3). But this would then contradict (2.24).

Hence $N_k \geq 2$ up to a subsequence. Define then

$$16 d_k := \min \Big(\min_{1 \leq i < j \leq N_k} d_g(x_{i,k}, x_{j,k}), \frac{1}{2} i_g(M) \Big), \tag{2.30}$$

and assume that the concentration points are ordered so that

$$d_g(x_{1,k}, x_{2,k}) \leq d_g(x_{1,k}, x_{3,k}) \leq \ldots \leq d_g(x_{1,k}, x_{N_k,k}). \tag{2.31}$$

Another important observation is that

$$d_k \longrightarrow 0 \quad \text{as} \quad k \longrightarrow +\infty. \tag{2.32}$$

Indeed, assume by contradiction that, up to a subsequence, $d_k \not\to 0$ as $k \to +\infty$. In this case, as a consequence of (2.30), N_k is uniformly bounded – that is, that there only are finitely many, isolated, possible concentration points $x_{i,k}$ – and we can assume that $N_k = N$ for all k. By the initial assumption (2.3), there exists then $i_0 \in \{1, \ldots, N\}$ such that

$$d_k^{\frac{2}{q_k - 2}} \max_{B_g(x_{i_0,k}, 8 d_k)} |u_k| \longrightarrow +\infty$$

as $k \to \infty$. But the sequences $(x_{i_0,k})_k$ and $\rho_k = d_k$ now satisfy (2.4) and (2.5), which again contradicts (2.24). This proves (2.32).

Let $R \geq 1$ and define, for any k, $N_{k,R}$ by

$$1 \leq i \leq N_{k,R} \iff d_g(x_{1,k}, x_{i,k}) \leq R d_k,$$

which is well-defined in view of (2.31). Clearly $N_{k,R} \geq 2$ for any $R \geq 16$. With (2.30) it is also easily seen that, for fixed R, $N_{k,R}$ is uniformly bounded in k. In what follows we will fix $R \geq 16$ and, up to a subsequence, we will therefore assume that $N_{k,R}$ is constant and equal to $N_R \geq 2$. Let now $1 \leq i \leq N_R$. At each point $x_{i,k}$ two alternatives can occur, up to a subsequence,

$$\begin{aligned} &\text{either} \quad d_k^{\frac{2}{q_k - 2}} \max_{B_g(x_{i,k}, 8 d_k)} |u_k| \leq C \qquad &\text{(Case one)} \\ &\text{or} \quad d_k^{\frac{2}{q_k - 2}} \max_{B_g(x_{i,k}, 8 d_k)} |u_k| \longrightarrow +\infty \qquad &\text{(Case two)} \end{aligned} \tag{2.33}$$

as $k \to +\infty$, where $C > 0$ is independent of k. It turns out that cases one and two cannot simultaneously occur among the points $x_{i,k}$ with $1 \leq i \leq N_R$:

Lemma 2.8. *Assume that, for some $i_0 \in \{1, \ldots, N_R\}$, $x_{i_0,k}$ satisfies the first case in* (2.33). *Then each other $x_{i,k}$, $i \in \{1, \ldots, N_R\} \backslash \{i_0\}$, also satisfies the first case in* (2.33).

Proof. Choose $R \geq 16$. We assume that there exists $i_0 \in \{1, \ldots, N_R\}$ for which Case one in (2.33) holds. We first remark that we then also have

$$d_k^{\frac{2}{q_k - 2}} \min_{B_g(x_{i_0,k}, 4 d_k)} u_k \geq C_{i_0} \tag{2.34}$$

for some positive constant $C_{i_0} > 0$ independent of k. Indeed, define

$$\check{u}_k(x) := d_k^{\frac{2}{q_k-2}} u_k\big(\exp_{x_{i_0,k}}(d_k x)\big)$$

for all $x \in B(0,8)$. By (2.1), (2.2), (2.32) and the assumption of Case one in (2.33) $\check{u}_k$ converges in $C^{2,\eta}_{loc}(B(0,7))$, for any $0 < \eta < 1$, towards a nonnegative solution $\check{u}_0$ of

$$\triangle \check{u}_0 = \check{u}_0^{2^*-1} \quad \text{in } B(0,7).$$

By Lemma 2.7 one has $\check{u}_0(0) \geq 1$, so that $\min_{x\in B(0,4)} \check{u}_0(x) \geq C_{i_0} > 0$ by the maximum principle, which proves (2.34).

Let now $\Lambda > \|h_0\|_{L^\infty(M)}$, $i \in \{1, \dots, N_R\}$, $i \neq i_0$ and let G_k denote the Green's function of $\triangle_g + \Lambda$, with Dirichlet boundary condition on $B_g(x_{i,k}, 3Rd_k)$. By standard properties of Green's functions (see again Robert [34]), for $x \in B(x_{i,k}, 2Rd_k)$ we have

$$\frac{1}{C} d_g(x,y)^{2-n} \leq G_k(x,y) \leq C d_g(x,y)^{2-n} \qquad \text{for any } y \in B_g(x_{i,k}, 3Rd_k)$$

and

$$|\partial_\nu G_k(x,y)| \leq C R^{1-n} d_k^{1-n} \qquad \text{for any } y \in \partial B_g(x_{i,k}, 3Rd_k)$$

for some $C > 0$ independent of k and R. Let $(z_k)_k$ be a sequence of points in $B_g(x_{i,k}, 2Rd_k)$. With (2.1), a representation formula for u_k on $B_g(x_{i,k}, 3Rd_k)$ gives

$$\begin{aligned}
u_k(z_k) &= \int_{B_g(x_{i,k},3Rd_k)} G_k(z_k,\cdot)|u_k|^{q_k-2}u_k dv_g \\
&\quad + \int_{B_g(x_{i,k},3Rd_k)} G_k(z_k,\cdot)\left(\Lambda - h_k\right) u_k dv_g - \int_{\partial B_g(x_{i,k},3Rd_k)} \partial_\nu G_k(z_k,\cdot) u_k d\sigma_g \\
&\geq \int_{B_g(x_{i_0,k},4d_k)} G_k(z_k,\cdot)|u_k|^{q_k-2}u_k dv_g - C \\
&\geq \frac{1}{C} d_k^{-\frac{2}{q_k-2}} - C
\end{aligned} \tag{2.35}$$

for some positive constant C depending on n and A, where A is the constant appearing in (2.2). Here we used (2.2) to estimate the integrals on the region of M where u_k is negative and (2.34) to estimate u_k from below on $B_g(x_{i_0,k}, 4d_k)$. With (2.32), (2.35) now becomes

$$\min_{B_g(x_{i,k},2Rd_k)} d_k^{\frac{2}{q_k-2}} u_k \geq C_0 + \mathrm{o}(1)$$

as $k \to \infty$, for some positive constant C_0 independent of k. In particular, by the analysis of subsection 2.1, Case two in (2.33) cannot be satisfied at $x_{i,k}$, since by (2.25) it would contradict the latter inequality. Hence Case one in (2.33) is satisfied at $x_{i,k}$ and this ends the proof of Lemma 2.8. □

Lemma 2.8 shows in particular that, for any $R \geq 16$, either all the concentration points $x_{i,k}$, $1 \leq i \leq N_R$ satisfy case one in (2.33) or they all satisfy case two.

End of the proof of Theorem 1.1. We first assume that, for any $R \geq 16$, all the $i \in \{1, \dots, N_R\}$ satisfy case one in (2.33). Then the function

$$w_k(x) := d_k^{\frac{2}{q_k-2}} u_k\big(\exp_{x_{1,k}}(d_k x)\big),$$

defined for $x \in B\left(0, i_g(M)/2d_k\right)$, is locally bounded and by (2.1), (2.2) and standard elliptic theory converges in $C^{2,\eta}_{loc}(\mathbb{R}^n)$, for any $0 < \eta < 1$, towards a nonnegative solution w_0 of

$$\triangle w_0 = w_0^{2^*-1} \quad \text{in } \mathbb{R}^n.$$

Also, w_0 is non-zero by Lemma 2.7, and by construction 0 and

$$\check{x}_2 := \lim_{k\to+\infty} \frac{1}{d_k} \exp^{-1}_{x_{1,k}}(x_{2,k})$$

are distinct critical points of w_0. But this is impossible by the classification result of [6].

Hence, all the points $x_{i,k}$, $1 \le i \le N_R$ satisfy case two in (2.33). By Lemma 2.7, and for all $i \in \{1, \dots, N_k\}$, the sequences $(x_{i,k})_k$ and $(d_k)_k$ satisfy (2.4) and (2.5) with $\rho_k = d_k$. Hence the analysis of subsection 2.1 applies and (2.25) shows that

$$\left|\left|\frac{u_k}{B_{i,k}} - 1\right|\right|_{L^\infty(B_g(x_{i,k},d_k))} = \mathrm{o}(1) \tag{2.36}$$

as $k \to +\infty$, for any $1 \le i \le N_R$, where

$$B_{i,k} := \mu_{i,k}^{n-2-\frac{2}{q_k-2}} \left(\mu_{i,k}^2 + \frac{d_g(x_{i,k},x)^2}{n(n-2)}\right)^{-\frac{n-2}{2}}$$

and $\mu_{i,k} := u_k(x_{i,k})^{-(q_k-2)/2}$. Let $0 < \delta < \frac{1}{2}$ be fixed and let $(z_k)_k$ be a sequence of points in M such that

$$d_g(x_{1,k}, z_k) = \delta d_k. \tag{2.37}$$

We let G_k be the Green's function with Dirichlet boundary condition of $\triangle_g + \Lambda$ on $B_g(x_{1,k}, Rd_k)$. A representation formula for u_k as in (2.35) gives, with (2.36),

$$\begin{aligned} u_k(z_k) \ge \big(1+\mathrm{o}(1)\big) &\int_{B_g(x_{1,k},d_k)} G_k(z_k,y) B_{1,k}^{q_k-1} dv_g \\ &+ \big(1+\mathrm{o}(1)\big) \int_{B_g(x_{2,k},d_k)} G_k(z_k,y) B_{2,k}^{q_k-1} dv_g - C \end{aligned} \tag{2.38}$$

as $k \to \infty$, for some positive constant C depending on n and A. Let $\tilde{G}$ denote the Green's function with Dirichlet boundary condition of the non-negative Euclidean Laplacian $\triangle$ on $B(0,R)$. Define, for any $z \in B(0,R)$,

$$f(z) := (n-2)\omega_{n-1}|z|^{n-2}\tilde{G}(0,z).$$

Using Fatou's lemma and standard properties of Green's functions we get that as $k \to \infty$,

$$\int_{B_g(x_{1,k},d_k)} G_k(z_k,y) B_{1,k}^{q_k-1} dv_g \ge \big(1+\mathrm{o}(1)\big) f(\check{z}) B_{1,k}(z_k) \tag{2.39}$$

holds, where $\check{z} := \lim_{k\to+\infty} \frac{1}{d_k} \exp^{-1}_{x_{1,k}}(z_k)$. Similarly, we obtain

$$\int_{B_g(x_{2,k},d_k)} G_k(z_k,y) B_{2,k}^{q_k-1} dv_g \ge \frac{1}{C} B_{2,k}(z_k) \tag{2.40}$$

for some $C > 0$ independent of k and R. Coming back to (2.38) with (2.36), (2.39) and (2.40) we obtain

$$\big(1 - f(\check{z}) + \mathrm{o}(1)\big) B_{1,k}(z_k) \ge \Big(\frac{1}{C} + \mathrm{o}(1)\Big) B_{2,k}(z_k)$$

as $k \to \infty$ which yields, with (2.37),

$$\limsup_{k\to+\infty}\left(\frac{\mu_{2,k}}{\mu_{1,k}}\right)^{n-2-\frac{2}{q_k-2}} \le C\frac{1-f(\check z)}{\delta^{n-2}} \tag{2.41}$$

for some $C>0$ independent of k and R. On the Euclidean ball $B(0,R)$ the Green's function $\tilde G$ is explicit, so that

$$f(\check z) = 1 - \frac{|\check z|^{n-2}}{R^{n-2}}.$$

Since $|\check z| = \delta$, after letting $R \to +\infty$ in (2.41), this yields

$$\limsup_{k\to+\infty}\frac{\mu_{2,k}}{\mu_{1,k}} = 0. \tag{2.42}$$

Now, by the choice of $x_{1,k}$ and $x_{2,k}$ in (2.31), the roles of $B_{1,k}$ and $B_{2,k}$ are symmetric. In particular, repeating the analysis from (2.37) to (2.42) by centering everything at $x_{2,k}$ yields in the same way

$$\limsup_{k\to+\infty}\frac{\mu_{1,k}}{\mu_{2,k}} = 0.$$

This is an obvious contradiction with (2.42), and concludes the proof of Theorem 1.1. □

The idea behind this last argument is as follows: since by (2.36) u_k is equivalent to $B_{1,k}$ on the whole ball $B_g(x_{1,k}, d_k)$, the bubble $B_{2,k}$ cannot interact at a pointwise level with $B_{1,k}$ on $B_g(x_{1,k}, d_k)$ – otherwise u_k would deviate at first order from $B_{1,k}$. As (2.42) shows, $B_{2,k}$ therefore has to concentrate much faster. But if (2.42) holds, then u_k cannot be equivalent at first order to $B_{2,k}$ on $B_g(x_{2,k}, d_k)$ anymore.

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Throughout this section, we assume that h is a constant and $h>0$ in case $n \in \{3,4,5\}$, $h>2$ in case $n=6$ and $h > n(n-2)/4$ in case $n \ge 7$. We assume moreover that $h \ne j(j+n-1)(n-2)/4$ for all $j \in \mathbb{N}$. By using the stereographic projection, we can write the equation (1.3) as

$$\begin{cases} \Delta u + h_0 u = |u|^{2^*-2} u & \text{in } \mathbb{R}^n \\ u \in D^{1,2}(\mathbb{R}^n), \end{cases} \tag{3.1}$$

where $\Delta u := -\operatorname{div}\nabla u$ is the Laplace operator for the Euclidean metric, h_0 is the function defined by

$$h_0(x) := \frac{4h - n(n-2)}{\left(1+|x|^2\right)^2} \quad \forall x \in \mathbb{R}^n$$

and $D^{1,2}(\mathbb{R}^n)$ is the completion of the set of smooth functions with compact support in $\mathbb{R}^n$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}.$$

For every $k \in \mathbb{N}$, we let H_k be the set of all functions $u \in D^{1,2}\left(\mathbb{R}^n\right)$ such that u is even in $x_2, \dots, x_n$ and

$$u\left(r\cos\left(\theta\right), r\sin\left(\theta\right), x_3, \dots, x_n\right) = u\left(r\cos\left(\theta + 2\pi/k\right), r\sin\left(\theta + 2\pi/k\right), x_3, \dots, x_n\right)$$

for all $\theta, x_3, \dots, x_n \in \mathbb{R}$ and $r > 0$. We equip H_k with the inner product

$$\left\langle u, v\right\rangle_h := \int_{\mathbb{R}^n} \left(\left\langle \nabla u, \nabla v\right\rangle + h_0 uv\right) dx \quad \forall u, v \in H_k$$

and the norm

$$\left\|u\right\|_h := \sqrt{\left\langle u, u\right\rangle_h} \quad \forall u \in H_k.$$

For every $k \geq 1$ and $r, t > 0$, we define

$$U_{k,r,t}\left(x\right) := \sum_{i=1}^{k} B_{i,k,r,t}\left(x\right) - u_0\left(x\right) \quad \forall x \in \mathbb{R}^n,$$

where

$$B_{i,k,r,t}\left(x\right) := \left(\frac{\sqrt{n\left(n-2\right)} t\delta_k}{\left(t\delta_k\right)^2 + \left|x - x_{i,k,r}\right|^2}\right)^{\frac{n-2}{2}},$$

$$x_{i,k,r} := \left(r\cos\left(2\left(i-1\right)\pi/k\right), r\sin\left(2\left(i-1\right)\pi/k\right), 0, \dots, 0\right),$$

$$\delta_k := \begin{cases} k^{-2}\left(\ln k\right)^{-2} & \text{if } n = 3 \\ k^{-2} & \text{if } n \in \{4,5,6\} \\ k^{-\frac{n-2}{n-4}} & \text{if } n \geq 7 \end{cases}$$

and

$$u_0\left(x\right) := \left(\frac{2\sqrt{h}}{1 + \left|x\right|^2}\right)^{\frac{n-2}{2}}.$$

Remark that the functions $B_{i,k,r,t}$ and u_0 are solutions to the problems

$$\begin{cases} \Delta B_{i,k,r,t} = B_{i,k,r,t}^{2^*-1} & \text{in } \mathbb{R}^n \\ B_{i,k,r,t} \in D^{1,2}\left(\mathbb{R}^n\right) \end{cases} \tag{3.2}$$

and

$$\begin{cases} \Delta u_0 + h_0 u_0 = u_0^{2^*-1} & \text{in } \mathbb{R}^n \\ u_0 \in D^{1,2}\left(\mathbb{R}^n\right). \end{cases} \tag{3.3}$$

We let $\left(\Delta + h_0\right)^{-1} : L^{\frac{2n}{n+2}}\left(\mathbb{R}^n\right) \to D^{1,2}\left(\mathbb{R}^n\right)$ be such that for every $v \in L^{\frac{2n}{n+2}}\left(\mathbb{R}^n\right)$, the function $u := \left(\Delta + h_0\right)^{-1} v$ is the unique solution to the problem

$$\begin{cases} \Delta u + h_0 u = v & \text{in } \mathbb{R}^n \\ u \in D^{1,2}\left(\mathbb{R}^n\right). \end{cases}$$

It follows from the Sobolev inequality that $\left(\Delta + h_0\right)^{-1}$ is a continuous operator from $L^{\frac{2n}{n+2}}\left(\mathbb{R}^n\right)$ to $D^{1,2}\left(\mathbb{R}^n\right)$. We then define

$$R_{k,r,t} := \left(\Delta + h_0\right)^{-1}\left(\left|U_{k,r,t}\right|^{2^*-2} U_{k,r,t}\right) - U_{k,r,t}. \tag{3.4}$$

As a first step, we prove the following estimate:

Lemma 3.1. *For every $a, b, c, d > 0$ such that $a < b$ and $c < d$, there exists a constant $C_0 > 0$ such that*

$$\|R_{k,r,t}\|_h \le C_0 \begin{cases} (\ln k)^{-1} & \text{if } n = 3 \\ k^{3-n} & \text{if } n \in \{4,5,6\} \\ k^{-\frac{(n+2)(n+4)}{2n(n-4)}} & \text{if } n \ge 7 \end{cases} \tag{3.5}$$

for all $k \ge 2$, $r \in [a,b]$ and $t \in [c,d]$.

Proof. By continuity of $(\Delta + h_0)^{-1} : L^{\frac{2n}{n+2}}(\mathbb{R}^n) \to D^{1,2}(\mathbb{R}^n)$, by using (3.2) and (3.3) and since $h_0 = \mathrm{O}\left(u_0^{2^*-2}\right)$, we obtain

$$\begin{aligned} &\left\| (\Delta + h_0)^{-1} \left(|U_{k,r,t}|^{2^*-2} U_{k,r,t} \right) - U_{k,r,t} \right\|_h \\ &= \mathrm{O}\left(\left\| |U_{k,r,t}|^{2^*-2} U_{k,r,t} - (\Delta + h_0) U_{k,r,t} \right\|_{\frac{2n}{n+2}} \right) \\ &= \mathrm{O}\left(\left\| |U_{k,r,t}|^{2^*-2} U_{k,r,t} - h_0 \sum_{i=1}^k B_{i,k,r,t} - \sum_{i=1}^k B_{i,k,r,t}^{2^*-1} + u_0^{2^*-1} \right\|_{\frac{2n}{n+2}} \right) \\ &= \mathrm{O}\left(\left\| u_0^{2^*-2} \sum_{i=1}^k B_{i,k,r,t} + u_0 \left(\sum_{i=1}^k B_{i,k,r,t} \right)^{2^*-2} \right\|_{\frac{2n}{n+2}} \right. \\ &\qquad \left. + \left\| \left(\sum_{i=1}^k B_{i,k,r,t} \right)^{2^*-1} - \sum_{i=1}^k B_{i,k,r,t}^{2^*-1} \right\|_{\frac{2n}{n+2}} \right). \end{aligned} \tag{3.6}$$

Moreover, by symmetry, we obtain

$$\begin{aligned} &\left\| \left(\sum_{i=1}^k B_{i,k,r,t} \right)^{2^*-1} - \sum_{i=1}^k B_{i,k,r,t}^{2^*-1} \right\|_{\frac{2n}{n+2}}^{\frac{2n}{n+2}} \\ &\qquad = \mathrm{O}\left(k \left\| \left(B_{1,k,r,t}^{2^*-2} \sum_{i=2}^k B_{i,k,r,t} + \left(\sum_{i=2}^k B_{i,k,r,t} \right)^{2^*-1} \right) \chi_{\Omega_1} \right\|_{\frac{2n}{n+2}}^{\frac{2n}{n+2}} \right), \end{aligned} \tag{3.7}$$

where χ_{Ω_1} is the characteristic function of Ω_1 and

$$\Omega_1 := \{ x \in \mathbb{R}^n : |x - x_{1,k,r}| < |x - x_{i,k,r}| \quad \forall i \in \{2, \dots, k\} \}. \tag{3.8}$$

Finally, we infer (3.5) from (3.6) and (3.7) by applying Lemma A.1 in the appendix and using the definition of δ_k. This ends the proof of Lemma 3.1. □

We define

$$P_{k,r,t} := \left\{ \phi \in H_k : \sum_{i=1}^k \langle \phi, Z_{i,j,k,r,t} \rangle_h = 0 \quad \forall j \in \{1,2\} \right\},$$

where

$$Z_{i,1,k,r,t} := \delta_k \frac{d}{dr} [B_{i,k,r,t}] \quad \text{and} \quad Z_{i,2,k,r,t} := \frac{d}{dt} [B_{i,k,r,t}].$$

We let $\Pi_{k,r,t}$ be the orthogonal projection of H_k onto $P_{k,r,t}$. We prove the following result:

Lemma 3.2. *For every $a,b,c,d>0$ such that $a<b$ and $c<d$, there exist $k_1\in\mathbb{N}$ and $C_1>0$ such that for every $k\geq k_1$, $r\in[a,b]$ and $t\in[c,d]$, the linear operator*

$$L_{k,r,t}:\phi\longmapsto\Pi_{k,r,t}\big(\phi-(\Delta+h_0)^{-1}\left((2^*-1)\left|U_{k,r,t}\right|^{2^*-2}\phi\right)\big) \tag{3.9}$$

is an isomorphism from $P_{k,r,t}$ to itself and

$$\frac{1}{C_1}\left\|\phi\right\|_h\leq\left\|L_{k,r,t}\left(\phi\right)\right\|_h\leq C_1\left\|\phi\right\|_h\quad\forall\phi\in P_{k,r,t}. \tag{3.10}$$

Proof. We begin with proving the second inequality in (3.10). Assume by contradiction that this inequality is not true. Then there exist sequences $(r_k)_k$ in $[a,b]$, $(t_k)_k$ in $[c,d]$ and $(\phi_k)_k$, $(\psi_k)_k$ in $D^{1,2}\left(\mathbb{R}^n\right)$ such that

$$\phi_k,\psi_k\in P_{k,r_k,t_k},\quad L_{k,r_k,t_k}\left(\phi_k\right).\psi_k=k,\quad\left\|\psi_k\right\|_h^2=k\ \text{ and }\ \left\|\phi_k\right\|_h^2=\mathrm{o}\left(k\right) \tag{3.11}$$

as $k\to\infty$. By symmetry, it follows from (3.11) that

$$\int_{\Omega_1}\Big(\left\langle\nabla\phi_k,\nabla\psi_k\right\rangle+h_0\phi_k\psi_k-(2^*-1)\left|U_{k,r_k,t_k}\right|^{2^*-2}\phi_k\psi_k\Big)dx=1, \tag{3.12}$$

$$\int_{\Omega_1}\Big(\left|\nabla\psi_k\right|^2+h_0\psi_k^2\Big)dx=1 \tag{3.13}$$

and

$$\int_{\Omega_1}\Big(\left|\nabla\phi_k\right|^2+h_0\phi_k^2\Big)dx\longrightarrow0 \tag{3.14}$$

as $k\to\infty$. In particular, it follows from (3.12)–(3.14) that

$$\int_{\Omega_1}\left|U_{k,r_k,t_k}\right|^{2^*-2}\phi_k\psi_kdx\longrightarrow\frac{1}{2^*-1} \tag{3.15}$$

as $k\to\infty$. On the other hand, straightforward estimates give

$$\int_{\Omega_1}\left|U_{k,r_k,t_k}\right|^{2^*-2}\phi_k\psi_kdx=\mathrm{O}\Bigg(\int_{\Omega_1}\Bigg[\Bigg(\sum_{i=2}^kB_{i,k,r_k,t_k}\Bigg)^{2^*-2}+B_{1,k,r_k,t_k}^{2^*-2}+u_0^{2^*-2}\Bigg]\left|\phi_k\psi_k\right|dx\Bigg). \tag{3.16}$$

By using Hölder's and Sobolev inequalities together with (3.11), Lemma A.1 in the appendix and the definition of δ_k, we obtain

$$\begin{aligned}\int_{\Omega_1}\Bigg(\sum_{i=2}^kB_{i,k,r_k,t_k}\Bigg)^{2^*-2}\left|\phi_k\psi_k\right|dx&\leq\Bigg\|\sum_{i=2}^kB_{i,k,r_k,t_k}\chi_{\Omega_1}\Bigg\|_{2^*}^{2^*-2}\left\|\phi_k\chi_{\Omega_1}\right\|_{2^*}\left\|\psi_k\chi_{\Omega_1}\right\|_{2^*}\\&=\mathrm{o}\Bigg(\Bigg\|\sum_{i=2}^kB_{i,k,r_k,t_k}\chi_{\Omega_1}\Bigg\|_{2^*}^{2^*-2}k^{\frac{2}{n}}\Bigg)=\mathrm{o}\left(1\right)\end{aligned} \tag{3.17}$$

as $k\to\infty$. Similarly, straightforward calculations give

$$\begin{aligned}&\int_{\Omega_1\backslash B(x_{1,k,r_k},r_k/k)}B_{1,k,r_k,t_k}^{2^*-2}\left|\phi_k\psi_k\right|dx\\&\quad\leq\left\|B_{1,k,r_k,t_k}\chi_{\Omega_1\backslash B(x_{1,k,r_k},r_k/k)}\right\|_{2^*}^{2^*-2}\left\|\phi_k\chi_{\Omega_1}\right\|_{2^*}\left\|\psi_k\chi_{\Omega_1}\right\|_{2^*}\\&\quad=\mathrm{o}\left(1\right)\end{aligned} \tag{3.18}$$

as $k \to \infty$, where $B(x_{1,k,r_k}, r_k/k)$ is the Euclidean ball of center x_{1,k,r_k} and radius r_k/k. It is easy to see that $B(x_{1,k,r_k}, r_k/k) \subset \Omega_1$. On the other hand, by rescaling, we obtain

$$\int_{B(x_{1,k,r_k}, r_k/k)} B_{1,k,r_k,t_k}^{2^*-2} \left|\phi_k \psi_k\right| dx = \int_{B(0, r_k/(k t_k \delta_k))} U^{2^*-2} \big|\tilde{\phi}_k \tilde{\psi}_k\big| dx, \tag{3.19}$$

where

$$U\left(x\right) := \left(\frac{\sqrt{n\left(n-2\right)}}{1+\left|x\right|^2}\right)^{\frac{n-2}{2}},$$

$$\tilde{\phi}_k\left(x\right) := \left(t_k \delta_k\right)^{\frac{n-2}{2}} \eta\left(k t_k \delta_k r_k^{-1} x\right) \phi_k\left(\left(t_k \delta_k\right) x + x_{1,r_k,t_k}\right),$$

$$\tilde{\psi}_k\left(x\right) := \left(t_k \delta_k\right)^{\frac{n-2}{2}} \eta\left(k t_k \delta_k r_k^{-1} x\right) \psi_k\left(\left(t_k \delta_k\right) x + x_{1,r_k,t_k}\right),$$

and $\eta : \mathbb{R}^n \to [0,\infty)$ is a smooth cutoff function such that $\eta \equiv 1$ in $B\left(0,1\right)$ and $\eta \equiv 0$ in $\mathbb{R}^n \backslash B\left(0,2\right)$. It follows from (3.13), (3.14) and the Sobolev inequality that $\big(\tilde{\psi}_k\big)_k$ and $\big(\tilde{\phi}_k\big)_k$ are bounded in $L^{2^*}\left(\mathbb{R}^n\right)$ and, up to a subsequence, $\big(\tilde{\phi}_k\big)_k$ converges weakly to 0 in $L^{2^*}\left(\mathbb{R}^n\right)$ and $\tilde{\phi}_k \to 0$ a.e. in $\mathbb{R}^n$ as $k \to \infty$. It then follows from (3.19) and standard integration theory that

$$\int_{B(x_{1,k,r_k}, r_k/k)} B_{1,k,r_k,t_k}^{2^*-2} \left|\phi_k \psi_k\right| dx \longrightarrow 0 \tag{3.20}$$

as $k \to \infty$. To estimate the last term in (3.16), we write

$$\int_{\Omega_1} u_0^{2^*-2} \left|\phi_k \psi_k\right| dx = \int_{\mathbb{R}^n} u_0^{2^*-2} \big|\phi_k/\sqrt{k}\big| \big|\psi_k/\sqrt{k}\big| dx. \tag{3.21}$$

By using (3.13), (3.14) and the Sobolev inequality, we obtain that $\big(\psi_k/\sqrt{k}\big)_k$ and $\big(\phi_k/\sqrt{k}\big)_k$ are bounded in $L^{2^*}\left(\mathbb{R}^n\right)$, up to a subsequence, $\big(\phi_k/\sqrt{k}\big)_k$ converges weakly to 0 in $L^{2^*}\left(\mathbb{R}^n\right)$ and $\phi_k/\sqrt{k} \to 0$ a.e. in $\mathbb{R}^n$ as $k \to \infty$. It then follows from (3.21) and standard integration theory that

$$\int_{\mathbb{R}^n} u_0^{2^*-2} \big|\phi_k/\sqrt{k}\big| \big|\psi_k/\sqrt{k}\big| dx \longrightarrow 0 \tag{3.22}$$

as $k \to \infty$. By putting together (3.16)–(3.18), (3.20) and (3.22), we obtain

$$\int_{\Omega_1} \left|U_{k,r_k,t_k}\right|^{2^*-2} \phi_k \psi_k dx \longrightarrow 0 \tag{3.23}$$

as $k \to \infty$, which is in contradiction with (3.15). This ends the proof of the second inequality in (3.10).

Now we prove the first inequality in (3.10). Assume by contradiction that this inequality is not true. Then there exist sequences $\left(r_k\right)_k$ in $[a,b]$, $\left(t_k\right)_k$ in $[c,d]$ and $\left(\phi_k\right)_k$ in $D^{1,2}\left(\mathbb{R}^n\right)$ such that

$$\phi_k \in P_{k,r_k,t_k}, \quad \left\|\phi_k\right\|_h^2 = k \quad \text{and} \quad \left\|L_{k,r_k,t_k}\left(\phi_k\right)\right\|_h^2 = \mathrm{o}\left(k\right) \tag{3.24}$$

as $k \to \infty$. By symmetry, it follows from (3.24) that

$$\int_{\Omega_1} \left(\left|\nabla \phi_k\right|^2 + h_0 \phi_k^2\right) dx = 1 \tag{3.25}$$

and

$$\int_{\Omega_1}\Big(\langle\nabla\phi_k,\nabla\psi\rangle+h_0\phi_k\psi-(2^*-1)\left|U_{k,r_k,t_k}\right|^{2^*-2}\phi_k\psi\Big)dx$$
$$=\mathrm{o}\left(\sqrt{\int_{\Omega_1}\Big(\left|\nabla\psi\right|^2+h_0\psi^2\Big)dx}\right)\quad(3.26)$$

as $k\to\infty$ uniformly in $\psi\in P_{k,r_k,t_k}$. In particular, it follows from (3.25) and (3.26) that

$$\int_{\Omega_1}\left|U_{k,r_k,t_k}\right|^{2^*-2}\phi_k^2dx\longrightarrow\frac{1}{2^*-1}\quad(3.27)$$

as $k\to\infty$. On the other hand, similarly as in (3.16)–(3.19), we obtain

$$\int_{\Omega_1}\left|U_{k,r_k,t_k}\right|^{2^*-2}\phi_k^2dx=\mathrm{O}\left(\int_{\Omega_1}\left[\sum_{i=1}^kB_{i,k,r_k,t_k}^{2^*-2}+u_0^{2^*-2}\right]\phi_k^2dx\right),\quad(3.28)$$

$$\int_{\Omega_1}\left(\sum_{i=2}^kB_{i,k,r_k,t_k}\right)^{2^*-2}\phi_k^2dx=\mathrm{o}\left(1\right),\quad(3.29)$$

$$\int_{\Omega_1\backslash B(x_{1,k,r_k},r_k/k)}B_{1,k,r_k,t_k}^{2^*-2}\phi_k^2dx=\mathrm{o}\left(1\right)\quad(3.30)$$

as $k\to\infty$ and

$$\int_{B(x_{1,k,r_k},r_k/k)}B_{1,k,r_k,t_k}^{2^*-2}\phi_k^2dx=\int_{B(0,r_k/(kt_k\delta_k))}U^{2^*-2}\tilde{\phi}_k^2dx.\quad(3.31)$$

It follows from (3.26) that

$$\int_{\Omega_1}\big(\langle\nabla\tilde{\phi}_k,\nabla\psi\rangle+\tilde{h}_k\tilde{\phi}_k\psi-(2^*-1)\left|\tilde{U}_{k,r_k,t_k}\right|^{2^*-2}\tilde{\phi}_k\psi\big)dx=\mathrm{o}\left(\left\|\psi\right\|_h\right)\quad(3.32)$$

as $k\to\infty$ uniformly in $\psi\in C_c^\infty\left(\mathbb{R}^n\right)$, where

$$\tilde{h}_k:=\left(t_k\delta_k\right)^2h\left(\left(t_k\delta_k\right)x+x_{1,r_k,t_k}\right)$$

and

$$\tilde{U}_{k,r_k,t_k}:=\left(t_k\delta_k\right)^{\frac{n-2}{2}}U_{k,r_k,t_k}\left(\left(t_k\delta_k\right)x+x_{1,r_k,t_k}\right).$$

It is easy to see that $\tilde{h}_k\to0$ and $\tilde{U}_{k,r_k,t_k}\to U$ in $C^0_{\mathrm{loc}}\left(\mathbb{R}^n\right)$ as $k\to\infty$. Hence it follows from (3.25) and (3.32) that, up to a subsequence, $\big(\tilde{\phi}_k\big)_k$ converges weakly in $D^{1,2}\left(\mathbb{R}^n\right)$ and pointwise almost everywhere in $\mathbb{R}^n$ to a solution $\tilde{\phi}_0$ of the equation

$$\Delta\tilde{\phi}_0=\left(2^*-1\right)U^{2^*-2}\tilde{\phi}_0\quad\text{in }\mathbb{R}^n.$$

Moreover, since $\phi_k\in P_{k,r_k,t_k}$, by passing to the limit as $k\to\infty$, we obtain that $\tilde{\phi}_0$ is even in $x_2,\dots,x_n$ and

$$\int_{\mathbb{R}^n}\Big\langle\nabla\tilde{\phi}_0,\nabla\left[\partial_{x_1}U\right]\Big\rangle dx=\int_{\mathbb{R}^n}\left\langle\nabla\tilde{\phi}_0,\nabla\left[\frac{d}{d\delta}\left[\delta^{\frac{n-2}{2}}U\left(\delta x\right)\right]_{\delta=1}\right]\right\rangle dx=0$$

and so $\tilde{\phi}_0=0$ (see Bianchi and Egnell [3] and Rey [33]). It then follows from (3.31) and standard integration theory that

$$\int_{B(x_{1,k,r_k},r_k/k)}B_{1,k,r_k,t_k}^{2^*-2}\phi_k^2dx\longrightarrow0\quad(3.33)$$

as $k\to\infty$. To estimate the last term in (3.28), we write

$$\int_{\Omega_1} u_0^{2^*-2}\phi_k^2 dx = \int_{\mathbb{R}^n} u_0^{2^*-2}\big(\phi_k/\sqrt{k}\big)^2 dx. \tag{3.34}$$

It follows from (3.24) that

$$\int_{\mathbb{R}^n} \big(\big\langle \nabla\big(\phi_k/\sqrt{k}\big), \nabla\psi\big\rangle + h_0\big(\phi_k/\sqrt{k}\big)\psi - (2^*-1)\left|U_{k,r_k,t_k}\right|^{2^*-2}\big(\phi_k/\sqrt{k}\big)\psi\big)dx = \mathrm{o}\left(\|\psi\|_h\right) \tag{3.35}$$

as $k\to\infty$ uniformly in $\psi\in P_{k,r_k,t_k}$. Moreover, straightforward estimates give

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big(\left|U_{k,r_k,t_k}\right|^{2^*-2} - u_0^{2^*-2}\big)\big(\phi_k/\sqrt{k}\big)\psi dx\\
&= \int_{\Omega_1} \big(\left|U_{k,r_k,t_k}\right|^{2^*-2} - u_0^{2^*-2}\big)\phi_k\psi_k dx\\
&= \begin{cases} \mathrm{O}\left(\displaystyle\int_{\Omega_1}\left[\left(\sum_{i=1}^k B_{i,k,r_k,t_k}\right)^{2^*-2} + u_0^{2^*-3}\sum_{i=1}^k B_{i,k,r_k,t_k}\right]\left|\phi_k\psi_k\right|dx\right) & \text{if } n\le 5\\ \mathrm{O}\left(\displaystyle\int_{\Omega_1}\left[\left(\sum_{i=2}^k B_{i,k,r_k,t_k}\right)^{2^*-2} + B_{1,k,r_k,t_k}^{2^*-2}\right]\left|\phi_k\psi_k\right|dx\right) & \text{if } n\ge 6,\end{cases}
\end{aligned} \tag{3.36}$$

where $\psi_k := \sqrt{k}\psi$. Similarly as in (3.17)–(3.19), we obtain

$$\int_{\Omega_1}\left(\sum_{i=2}^k B_{i,k,r_k,t_k}\right)^{2^*-2}\left|\phi_k\psi_k\right|dx = \mathrm{o}(1), \tag{3.37}$$

$$\int_{\Omega_1\backslash B(x_{1,k,r_k},r_k/k)} B_{1,k,r_k,t_k}^{2^*-2}\left|\phi_k\psi_k\right|dx = \mathrm{o}(1), \tag{3.38}$$

$$\sum_{i=1}^k\int_{\Omega_1} u_0^{2^*-3}B_{i,k,r_k,t_k}\left|\phi_k\psi_k\right|dx = \mathrm{o}(1) \quad \text{for } n\le 5 \tag{3.39}$$

as $k\to\infty$ and

$$\int_{B(x_{1,k,r_k},r_k/k)} B_{1,k,r_k,t_k}^{2^*-2}\left|\phi_k\psi_k\right|dx = \int_{B(0,r_k/(kt_k\delta_k))} U^{2^*-2}\big|\tilde\phi_k\tilde\psi_k\big|dx. \tag{3.40}$$

Since $\big(\tilde\phi_k\big)_k$ and $\big(\tilde\psi_k\big)_k$ are bounded in $L^{2^*}(\mathbb{R}^n)$ and $\tilde\phi_k\to 0$ a.e. in $\mathbb{R}^n$, it follows from (3.40) and standard integration theory that

$$\int_{B(x_{1,k,r_k},r_k/k)} B_{1,k,r_k,t_k}^{2^*-2}\left|\phi_k\psi_k\right|dx \longrightarrow 0 \tag{3.41}$$

as $k\to\infty$. By putting together (3.36)–(3.39) and (3.41), we obtain

$$\int_{\mathbb{R}^n}\big(\left|U_{k,r_k,t_k}\right|^{2^*-2} - u_0^{2^*-2}\big)\big(\phi_k/\sqrt{k}\big)\psi dx \longrightarrow 0 \tag{3.42}$$

as $k\to\infty$. It follows from (3.24), (3.35) and (3.42) that, up to a subsequence, $\big(\phi_k/\sqrt{k}\big)_k$ converges weakly in $D^{1,2}(\mathbb{R}^n)$ and pointwise almost everywhere in $\mathbb{R}^n$ to a solution ϕ_0 of the equation

$$\Delta\phi_0 + h_0\phi_0 = (2^*-1)\,u_0^{2^*-2}\phi_0 \quad \text{in } \mathbb{R}^n. \tag{3.43}$$

By letting $\varphi_P : \mathbb{S}^n\backslash\{P\} \to \mathbb{R}^n$ be the stereographic projection with respect to the point $P := (0, \dots, 0, 1)$, we can write (3.43) as

$$\Delta_{g_0} \hat{\phi}_0 = \frac{4h\hat{\phi}_0}{n-2} \quad \text{in } \mathbb{S}^n, \tag{3.44}$$

where

$$\hat{\phi}_0 := \left(\frac{1+|\varphi_P(x)|^2}{2}\right)^{\frac{n-2}{2}} \phi_0(\varphi_P(x)) \quad \forall x \in \mathbb{R}^n.$$

Since $h \neq j(j+n-1)(n-2)/4$ for all $j \geq 1$, it follows from (3.44) that $\hat{\phi}_0 = 0$ and so $\phi_0 = 0$. It then follows from (3.34) and standard integration theory that

$$\int_{\Omega_1} u_0^{2^*-2} \phi_k^2 dx \longrightarrow 0 \tag{3.45}$$

as $k \to \infty$. By putting together (3.28)–(3.30), (3.33) and (3.45), we obtain

$$\int_{\Omega_1} |U_{k,r_k,t_k}|^{2^*-2} \phi_k^2 dx \longrightarrow 0 \tag{3.46}$$

as $k \to \infty$, which is in contradiction with (3.27). This completes the proof of Lemma 3.2. ☐

By using Lemmas 3.1 and 3.2, we prove the following result:

Lemma 3.3. *Let $a, b, c, d > 0$ be such that $a < b$ and $c < d$. Let k_1 be as in Lemma 3.2. Then there exist $k_2 \geq k_1$ and $C_2 > 0$ such that for every $k \geq k_2$, $r \in [a, b]$ and $t \in [c, d]$, there exists a unique solution $\phi_{k,r,t} \in P_{k,r,t}$ of the equation*

$$\Pi_{k,r,t}\big(U_{k,r,t}+\phi_{k,r,t}-(\Delta+h_0)^{-1}\left(|U_{k,r,t}+\phi_{k,r,t}|^{2^*-2}(U_{k,r,t}+\phi_{k,r,t})\right)\big) = 0 \tag{3.47}$$

such that

$$\|\phi_{k,r,t}\|_h^2 + \int_{\mathbb{R}^n} |U_{k,r,t}|^{2^*-2} \phi_{k,r,t}^2 dx \leq C_2 \begin{cases} (\ln k)^{-2} & \text{if } n = 3 \\ k^{-2(n-3)} & \text{if } n \in \{4,5,6\} \\ k^{-\frac{(n+2)(n+4)}{n(n-4)}} & \text{if } n \geq 7. \end{cases} \tag{3.48}$$

Moreover, the map $(r, t) \mapsto \phi_{k,r,t}$ is continuously differentiable.

Proof. We define the operators

$$N_{k,r,t} : \phi \longmapsto (\Delta+h_0)^{-1}\Big(|U_{k,r,t}+\phi|^{2^*-2}(U_{k,r,t}+\phi) - |U_{k,r,t}|^{2^*-2} U_{k,r,t} - (2^*-1)|U_{k,r,t}|^{2^*-2}\phi\Big)$$

and

$$T_{k,r,t} : \phi \longmapsto L_{k,r,t}^{-1}\left(\Pi_{k,r,t}\left(N_{k,r,t}(\phi) + R_{k,r,t}\right)\right),$$

where $R_{k,r,t}$ and $L_{k,r,t}$ are as in (3.4) and (3.9). For every $C>0$, $k\in\mathbb{N}$, $r\in[a,b]$ and $t\in[c,d]$, we define the set

$$V_{k,r,t}(C):=\left\{\phi\in P_{k,r,t}:\ \|\phi\|_h^2+\int_{\mathbb{R}^n}|U_{k,r,t}|^{2^*-2}\phi^2dx \le C\begin{cases}(\ln k)^{-2} & \text{if } n=3\\ k^{-2(n-3)} & \text{if } n\in\{4,5,6\}\\ k^{-\frac{(n+2)(n+4)}{n(n-4)}} & \text{if } n\ge 7.\end{cases}\right\}.$$

We will prove that if C is chosen large enough, then $T_{k,r,t}$ has a fixed point in $V_{k,r,t}(C)$, which is equivalent to solving the equation (3.47). It follows from Lemma 3.2 that

$$\|T_{k,r,t}(\phi)\|_h\le C_1\big(\|N_{k,r,t}(\phi)\|_h+\|R_{k,r,t}\|_h\big) \tag{3.49}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}|U_{k,r,t}|^{2^*-2}T_{k,r,t}(\phi)^2dx&=\frac{1}{2^*-1}\left\langle T_{k,r,t}(\phi)-L_{k,r,t}(T_{k,r,t}(\phi)),T_{k,r,t}(\phi)\right\rangle_h\\&=\frac{1}{2^*-1}\left\langle T_{k,r,t}(\phi)-N_{k,r,t}(\phi)-R_{k,r,t},T_{k,r,t}(\phi)\right\rangle_h\\&\le\frac{C_1(C_1+1)}{2^*-1}\big(\|N_{k,r,t}(\phi)\|_h+\|R_{k,r,t}\|_h\big)^2\end{aligned} \tag{3.50}$$

for all $k\ge k_1$, $r\in[a,b]$, $t\in[c,d]$ and $\phi\in P_{k,r,t}$. By continuity of $(\Delta+h_0)^{-1}:L^{\frac{2n}{n+2}}(\mathbb{R}^n)\to D^{1,2}(\mathbb{R}^n)$ and using Hölder's inequality and Sobolev inequality, we obtain

$$\begin{aligned}\|N_{k,r,t}(\phi)\|_h&=\mathrm{O}\Big(\big\||U_{k,r,t}+\phi|^{2^*-2}(U_{k,r,t}+\phi)-|U_{k,r,t}|^{2^*-2}U_{k,r,t}\\&\qquad-(2^*-1)|U_{k,r,t}|^{2^*-2}\phi\big\|_{\frac{2n}{n+2}}\Big)\\&=\begin{cases}\mathrm{O}\Big(\big\||U_{k,r,t}|^{2^*-3}\phi^2+|\phi|^{2^*-1}\big\|_{\frac{2n}{n+2}}\Big) & \text{if } n\le 6\\ \mathrm{O}\Big(\big\||\phi|^{2^*-1}\big\|_{\frac{2n}{n+2}}\Big) & \text{if } n\ge 7\end{cases}\\&=\begin{cases}\mathrm{O}\Big(\big\||U_{k,r,t}|^{2^*-2}\phi^2\big\|_1^{\frac{6-n}{4}}\|\phi\|_h^{\frac{n-2}{2}}+\|\phi\|_h^{2^*-1}\Big) & \text{if } n\le 6\\ \mathrm{O}\Big(\|\phi\|_h^{2^*-1}\Big) & \text{if } n\ge 7\end{cases}\\&=\begin{cases}\mathrm{O}\Big((\ln k)^{-2}\Big) & \text{if } n=3\\ \mathrm{O}\Big(k^{-2(n-3)}\Big) & \text{if } n\in\{4,5,6\}\\ \mathrm{O}\Big(k^{-\frac{(n+2)^2(n+4)}{2n(n-2)(n-4)}}\Big) & \text{if } n\ge 7\end{cases}\end{aligned} \tag{3.51}$$

uniformly in $k\ge k_1$, $r\in[a,b]$, $t\in[c,d]$ and $\phi\in V_{k,r,t}(C)$. It follows from (3.5) and (3.49)–(3.51) that there exists $k_2\ge k_1$ and $C_2>0$ such that $T_{k,r,t}(V_{k,r,t}(C_2))\subset V_{k,r,t}(C_2)$ for all $k\ge k_2$, $r\in[a,b]$ and $t\in[c,d]$. It remains to prove that $T_{k,r,t}$ is a contraction mapping on $V_{k,r,t}(C_2)$. It follows from Lemma 3.2 that

$$\|T_{k,r,t}(\phi_1)-T_{k,r,t}(\phi_2)\|_h\le C_1\|N_{k,r,t}(\phi_1)-N_{k,r,t}(\phi_2)\|_h \tag{3.52}$$

for all $\phi \in P_{k,r,t}$. By continuity of $(\Delta + h_0)^{-1} : L^{\frac{2n}{n+2}}(\mathbb{R}^n) \to D^{1,2}(\mathbb{R}^n)$ and using Hölder's and Sobolev inequalities, we obtain

$$\begin{aligned}
&\left\| N_{k,r,t}(\phi_1) - N_{k,r,t}(\phi_2) \right\|_h \\
&\quad = \mathrm{O}\Big(\big\| \left|U_{k,r,t} + \phi_1\right|^{2^*-2} (U_{k,r,t} + \phi_1) - \left|U_{k,r,t} + \phi_2\right|^{2^*-2} (U_{k,r,t} + \phi_2) \\
&\qquad\quad - (2^* - 1) \left|U_{k,r,t}\right|^{2^*-2} (\phi_1 - \phi_2) \big\|_{\frac{2n}{n+2}} \Big) \\
&\quad = \begin{cases} \mathrm{O}\Big(\big\| \big(\left|U_{k,r,t}\right|^{2^*-3} (|\phi_1| + |\phi_2|) \\ \qquad + |\phi_1|^{2^*-2} + |\phi_2|^{2^*-2} \big) |\phi_1 - \phi_2| \big\|_{\frac{2n}{n+2}} \Big) & \text{if } n \le 6 \\ \mathrm{O}\Big(\big\| \big(|\phi_1|^{2^*-2} + |\phi_2|^{2^*-2} \big) |\phi_1 - \phi_2| \big\|_{\frac{2n}{n+2}} \Big) & \text{if } n \ge 7 \end{cases} \\
&\quad = \begin{cases} \mathrm{O}\Big(\big(\big\| \left|U_{k,r,t}\right|^{2^*-2} (|\phi_1| + |\phi_2|) |\phi_1 - \phi_2| \big\|_1^{\frac{6-n}{4}} \\ \qquad \times \left(\|\phi_1\|_h + \|\phi_2\|_h \right)^{\frac{n-2}{4}} \left\|\phi_1 - \phi_2\right\|_h^{\frac{n-2}{4}} \\ \qquad + \big(\|\phi_1\|_h^{2^*-2} + \|\phi_2\|_h^{2^*-2} \big) \|\phi_1 - \phi_2\|_h \Big) & \text{if } n \le 6 \\ \mathrm{O}\Big(\big(\|\phi_1\|_h^{2^*-2} + \|\phi_2\|_h^{2^*-2} \big) \|\phi_1 - \phi_2\|_h \Big) & \text{if } n \ge 7 \end{cases}
\end{aligned} \tag{3.53}$$

uniformly in $k \ge k_1$, $r \in [a,b]$, $t \in [c,d]$ and $\phi \in P_{k,r,t}$. Moreover, we obtain

$$\begin{aligned}
&\big\| \left|U_{k,r,t}\right|^{2^*-2} (|\phi_1| + |\phi_2|) |\phi_1 - \phi_2| \big\|_1 \\
&\qquad = \frac{1}{2^* - 1} \left\langle |\phi_1| + |\phi_2| - L_{k,r,t}(|\phi_1| + |\phi_2|), |\phi_1 - \phi_2| \right\rangle_h \\
&\qquad \le \frac{C_1 + 1}{2^* - 1} \left(\|\phi_1\|_h + \|\phi_2\|_h \right) \|\phi_1 - \phi_2\|_h .
\end{aligned} \tag{3.54}$$

It follows from (3.52)–(3.54) that

$$\left\| T_{k,r,t}(\phi_1) - T_{k,r,t}(\phi_2) \right\|_h = \mathrm{o}\left(\|\phi_1 - \phi_2\|_h \right) \tag{3.55}$$

as $k \to \infty$ uniformly in $k \ge k_1$, $r \in [a,b]$, $t \in [c,d]$ and $\phi \in V_{k,r,t}(C_2)$. By using (3.55) and increasing if necessary the values of k_2 and C_2, we obtain that $T_{k,r,t}$ is a contraction mapping on $V_{k,r,t}$ for all $k \ge k_2$, $r \in [a,b]$ and $t \in [c,d]$. We can then apply the fixed point theorem, which gives the existence of a unique solution $\phi_{k,r,t} \in V_{k,r,t}(C_2)$ to the equation (3.47). The continuous differentiability of $(r,t) \mapsto \phi_{k,r,t}$ is standard (see for instance Robert and Vétois [36]). This ends the proof of Lemma 3.3. □

We define

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + h_0 u^2 \right) dx - \frac{1}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} dx \qquad \forall u \in D^{1,2}(\mathbb{R}^n).$$

We prove the following result:

Lemma 3.4. *Let $a,b,c,d>0$ be such that $a<b$ and $c<d$. Let $\phi_{k,r,t}$ be as in Lemma 3.1. Then*

$$I\left(U_{k,r,t}+\phi_{k,r,t}\right)=\frac{1}{n}\int_{\mathbb{R}^n}\left|u_0\right|^{2^*}+\frac{k}{n}K_n^{-n}+\frac{2}{n}K_n^{-n}E\left(r,t\right)\times\begin{cases}\left(\ln k\right)^{-1}\left(1+\mathrm{o}\left(1\right)\right) & \text{if } n=3\\ k^{3-n}\left(1+\mathrm{o}\left(1\right)\right) & \text{if } n\in\{4,5,6\}\\ k^{-\frac{n}{n-4}}\left(1+\mathrm{o}\left(1\right)\right) & \text{if } n\geq 7\end{cases}\tag{3.56}$$

as $k\to\infty$ uniformly in $r\in[a,b]$ and $t\in[c,d]$, where

$$E\left(r,t\right):=\begin{cases}\frac{8}{\pi}\left(\frac{u_0\left(r\right)t^{1/2}}{3^{1/4}}-\frac{t}{4\pi r}\right) & \text{if } n=3\\ \frac{\omega_{n-1}}{\omega_n}\left(\frac{2^{n-1}u_0\left(r\right)t^{\frac{n-2}{2}}}{\left[n\left(n-2\right)\right]^{\frac{n-2}{4}}}-\left(\frac{2t}{\pi r}\right)^{n-2}S_n\right) & \text{if } n\in\{4,5\}\\ \frac{5}{8}\left(\left(2u_0\left(r\right)+h_0\left(r\right)\right)t^2-\frac{3t^4}{2\left(\pi r\right)^4}S_6\right) & \text{if } n=6\\ \frac{\left(n-1\right)h_0\left(r\right)t^2}{\left(n-2\right)\left(n-4\right)}-\frac{\omega_{n-1}t^{n-2}}{\omega_n\left(\pi r\right)^{n-2}}S_n & \text{if } n\geq 7,\end{cases}\tag{3.57}$$

ω_{n-1} and ω_n are the areas of the unit spheres in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively,

$$S_n:=\sum_{i=1}^{\infty}i^{2-n}\quad\text{and}\quad K_n:=\frac{2\omega_n^{-1/n}}{\sqrt{n\left(n-2\right)}}$$

i.e. K_n is the best constant for the embedding $D^{1,2}\left(\mathbb{R}^n\right)\hookrightarrow L^{2^*}\left(\mathbb{R}^n\right)$ (see Aubin [1], *Rodemich* [38] *and Talenti* [42]*).*

Proof. By integrating by parts and using (3.2) and (3.3), we obtain

$$\int_{\mathbb{R}^n}\left|\nabla\left(U_{k,r,t}+\phi_{k,r,t}\right)\right|^2dx=\int_{\mathbb{R}^n}\Bigg[u_0^{2^*}-h_0u_0^2+2\left(h_0u_0-u_0^{2^*-1}\right)\sum_{i=1}^{k}B_{i,k,r,t}$$
$$+\sum_{i,j=1}^{k}B_{i,k,r,t}^{2^*-1}B_{j,k,r,t}+2\bigg(h_0u_0-u_0^{2^*-1}+\sum_{i=1}^{k}B_{i,k,r,t}^{2^*-1}\bigg)\phi_{k,r,t}+\left|\nabla\phi_{k,r,t}\right|^2\Bigg]dx.\tag{3.58}$$

If follows from (3.58) that

$$I\left(U_{k,r,t}+\phi_{k,r,t}\right)=\int_{\mathbb{R}^n}\Bigg[\frac{1}{n}u_0^{2^*}+\frac{1}{n}\sum_{i=1}^{k}B_{i,k,r,t}^{2^*}+u_0\sum_{i=1}^{k}B_{i,k,r,t}^{2^*-1}$$
$$-\frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}B_{i,k,r,t}^{2^*-1}B_{j,k,r,t}+\frac{1}{2}h_0\sum_{i,j=1}^{k}B_{i,k,r,t}B_{j,k,r,t}-\frac{1}{2^*}\bigg(\left|U_{k,r,t}+\phi_{k,r,t}\right|^{2^*}-u_0^{2^*}$$

$$- \sum_{i=1}^{k} B_{i,k,r,t}^{2^*} + 2^* u_0 \sum_{i=1}^{k} B_{i,k,r,t}^{2^*-1} + 2^* u_0^{2^*-1} \sum_{i=1}^{k} B_{i,k,r,t} - 2^* \sum_{i=1}^{k} \sum_{j \neq i} B_{i,k,r,t}^{2^*-1} B_{j,k,r,t}$$
$$- 2^* \left|U_{k,r,t}\right|^{2^*-2} U_{k,r,t}\phi_{k,r,t} \Big) \Big] dx + \frac{1}{2} \left\|\phi_{k,r,t}\right\|_h^2 - \left\langle R_{k,r,t}, \phi_{k,r,t}\right\rangle_h , \quad (3.59)$$

where $R_{k,r,t}$ is as in (3.4). Moreover, straightforward estimates give

$$\sum_{i=1}^{k} \sum_{j \neq i} \int_{\mathbb{R}^n} B_{i,k,r,t}^{2^*-1} B_{j,k,r,t} dx = k \sum_{i=1}^{k} \sum_{j \neq i} \int_{\Omega_1} B_{i,k,r,t}^{2^*-1} B_{j,k,r,t} dx$$
$$= k \sum_{i=2}^{k} \int_{\Omega_1} B_{1,k,r,t}^{2^*-1} B_{i,k,r,t} dx + \mathrm{O}\left(k \int_{\Omega_1} \left[\left(\sum_{i=2}^{k} B_{i,k,r,t} \right)^{2^*} \right.\right.$$
$$\left.\left. + B_{1,k,r,t} \sum_{i=2}^{k} B_{i,k,r,t}^{2^*-1} \right] dx \right), \quad (3.60)$$

$$\int_{\mathbb{R}^n} \left(\left|U_{k,r,t} + \phi_{k,r,t}\right|^{2^*} - \left|U_{k,r,t}\right|^{2^*} - 2^* \left|U_{k,r,t}\right|^{2^*-2} U_{k,r,t}\phi_{k,r,t} \right) dx$$
$$= \mathrm{O}\left(\int_{\mathbb{R}^n} \left(\left|U_{k,r,t}\right|^{2^*-2} \phi_{k,r,t}^2 + \left|\phi_{k,r,t}\right|^{2^*} \right) dx \right), \quad (3.61)$$

$$\int_{\mathbb{R}^n} \left[\left|U_{k,r,t}\right|^{2^*} - u_0^{2^*} - \left(\sum_{i=1}^{k} B_{i,k,r,t} \right)^{2^*} + 2^* u_0 \left(\sum_{i=1}^{k} B_{i,k,r,t} \right)^{2^*-1} \right.$$
$$\left. + 2^* u_0^{2^*-1} \sum_{i=1}^{k} B_{i,k,r,t} \right] dx$$
$$= \begin{cases} \mathrm{O}\left(\int_{\mathbb{R}^n} \left[u_0^{2^*-2} \left(\sum_{i=1}^{k} B_{i,k,r,t} \right)^2 + u_0^2 \left(\sum_{i=1}^{k} B_{i,k,r,t} \right)^{2^*-2} \right] dx \right) & \\ \qquad \text{if } n \leq 5 \text{ (i.e. } 2^*-2 > 1) & \\ 0 \qquad \text{if } n = 6 \text{ (i.e. } 2^*-2 = 1) & \\ \mathrm{O}\left(\int_{\mathbb{R}^n} \min\left[u_0^{2^*-2} \left(\sum_{i=1}^{k} B_{i,k,r,t} \right)^2, u_0^2 \left(\sum_{i=1}^{k} B_{i,k,r,t} \right)^{2^*-2} \right] dx \right) & \\ \qquad \text{if } n \geq 7 \text{ (i.e. } 2^*-2 < 1), & \end{cases} \quad (3.62)$$

$$\int_{\mathbb{R}^n} \left[\left(\sum_{i=1}^{k} B_{i,k,r,t} \right)^{2^*} - \sum_{i=1}^{k} B_{i,k,r,t}^{2^*} - 2^* \sum_{i=1}^{k} \sum_{j \neq i} B_{i,k,r,t}^{2^*-1} B_{j,k,r,t} \right] dx$$
$$= \mathrm{O}\left(k \int_{\Omega_1} \left[B_{1,k,r,t} \sum_{i=2}^{k} B_{i,k,r,t}^{2^*-1} + B_{1,k,r,t}^{2^*-2} \left(\sum_{i=2}^{k} B_{i,k,r,t} \right)^2 + \left(\sum_{i=2}^{k} B_{i,k,r,t} \right)^{2^*} \right] dx \right) \quad (3.63)$$

and

$$\int_{\mathbb{R}^n} u_0 \left[\left(\sum_{i=1}^k B_{i,k,r,t} \right)^{2^*-1} - \sum_{i=1}^k B_{i,k,r,t}^{2^*-1} \right] dx$$
$$= \mathrm{O} \left(k \int_{\Omega_1} u_0 \left[B_{1,k,r,t}^{2^*-2} \sum_{i=2}^k B_{i,k,r,t} + \left(\sum_{i=2}^k B_{i,k,r,t} \right)^{2^*-1} \right] dx \right), \quad (3.64)$$

where Ω_1 is as in (3.8). By remarking that $h_0 = \mathrm{O}\left(u_0^{2^*-2}\right)$ and putting together (3.59)–(3.64), we obtain

$$I\left(U_{k,r,t} + \phi_{k,r,t}\right) = \int_{\mathbb{R}^n} \left[\frac{1}{n} u_0^{2^*} + \frac{1}{n} \sum_{i=1}^k B_{i,k,r,t}^{2^*} + u_0 \sum_{i=1}^k B_{i,k,r,t}^{2^*-1} + \frac{1}{2} h_0 \sum_{i=1}^k B_{i,k,r,t}^2 \right.$$
$$\left. + \mathrm{O}\left(\left|U_{k,r,t}\right|^{2^*-2} \phi_{k,r,t}^2 + \left|\phi_{k,r,t}\right|^{2^*} \right) \right] dx + \mathrm{O}\left(\left\|\phi_{k,r,t}\right\|_h^2 + \left\|R_{k,r,t}\right\|_h \left\|\phi_{k,r,t}\right\|_h \right)$$
$$- \frac{k}{2} \int_{\Omega_1} B_{1,k,r,t}^{2^*-1} \sum_{i=2}^k B_{i,k,r,t} dx + \mathrm{O}\left(k J_{k,r,t}\right), \quad (3.65)$$

where

$$J_{k,r,t} = \int_{\Omega_1} \left[\left(\sum_{i=2}^k B_{i,k,r,t} \right)^{2^*} + B_{1,k,r,t} \sum_{i=2}^k B_{i,k,r,t}^{2^*-1} + B_{1,k,r,t}^{2^*-2} \left(\sum_{i=2}^k B_{i,k,r,t} \right)^2 \right.$$
$$\left. + u_0^{2^*-2} \left(\sum_{i=2}^k B_{i,k,r,t} \right)^2 \right] dx$$
$$+ \begin{cases} \displaystyle\int_{\Omega_1} \left[u_0^{2^*-2} B_{1,k,r,t}^2 + u_0^2 B_{1,k,r,t}^{2^*-2} + u_0^2 \left(\sum_{i=2}^k B_{i,k,r,t} \right)^{2^*-2} \right] dx & \text{if } n \le 5 \\ \displaystyle\int_{\Omega_1} u_0 B_{1,k,r,t} \sum_{i=2}^k B_{i,k,r,t} dx & \text{if } n = 6 \\ \displaystyle\int_{\Omega_1 \setminus B(x_{1,k,r}, \sqrt{\delta_k})} u_0^{2^*-2} B_{1,k,r,t}^2 dx + \int_{B(x_{1,k,r}, \sqrt{\delta_k})} u_0^2 B_{1,k,r,t}^{2^*-2} dx & \\ \displaystyle\quad + \int_{\Omega_1} \left[u_0 B_{1,k,r,t}^{2^*-2} \sum_{i=2}^k B_{i,k,r,t} + u_0^{2^*-2} B_{1,k,r,t} \sum_{i=2}^k B_{i,k,r,t} \right. & \\ \displaystyle\qquad \left. + u_0 \left(\sum_{i=2}^k B_{i,k,r,t} \right)^{2^*-1} \right] dx & \text{if } n \ge 7, \end{cases}$$
$$(3.66)$$

where $B(x_{1,k,r}, \sqrt{\delta_k})$ is the Euclidean ball of center x_{1,k,r_k} and radius $\sqrt{\delta_k}$. Straightforward calculations (see for instance Aubin [2] and Robert and Vétois [37]) give

$$\int_{\mathbb{R}^n} B_{i,k,r,t}^{2^*} dx = K_n^{-n}, \quad (3.67)$$

$$\int_{\mathbb{R}^n} u_0 B_{i,k,r,t}^{2^*-1} dx = \frac{2^n \omega_{n-1} K_n^{-n} u_0\left(r\right) \left(t\delta_k\right)^{\frac{n-2}{2}}}{n^{\frac{n+2}{4}} \left(n-2\right)^{\frac{n-2}{4}} \omega_n} \left(1 + \mathrm{o}\left(1\right)\right) \quad (3.68)$$

and

$$\int_{\mathbb{R}^n} h_0 B_{i,k,r,t}^2 dx = \begin{cases} \mathrm{O}\left(\delta_k\right) & \text{if } n=3 \\ \mathrm{O}\left(\delta_k^2 \left|\ln \delta_k\right|\right) & \text{if } n=4 \\ \dfrac{4\left(n-1\right)K_n^{-n} h_0\left(r\right)\left(t\delta_k\right)^2}{n\left(n-2\right)\left(n-4\right)} + \mathrm{o}\left(\delta_k^2\right) & \text{if } n \geq 5 \end{cases} \tag{3.69}$$

as $k \to \infty$. For every $i \in \{2, \dots, k\}$, by applying the mean value theorem and observing that $|x_{1,k,r} - x_{i,k,r}| \geq 4r/k$, we obtain

$$B_{i,k,r,t} = \frac{\left[\sqrt{n\left(n-2\right)}t\delta_k\right]^{\frac{n-2}{2}}}{\left|x_{1,k,r} - x_{i,k,r}\right|^{n-2}} + \mathrm{O}\left(\frac{\delta_k^{\frac{n+2}{2}}}{\left|x_{1,k,r} - x_{i,k,r}\right|^n} + \frac{\delta_k^{\frac{n-2}{2}}\left|x - x_{1,k,r}\right|}{\left|x_{1,k,r} - x_{i,k,r}\right|^{n-1}}\right) \tag{3.70}$$

in $B(x_{1,k,r}, r/k)$. Direct calculations give

$$\int_{B(x_{1,k,r},r/k)} B_{1,k,r,t}^{2^*-1} dx = \frac{2^n \omega_{n-1} K_n^{-n} \left(t\delta_k\right)^{\frac{n-2}{2}}}{n^{\frac{n+2}{4}} \left(n-2\right)^{\frac{n-2}{4}} \omega_n} \left(1 + \mathrm{o}\left(1\right)\right), \tag{3.71}$$

$$\int_{B(x_{1,k,r},r/k)} B_{1,k,r,t}^{2^*-1} \left|x - x_{1,k,r}\right| dx = \mathrm{O}\left(\delta_k^{n/2}\right) \tag{3.72}$$

and

$$\sum_{i=2}^{k} \frac{1}{\left|x_{1,k,r} - x_{i,k,r}\right|^p} = \begin{cases} \dfrac{k}{2\pi r}\left(\ln k + \mathrm{o}\left(\ln k\right)\right) & \text{if } p=1 \\ \left(\dfrac{k}{2\pi r}\right)^p \displaystyle\sum_{i=1}^{\infty} i^{-p} + \mathrm{o}\left(1\right) & \text{if } p>1 \end{cases} \tag{3.73}$$

as $k \to \infty$ since $|x_{1,k,r} - x_{i,k,r}| = 2r \sin\left(\left(i-1\right)\pi/k\right)$. By putting together (3.70)–(3.73), we obtain

$$\int_{B(x_{1,k,r},r/k)} B_{1,k,r,t}^{2^*-1} \sum_{i=2}^{k} B_{i,k,r,t} dx = \frac{4\omega_{n-1} K_n^{-n}}{n\omega_n}\left(\frac{kt\delta_k}{\pi r}\right)^{n-2} \times \begin{cases} \ln k\left(1 + \mathrm{o}\left(1\right)\right) & \text{if } n=3 \\ S_n + \mathrm{o}\left(1\right) & \text{if } n \geq 4 \end{cases} \tag{3.74}$$

as $k \to \infty$. On the other hand, by similar calculations as in the proof of Lemma A.1, we obtain

$$\begin{aligned} &\int_{\Omega_1 \backslash B(x_{1,k,r},r/k)} B_{1,k,r,t}^{2^*-1} \sum_{i=2}^{k} B_{i,k,r,t} dx \\ &\quad = \mathrm{O}\left(\delta_k^n \sum_{i=2}^{k} \int_{\Omega_1 \backslash B(x_{1,k,r},r/k)} \frac{dx}{\left|x - x_{i,k,r}\right|^{n-2}\left|x - x_{1,k,r}\right|^{n+2}}\right) \\ &\quad = \begin{cases} \mathrm{O}\left(\left(k\delta_k\right)^3 \ln k\right) & \text{if } n=3 \\ \mathrm{O}\left(\left(k\delta_k\right)^n\right) & \text{if } n \geq 4. \end{cases} \end{aligned} \tag{3.75}$$

Now we estimate the integrals in the remainder terms of (3.65). By using (3.5) and (3.48) together with the Sobolev inequality, we obtain

$$\int_{\mathbb{R}^n}\left(\left|U_{k,r,t}\right|^{2^*-2}\phi_{k,r,t}^2+\left|\phi_{k,r,t}\right|^{2^*}\right)dx+\left\|\phi_{k,r,t}\right\|_h^2+\left\|R_{k,r,t}\right\|_h\left\|\phi_{k,r,t}\right\|_h$$
$$=\begin{cases}\mathrm{O}\left(\left(\ln k\right)^{-2}\right) & \text{if } n=3\\ \mathrm{O}\left(k^{-2(n-3)}\right) & \text{if } n\in\{4,5,6\}\\ \mathrm{O}\left(k^{-\frac{(n+2)(n+4)}{n(n-4)}}\right) & \text{if } n\geq 7.\end{cases} \tag{3.76}$$

Finally, by applying Lemma A.1 in the appendix and using the definition of δ_k, we obtain

$$J_{k,r,t}=\begin{cases}\mathrm{o}\left(\left(k\ln k\right)^{-1}\right) & \text{if } n=3\\ \mathrm{o}\left(k^{2-n}\right) & \text{if } n\in\{4,5,6\}\\ \mathrm{o}\left(k^{-\frac{2(n-2)}{n-4}}\right) & \text{if } n\geq 7.\end{cases} \tag{3.77}$$

We then obtain (3.56) by combining (3.65)–(3.69) and (3.74)–(3.77) and using the definition of δ_k. This ends the proof of Lemma 3.4. □

Now we can end the proof of Theorem 1.2.

Proof of Theorem 1.2. We let E be the function defined in (3.57). Observe that the assumptions of Theorem 1.2 imply that $2u_0+h_0>0$ in case $n=6$ and $h_0>0$ in case $n\geq 7$. Then it is easy to check that E attains a strict maximum at the point $(1,t_0)$, where

$$t_0:=\begin{cases}\dfrac{\pi^2u_0\left(1\right)^2}{\sqrt{3}} & \text{if } n=3\\ \dfrac{4\pi^2}{\sqrt{n\left(n-2\right)}}\left(\dfrac{u_0\left(1\right)}{S_n}\right)^{\frac{2}{n-2}} & \text{if } n\in\{4,5\}\\ \pi^2\sqrt{\dfrac{2u_0\left(1\right)+h_0\left(1\right)}{3S_6}} & \text{if } n=6\\ \left(\dfrac{2\pi^{n-2}\left(n-1\right)\omega_nh_0\left(1\right)}{\left(n-2\right)^2\left(n-4\right)\omega_{n-1}S_n}\right)^{\frac{1}{n-4}} & \text{if } n\geq 7.\end{cases}$$

It then follows from (3.56) that for k large, there exists a critical point (r_k,t_k) of the function $(r,t)\mapsto I\left(U_{k,r,t}+\phi_{k,r,t}\right)$ such that $(r_k,t_k)\to(1,t_0)$ as $k\to\infty$. We then define $u_k:=U_{k,r,t_k}+\phi_{k,r,t_k}$.

Since u_k is also a solution of the equation (3.47), we obtain that there exist real numbers $c_{1,k}$ and $c_{2,k}$ such that

$$DI\left(u_k\right)=\sum_{j=1}^{2}c_{j,k}\sum_{i=1}^{k}\left\langle Z_{i,j,k,r_k,t_k},\cdot\right\rangle_h. \tag{3.78}$$

Since (r_k,t_k) is a critical point of $(r,t)\mapsto I\left(U_{k,r,t}+\phi_{k,r,t}\right)$, we obtain

$$DI\left(u_k\right).\frac{d}{dr}\left[U_{k,r,t_k}+\phi_{k,r,t_k}\right]_{r=r_k}=DI\left(u_k\right).\frac{d}{dt}\left[U_{k,r_k,t}+\phi_{k,r_k,t}\right]_{t=t_k}=0. \tag{3.79}$$

Direct calculations give

$$\left\langle Z_{i,j,k,r_k,t_k}, \frac{d}{dr}\left[U_{k,r,t_k}\right]_{r=r_k}\right\rangle_h = \frac{1}{\delta_k}\sum_{\alpha=1}^{k}\left\langle Z_{i,j,k,r_k,t_k}, Z_{\alpha,1,k,r_k,t_k}\right\rangle_h = \begin{cases} \frac{1}{\delta_k}\left(\left\|\nabla V_1\right\|_2^2 + \mathrm{o}(1)\right) & \text{if } j=1\\ \mathrm{o}\left(\frac{1}{\delta_k}\right) & \text{if } j=2\end{cases} \tag{3.80}$$

and

$$\left\langle Z_{i,j,k,r_k,t_k}, \frac{d}{dt}\left[U_{k,r_k,t}\right]_{t=t_k}\right\rangle_h = \sum_{\alpha=1}^{k}\left\langle Z_{i,j,k,r_k,t_k}, Z_{\alpha,2,k,r_k,t_k}\right\rangle_h = \begin{cases} \mathrm{o}(1) & \text{if } j=1\\ \left\|\nabla V_2\right\|_2^2 + \mathrm{o}(1) & \text{if } j=2\end{cases} \tag{3.81}$$

as $k\to\infty$, where

$$V_1 := \frac{n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}x_1}{\left(1+|x|^2\right)^{\frac{n}{2}}} \quad\text{and}\quad V_2 := \frac{n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}\left(|x|^2-1\right)}{2\left(1+|x|^2\right)^{\frac{n}{2}}}.$$

Moreover, since $\phi_{k,r,t}\in P_{k,r,t}$, we obtain

$$\sum_{i=1}^{k}\left\langle Z_{i,j,k,r_k,t_k}, \frac{d}{dr}\left[\phi_{k,r,t_k}\right]_{r=r_k}\right\rangle_h = -\sum_{i=1}^{k}\left\langle \frac{d}{dr}\left[Z_{i,j,k,r,t_k}\right]_{r=r_k}, \phi_{k,r_k,t_k}\right\rangle_h \tag{3.82}$$

and

$$\sum_{i=1}^{k}\left\langle Z_{i,j,k,r_k,t_k}, \frac{d}{dt}\left[\phi_{k,r_k,t}\right]_{t=t_k}\right\rangle_h = -\sum_{i=1}^{k}\left\langle \frac{d}{dt}\left[Z_{i,j,k,r_k,t}\right]_{t=t_k}, \phi_{k,r_k,t_k}\right\rangle_h. \tag{3.83}$$

By using Cauchy–Schwartz inequality together with (3.48) and direct calculations, we obtain

$$\left|\left\langle \frac{d}{dr}\left[Z_{i,j,k,r,t_k}\right]_{r=r_k}, \phi_{k,r_k,t_k}\right\rangle_h\right| \le \left\|\frac{d}{dr}\left[Z_{i,j,k,r,t_k}\right]_{r=r_k}\right\|_h \left\|\phi_{k,r_k,t_k}\right\|_h = \mathrm{o}\left(\left\|\frac{d}{dr}\left[Z_{i,j,k,r,t_k}\right]_{r=r_k}\right\|_h\right) = \mathrm{o}\left(\frac{1}{\delta_k}\right) \tag{3.84}$$

and

$$\left|\left\langle \frac{d}{dt}\left[Z_{i,j,k,r_k,t}\right]_{t=t_k}, \phi_{k,r_k,t_k}\right\rangle_h\right| \le \left\|\frac{d}{dt}\left[Z_{i,j,k,r_k,t}\right]_{t=t_k}\right\|_h \left\|\phi_{k,r_k,t_k}\right\|_h = \mathrm{o}\left(\left\|\frac{d}{dt}\left[Z_{i,j,k,r_k,t}\right]_{t=t_k}\right\|_h\right) = \mathrm{o}(1) \tag{3.85}$$

as $k\to\infty$. It follows from (3.78)–(3.84) that if k is large enough, then $c_{1,k}=c_{2,k}=0$, i.e. the function u_k is a solution of the equation (3.1).

It remains to verify that (1.4) holds true. For this, we define

$$v_k := u_k^- = -\min\left(0,u_k\right).$$

By remarking that $v_k \to u_0$ a.e. in $\mathbb{R}^n$, we obtain

$$\liminf_{k\to\infty} \min_M u_k = -\limsup_{k\to\infty} \max_M v_k \le -\max_M u_0 < 0. \tag{3.86}$$

Moreover, since $0 \le v_k < u_0 - \phi_{k,r_k,t_k}$ and $\phi_{k,r_k,t_k} \to 0$ in $L^{2^*}(\mathbb{R}^n)$, we obtain that $v_k \to u_0$ in $L^{2^*}(\mathbb{R}^n)$ as $k \to \infty$. For every $p \ge 2$, by multiplying (3.1) by v_k^{p-1} and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} v_k^{p+2^*-2} dx &= \int_{\mathbb{R}^n} \Big((p-1)\, v_k^{p-2} \left|\nabla v_k\right|^2 + h_0 v_k^p \Big)\, dx \\ &= \int_{\mathbb{R}^n} \left(\frac{4\,(p-1)}{p^2} \left|\nabla\big[v_k^{p/2}\big]\right|^2 + h_0 v_k^p \right) dx \ge \frac{4\,(p-1)}{p^2} \big\|v_k^{p/2}\big\|_h^2. \end{aligned} \tag{3.87}$$

It follows from the Sobolev inequality and (3.86) that

$$\left\|v_k\right\|_{2^*p/2}^{\frac{p}{p+2^*-2}} = \big\|v_k^{p/2}\big\|_{2^*}^{\frac{2}{p+2^*-2}} \le C \left\|v_k\right\|_{p+2^*-2} \tag{3.88}$$

for some constant $C > 0$ independent of k. On the other hand, by applying the triangle inequality together with an interpolation inequality, we obtain

$$\begin{aligned}\left\|v_k\right\|_{p+2^*-2} &\le \left\|u_0\right\|_{p+2^*-2} + \left\|v_k - u_0\right\|_{p+2^*-2} \\ &\le \left\|u_0\right\|_{p+2^*-2} + \left\|v_k - u_0\right\|_{2^*p/2}^{\frac{p}{p+2^*-2}} \left\|v_k - u_0\right\|_{2^*}^{\frac{2^*-2}{p+2^*-2}} \\ &\le \left\|u_0\right\|_{p+2^*-2} + \big(\left\|v_k\right\|_{2^*p/2} + \left\|u_0\right\|_{2^*p/2} \big)^{\frac{p}{p+2^*-2}} \left\|v_k - u_0\right\|_{2^*}^{\frac{2^*-2}{p+2^*-2}}. \end{aligned} \tag{3.89}$$

Since $v_k \to u_0$ in $L^{2^*}(\mathbb{R}^n)$ as $k \to \infty$, it follows from (3.87)–(3.89) that $\left(v_k\right)_k$ is bounded in $L^{2^*p/2}(\mathbb{R}^n)$. By applying Theorem 4.1 in the book of Han and Lin [21], we then obtain that $\left(v_k\right)_k$ is bounded in $L^\infty(M)$. This, together with (3.85), implies the first part of (1.4).

Now we prove the second part of (1.4). Assume by contradiction that $\left\|u_k\right\|_\infty \le C$ for some constant $C > 0$ independent of k. We then obtain

$$-\phi_{k,r_k,t_k} \ge B_{1,k,r_k,t_k} - u_0 - C \ge \left(\frac{\sqrt{n\,(n-2)}}{2t_k\delta_k} \right)^{\frac{n-2}{2}} - u_0 - C \quad \text{in } B\left(x_{1,k,r_k}, t_k\delta_k\right)$$

and so

$$\left\|\phi_{k,r_k,t_k}\right\|_{2^*} \ge \left(\frac{\sqrt{n\,(n-2)}}{2} \right)^n + \mathrm{o}(1)$$

as $k \to \infty$, which is in contradiction with (3.48). This completes the proof of Theorem 1.2. □

Appendix A. Integral estimates

In this appendix, we prove the following lemma, which we used several times in the proof of Theorem 1.2.

Lemma A.1. *Let Ω_1 and $B_{i,k,r,t}$ be as in the previous section. For every $\alpha, \beta \ge 0$ and $a, b, c, d > 0$ such that $\alpha + \beta \le 2^*$, $a < b$ and $c < d$, there exists a constant*

$C>0$ such that

$$\int_{\Omega_1} u_0^{2^*-\alpha-\beta} B_{1,k,r,t}^{\alpha}\bigg(\sum_{i=2}^{k} B_{i,k,r,t}\bigg)^{\beta} dx$$

$$\le C\left(\delta_k^{\frac{(\alpha+\beta)(n-2)}{2}} \times \begin{cases} k^{\beta-1} & \text{if } \alpha(n-2)+\beta(n-3)<n-1\\ k^{\beta-1}(\ln k)^{\beta+1} & \text{if } \alpha=2 \text{ and } n=3\\ k^{\alpha+\beta-3}(\ln k)^{\beta} & \text{if } \alpha>2 \text{ and } n=3\\ k^{\beta-1}\ln k & \text{if } \alpha(n-2)+\beta(n-3)=n-1 \text{ and } n\ge 4\\ k^{(\alpha+\beta)(n-2)-n} & \text{if } \alpha(n-2)+\beta(n-3)>n-1 \text{ and } n\ge 4 \end{cases}\right.$$

$$\left. + \begin{Bmatrix} (k\ln k)^{\beta} & \text{if } n=3\\ k^{\beta(n-2)} & \text{if } n\ge 4 \end{Bmatrix} \times \begin{Bmatrix} \delta_k^{\frac{(\alpha+\beta)(n-2)}{2}} k^{\alpha(n-2)-n} & \text{if } \alpha<\frac{n}{n-2}\\ \delta_k^{\frac{n+\beta(n-2)}{2}}\left|\ln \delta_k\right| & \text{if } \alpha=\frac{n}{n-2}\\ \delta_k^{n+\frac{(\beta-\alpha)(n-2)}{2}} & \text{if } \alpha>\frac{n}{n-2} \end{Bmatrix}\right) \tag{A.1}$$

for all $k\ge 2$, $r\in[a,b]$ and $t\in[c,d]$.

Proof. By splitting the integral into three parts, in the domains $\Omega_1' := B(x_{1,k,r},\delta_k)$, $\Omega_1'' := \Omega_1\cap B(0,2r)\backslash B(x_{1,k,r},\delta_k)$ and $\Omega_1''' := \Omega_1\backslash B(0,2r)$ and remarking that $|x-x_{1,k,r}|\ge |x|/2$ for all $x\in\Omega_1'''$, we obtain

$$\begin{aligned}\int_{\Omega_1} u_0^{2^*-\alpha-\beta} B_{1,k,r,t}^{\alpha}\bigg(\sum_{i=2}^{k} B_{i,k,r,t}\bigg)^{\beta} dx\\ = \mathrm{O}\Bigg(\delta_k^{\frac{(\beta-\alpha)(n-2)}{2}}\int_{\Omega_1'}\bigg(\sum_{i=2}^{k}\frac{1}{|x-x_{i,k,r}|^{n-2}}\bigg)^{\beta} dx\\ + \delta_k^{\frac{(\alpha+\beta)(n-2)}{2}}\int_{\Omega_1''}\bigg(\sum_{i=2}^{k}\frac{1}{|x-x_{i,k,r}|^{n-2}}\bigg)^{\beta}\frac{dx}{|x-x_{1,k,r}|^{\alpha(n-2)}}\\ + \delta_k^{\frac{(\alpha+\beta)(n-2)}{2}}\int_{\Omega_1'''}\bigg(\sum_{i=2}^{k}\frac{1}{|x-x_{i,k,r}|^{n-2}}\bigg)^{\beta}\frac{dx}{|x|^{2n-\beta(n-2)}}\Bigg).\end{aligned} \tag{A.2}$$

For every $x\in\Omega_1$, since $|x-x_{1,k,r}|<|x-x_{i,k,r}|$, we obtain

$$\langle x_{1,k,r}-x, x_{1,k,r}-x_{i,k,r}\rangle < \frac{1}{2}\left|x_{1,k,r}-x_{i,k,r}\right|^2. \tag{A.3}$$

Moreover, by applying Young's inequality, we obtain

$$\langle x_{1,k,r}-x, x_{1,k,r}-x_{i,k,r}\rangle \le \frac{3}{4}\left|x-x_{1,k,r}\right|^2+\frac{1}{3}\left|x_{1,k,r}-x_{i,k,r}\right|^2. \tag{A.4}$$

It follows from (A.3) and (A.4) that

$$\begin{aligned}\left|x-x_{i,k,r}\right|^2 &> \frac{1}{4}\left|x-x_{1,k,r}\right|^2+\frac{1}{6}\left|x_{1,k,r}-x_{i,k,r}\right|^2\\ &= \frac{1}{4}\left|x-x_{1,k,r}\right|^2+\frac{2}{3}r^2\sin\left((i-1)\pi/k\right)^2\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{i=2}^{k}\frac{1}{\left|x-x_{i,k,r}\right|^{n-2}} &\leq \sum_{i=2}^{k}\frac{1}{\left(\frac{1}{4}\left|x-x_{1,k,r}\right|^{2}+\frac{2}{3}r^{2}\sin\left(\left(i-1\right)\pi/k\right)^{2}\right)^{\frac{n-2}{2}}}\\
&\leq 2k\int_{1/k}^{1/2}\frac{ds}{\left(\frac{1}{4}\left|x-x_{1,k,r}\right|^{2}+\frac{2}{3}r^{2}\sin\left(\pi s\right)^{2}\right)^{\frac{n-2}{2}}}\\
&\quad+\frac{2}{\left(\frac{1}{4}\left|x-x_{1,k,r}\right|^{2}+\frac{2}{3}r^{2}\sin\left(\pi/k\right)^{2}\right)^{\frac{n-2}{2}}}\\
&\leq 2k\int_{1/k}^{1/2}\frac{ds}{\left(\frac{1}{4}\left|x-x_{1,k,r}\right|+\frac{\sqrt{2}rs}{\sqrt{3}}\right)^{n-2}}+\frac{2}{\left(\frac{1}{4}\left|x-x_{1,k,r}\right|+\frac{\sqrt{2}r}{\sqrt{3}k}\right)^{n-2}}\\
&\leq\begin{cases}Ck\ln\left(\dfrac{\left|x-x_{1,k,r}\right|+1}{\left|x-x_{1,k,r}\right|+1/k}\right) & \text{if } n=3\\ Ck\left(\dfrac{1}{\left(\left|x-x_{1,k,r}\right|+1/k\right)^{n-3}}-\dfrac{1}{\left(\left|x-x_{1,k,r}\right|+1\right)^{n-3}}\right) & \text{if } n\geq 4\end{cases} \qquad \text{(A.5)}
\end{aligned}$$

for some constant $C>0$ independent of k. It follows from (A.5) that

$$\sum_{i=2}^{k}\frac{1}{\left|x-x_{i,k,r}\right|^{n-2}}\leq\begin{cases}Ck\ln k & \text{if } n=3\\ Ck^{n-2} & \text{if } n\geq 4\end{cases} \qquad \text{(A.6)}$$

and

$$\sum_{i=2}^{k}\frac{1}{\left|x-x_{i,k,r}\right|^{n-2}}\leq\begin{cases}Ck\ln\left(1+\dfrac{1}{\left|x-x_{1,k,r}\right|}\right) & \text{if } n=3\\ \dfrac{Ck}{\max\left(\left|x-x_{1,k,r}\right|^{n-3},\left|x-x_{1,k,r}\right|^{n-2}\right)} & \text{if } n\geq 4.\end{cases} \qquad \text{(A.7)}$$

It follows from (A.6) that

$$\int_{\Omega_1'}\left(\sum_{i=2}^{k}\frac{1}{\left|x-x_{i,k,r}\right|^{n-2}}\right)^{\beta}dx=\begin{cases}\mathrm{O}\left(\delta_k^n\left(k\ln k\right)^{\beta}\right) & \text{if } n=3\\ \mathrm{O}\left(\delta_k^n k^{\beta(n-2)}\right) & \text{if } n\geq 4.\end{cases} \qquad \text{(A.8)}$$

We define

$$\Gamma_1:=\left\{(x_1,\dots,x_n)\in\Omega_1:\,\left|x_1-r\right|^2+\sum_{i=3}^{n}\left|x_i\right|^2<\frac{1}{k^2}\right\}.$$

By remarking that

$$\Omega_1''\subset\left\{(x_1,\dots,x_n)\in\mathbb{R}^n:\,\left|x_2\right|<2r\pi/k\right\} \qquad \text{(A.9)}$$

and since $k\delta_k \to 0$ as $k \to \infty$, it follows from (A.6) that

$$\int_{\Omega_1''\cap\Gamma_1} \left(\sum_{i=2}^{k} \frac{1}{|x - x_{i,k,r}|^{n-2}} \right)^{\beta} \frac{dx}{|x - x_{1,k,r}|^{\alpha(n-2)}}$$
$$= \left. \begin{cases} \mathrm{O}\left((k \ln k)^{\beta} \right) & \text{if } n = 3 \\ \mathrm{O}\left(k^{\beta(n-2)} \right) & \text{if } n \geq 4 \end{cases} \right\} \times \begin{cases} \mathrm{O}\left(k^{\alpha(n-2)-n} \right) & \text{if } \alpha < \dfrac{n}{n-2} \\ \mathrm{O}\left(|\ln \delta_k| \right) & \text{if } \alpha = \dfrac{n}{n-2} \\ \mathrm{O}\left(\delta_k^{n-\alpha(n-2)} \right) & \text{if } \alpha > \dfrac{n}{n-2}\,. \end{cases} \tag{A.10}$$

By using (A.7) and (A.9), straightforward estimates give

$$\int_{\Omega_1''\setminus\Gamma_1} \left(\sum_{i=2}^{k} \frac{1}{|x - x_{i,k,r}|^{n-2}} \right)^{\beta} \frac{dx}{|x - x_{1,k,r}|^{\alpha(n-2)}}$$
$$= \begin{cases} \mathrm{O}\left(k^{\beta-1} \right) & \text{if } \alpha(n-2) + \beta(n-3) < n-1 \\ \mathrm{O}\left(k^{\beta-1} (\ln k)^{\beta+1} \right) & \text{if } \alpha = 2 \text{ and } n = 3 \\ \mathrm{O}\left(k^{\alpha+\beta-3} (\ln k)^{\beta} \right) & \text{if } \alpha > 2 \text{ and } n = 3 \\ \mathrm{O}\left(k^{\beta-1} \ln k \right) & \text{if } \alpha(n-2) + \beta(n-3) = n-1 \text{ and } n \geq 4 \\ \mathrm{O}\left(k^{(\alpha+\beta)(n-2)-n} \right) & \text{if } \alpha(n-2) + \beta(n-3) > n-1 \text{ and } n \geq 4. \end{cases} \tag{A.11}$$

By using (A.7), we obtain

$$\int_{\Omega_1'''} \left(\sum_{i=2}^{k} \frac{1}{|x - x_{i,k,r}|^{n-2}} \right)^{\beta} \frac{dx}{|x|^{2n-\beta(n-2)}} = \mathrm{O}\left(k^{\beta-1} \right). \tag{A.12}$$

Finally, (A.1) follows from (A.2), (A.8) and (A.10)–(A.12). This ends the proof of Lemma A.1. □

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