A POINTWISE FINITE-DIMENSIONAL REDUCTION METHOD FOR EINSTEIN-LICHNEROWICZ-TYPE SYSTEMS: THE SIX-DIMENSIONAL CASE.

BRUNO PREMOSELLI

Abstract. We construct blowing-up solutions for smooth perturbations of a focusing Einstein-Lichnerowicz system on a closed 6-dimensional manifold. Our construction combines asymptotic analysis techniques with a finite-dimensional reduction approach, following the scheme of proof developed in [24]. The blow-up examples constructed here highlight in particular the role of the coupling field in dimension 6.

1. Introduction

1.1. **Statement of the results.** Let (M, g) be a closed (that is, compact without boundary) Riemannian manifold of dimensions $n \geq 3$. Let $2^* = \frac{2n}{n-2}$. The Einstein-Lichnerowicz system in M, of unknowns (u, T), which we shall investigate in this paper writes as follows:

$$\begin{cases} \triangle_g u + hu = f u^{2^*-1} + \dfrac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g T = u^{2^*} X + Y. \end{cases} \tag{1.1}$$

Here u is a smooth positive function and T is a smooth 1-form. Denoting by ∇ the Levi-Civita connection of g, the conformal Killing derivative $\mathcal{L}_g T$ writes as

$$\mathcal{L}_g T_{ij} = \nabla_i T_j + \nabla_j T_i - \frac{2}{n}\big(\mathrm{div}_g T\big) g_{ij}, \tag{1.2}$$

while the Laplace-Beltrami and the Lamé operator are respectively given by $\triangle_g u = -\mathrm{div}_g(\nabla u)$ and $\overrightarrow{\triangle}_g T = -\mathrm{div}_g\big(\mathcal{L}_g T\big)$. Also, in (1.1), h, f, π are smooth functions in M with $f > 0$ and $\pi \not\equiv 0$, σ is a smooth bilinear form with $\mathrm{tr}_g \sigma = 0$ and $\mathrm{div}_g \sigma = 0$ and X and Y are smooth 1-form. The $f > 0$ assumption of (1.1) is referred to as the *focusing case*.

System (1.1) was introduced in [19] as a conformal reformulation of the so-called *constraint equations*, which constrain admissible initial-data sets for the evolution problem of the Einstein equations in General Relativity (see also [1]). As such, solutions of (1.1) explicitly parameterize admissible initial-data sets. The coefficients of (1.1) depend on the physics data of the problem and on the chosen matter model.

In the scalar-field setting, for instance, they take the following form:

$$h = \frac{n-2}{4(n-1)}\left(S_g - |\nabla\psi|_g^2\right),\ f = \frac{n-2}{4(n-1)}\left(2V(\psi) - \frac{n-1}{n}\tau^2\right),\ X = -\frac{n-1}{n}\nabla\tau,$$

where (ψ, π) are scalar-field data, τ is a mean curvature and V is a potential. Also, S_g denotes the scalar curvature of (M, g). If (u, T) is a solution of (1.1), an explicit initial-data set for the scalar-field Einstein equations is then given by the following parametrization:

$$(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}) = \left(u^{\frac{4}{n-2}}g, \frac{\tau}{n}u^{\frac{4}{n-2}}g + u^{-2}(\sigma + \mathcal{L}_gT), \psi, u^{-\frac{2n}{n-2}}\pi\right). \tag{1.3}$$

In recent years system (1.1) proved extremely effective to construct admissible initial-data sets by (1.3): see for instance [6, 14, 16, 20, 22] for existence results and [23] for multiplicity results.

The relevance of system (1.1) as a physical reformulation of the initial-value problem in General Relativity has been investigated in [12, 25], where a notion of *elliptic stability* for (1.1) has been introduced. Let $(h, f, \pi, \sigma, X, Y)$ be a set of coefficients of (1.1). Following [9], we say that (1.1) is *stable* if, for any sequence $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k$ converging towards $(h, f, \pi, \sigma, X, Y)$ in $C^2(M)$ as $k \to +\infty$, and for any sequence $(u_k, T_k)_k$, $u_k > 0$ of solutions of:

$$\begin{cases} \triangle_g u_k + h_k u_k = f_k {u_k}^{2^*-1} + \dfrac{|\mathcal{L}_gT_k + \sigma_k|_g^2 + \pi_k^2}{u_k^{2^*+1}} \\ \overrightarrow{\triangle}_g T_k = u_k^{2^*}X_k + Y_k \end{cases}, \tag{1.4}$$

then $(u_k, T_k)_k$ converges to a solution (u_0, T_0), $u_0 > 0$ of (1.1) in $C^{2,\eta}(M)$ for all $0 < \eta < 1$ (up to a subsequence and up to the kernel of $\mathcal{L}_g$). Similarly, (1.1) is said to be *compact* if it is stable for constants perturbations where $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k \equiv (h, f, \pi, \sigma, X, Y)$ for all k – or equivalently, if its set of solutions is compact in the $C^2(M)$ topology. In other words, this notion of stability captures the structure of the set of solutions of (1.1).

Stability issues for (1.1) have been thoroughly investigated in recent years. It turns out that, on locally conformally flat manifolds, (1.1) is always stable when $3 \le n \le 5$ and that explicit conditions on $(h, f, \pi, \sigma, X, Y)$ guaranteeing the stability can be given when $n \ge 6$ (see [12, 25]). These conditions have later been shown to be sharp when $n \ge 7$ in [24], where non-compact sequences of solutions of (1.1) have been constructed. Loosely speaking, these results show that when system (1.1) is *not* stable it exhibits pathological – and unphysical – examples of blowing-up sequences of solutions. Let us also mention that in the particular case where $X \equiv 0$ the conformal flatness assumption can be dropped (see [10, 17, 23]) and the stability of (1.1) yields multiplicity results ([23]).

In this article we investigate the instability behavior of (1.1) in dimension 6, where we extend the analysis of [24] to produce smooth instability examples for (1.1). Assume that (M, g) is of positive Yamabe type and $\overrightarrow{\triangle}_g$ has no kernel (it is a generic assumption in g by [2]). Let:

(1) $f_0 > 0$ be a a positive constant
(2) $\pi \not\equiv 0$ be a smooth function
(3) Y be a smooth vector field and $\tilde{Y}$ be such that $\overrightarrow{\triangle}_g \tilde{Y} = Y$
(4) σ be a smooth traceless and divergence-free $(2,0)$-tensor with $|\sigma|_g \not\equiv 0$

Our main result constructs blowing-up sequences of solutions for arbitrarily regular perturbations of (1.1):

Theorem 1.1. *Let (M, g) be a closed 6-dimensional manifold of positive Yamabe type such that $\overrightarrow{\triangle}_g$ has no kernel. Let (f_0, π, σ, Y) be as in (1)–(4) above. Assume in addition that there exists $\xi_0 \in M$ such that $\big(\mathcal{L}_g \tilde{Y} + \sigma\big)(\xi_0)$ is invertible on $T_{\xi_0}M$ and let X_0 be a smooth 1-form in M, vanishing in a neighbourhood of ξ_0, with $\|X_0\|_{C^0(M)}$ small enough. Then there exist smooth families $(h_\varepsilon)_{\varepsilon>0}$ and $(X_\varepsilon)_{\varepsilon>0}$ and there exists a family $(u_\varepsilon, T_\varepsilon)_\varepsilon$, $u_\varepsilon > 0$, of solutions of bounded energy of:*

$$\begin{cases} \triangle_g u_\varepsilon + h_\varepsilon u_\varepsilon = f_0 u_\varepsilon^2 + \dfrac{|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 + \pi^2}{u_\varepsilon^4} \\ \overrightarrow{\triangle}_g T_\varepsilon = u_\varepsilon^3 X_\varepsilon + Y \end{cases} \tag{1.5}$$

which blows-up at ξ_0. In addition, for any $r \in \mathbb{N}^*$, $\lim_{\varepsilon\to 0} X_\varepsilon = X_0$ in $C^r(M)$ and $\lim_{\varepsilon\to 0} h_\varepsilon = \frac{1}{5} S_g + 2 f_0 u_0$ in $C^r(M)$, where $u_0 > 0$ is the weak limit of $(u_\varepsilon)_\varepsilon$ and S_g denotes the scalar curvature of (M, g).

Theorem 1.1 generalizes the instability result of [24] that only allowed for little regularity in the perturbations $(h_\varepsilon)_\varepsilon$ and required more restrictive assumptions on the coefficients. It highlights in particular the role of the coupling field X_0 and of the gauge term $\mathcal{L}_g \tilde{Y} + \sigma$ in the appearance of instability phenomena for (1.1). Smallness assumptions on X_0 are necessary for the existence of solutions of (1.5) as shown in [22]. The explicit expression of h_ε and X_ε is given in Section 2. More informations on the family $(u_\varepsilon, T_\varepsilon)$ are obtained in the course of the proof. Examples of manifolds (M, g) satisfying the assumptions of Theorem 1.1 are given in Section 2. Finally note that we did not aim here at constructing physical coefficients in (1.5). For non-compactness examples for (1.1) in the physical scalar-field case we refer to [26].

We deal with the full coupling $X_0 \not\equiv 0$ by following the approach developed in [24]. Section 2 introduces the exact form of the coefficients and of the defects of compactness considered. Section 2.3 applies the general constructive result of [24] to the current setting. Section 3 is devoted to the asymptotic expansion of the kernel coefficients. The expansions rely here on an asymptotic pointwise control on the remainder of the finite-dimensional reduction and not on the usual reduced-energy approach. Finally, Section 4 contains the concluding argument in the proof of Theorem 1.1 and Section 5 gathers a few technical results used in the course of the proof.

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2. A pointwise finite-dimensional reduction

2.1. **Construction of the coefficients.** In what follows we let (M, g) be a closed 6-dimensional manifold of positive Yamabe type, such that $\overrightarrow{\triangle}_g$ has no kernel. Examples of such manifolds (M, g) are easily obtained, for instance as small perturbations of the standard sphere $(\mathbb{S}^6, g_{std})$. Indeed, by [2], the set of smooth Riemannian metrics h on M for which $\overrightarrow{\triangle}_h$ has no kernel is a dense subset in the $C^k(M)$ topology for large enough k.

Lee-Parker's conformal normal coordinates result [18] asserts that there exists $\Lambda \in C^\infty(M \times M)$ such that for any $\xi \in M$ there holds, for an arbitrarily large given N:

$$\left| \left(\exp_\xi^{g_\xi}\right)^* g_\xi \right| (y) = 1 + O(|y|^N), \tag{2.1}$$

C^1-uniformly in $\xi \in M$ and in $y \in T_\xi M$. In (2.1) we have let

$$g_\xi = \Lambda_\xi g, \tag{2.2}$$

where the conformal factor $\Lambda_\xi = \Lambda(\xi, \cdot)$ satisfies in addition:

$$\Lambda_\xi(\xi) = 1, \quad \nabla \Lambda_\xi(\xi) = 0. \tag{2.3}$$

In (2.1), $\exp_\xi^{g_\xi}$ is the exponential map for the metric g_ξ at ξ with the identification of $T_\xi M$ to $\mathbb{R}^n$ via a smooth, local orthonormal basis $(e_1, \cdots, e_6)$ of $T_\xi M$. The Lee-Parker result also states that for any $\xi \in M$:

$$S_{g_\xi}(x) = O\big(d_{g_\xi}(\xi, x)^2\big), \tag{2.4}$$

where S_{g_ξ} is the scalar curvature of (M, g_ξ).

Let $(\mu_\varepsilon)_\varepsilon$ be a family of positive real numbers with $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. Let (f_0, π, σ, Y) be as in the assumptions of Theorem 1.1. Let $\tilde{Y}$ be the only solution of $\overrightarrow{\triangle}_g \tilde{Y} = Y$ in M. We assume that there exists $\xi_0 \in M$ such that

$$\big(\sigma + \mathcal{L}_g \tilde{Y}\big)(\xi_0) \quad \text{– viewed as a symmetric matrix – is invertible on } T_{\xi_0} M. \tag{2.5}$$

Examples of such coefficients can always be found. Let H be a smooth nonnegative function in $\mathbb{R}^6$, compactly supported in $B_0(1)$, and satisfying $H \equiv 1$ on $B_0(\frac{3}{4})$. Let $(\beta_\varepsilon)_\varepsilon$ be a family of positive numbers satisfying:

$$\mu_\varepsilon << \beta_\varepsilon^r << 1 \text{ for any } r \in \mathbb{N}^* \text{ as } \varepsilon \to 0. \tag{2.6}$$

Let X_0 be a 1-form in M which vanishes in a neighbourhood of ξ_0. Let Z be a smooth 1-form in $\mathbb{R}^6$, compactly supported in $B_0(1)$, with $|Z_0(0)| > 0$. Assume that there exists $p_0 \in B_0(1)$, $|p_0| = \frac{1}{2}$, such that $Z(p_0) = 0$ and $\nabla Z(p_0)$ is an invertible matrix. Define, for any $x \in M$:

$$X_\varepsilon(x) = X_0(x) + \mu_\varepsilon^{\frac{5}{2}} Z\left(\frac{1}{\beta_\varepsilon}\left(\exp_{\xi_0}^{g_{\xi_0}}\right)^{-1}(x)\right), \tag{2.7}$$

where β_ε is as in (2.6). By (2.6), X_ε is smooth. Assume that X_0 satisfies:

$$\eta := \|X_0\|_{L_\infty(M)} \le \eta_0, \tag{2.8}$$

for some positive η_0, and let

$$\alpha := |Z(0)| = \frac{|X_\varepsilon(\xi_0)|_{g_{\xi_0}}}{\mu_\varepsilon^{\frac{5}{2}}}, \tag{2.9}$$

to be chosen small in the proof. Up to choosing η_0 in (2.8) small enough, the implicit function theorem provides us with a smooth family $(u_\varepsilon, T_\varepsilon)$ satisfying:

$$\begin{cases} \triangle_g u_\varepsilon + h_\varepsilon u_\varepsilon = f_0 u_\varepsilon^2 + \dfrac{|\mathcal{L}_g T_\varepsilon + \sigma|^2 + \pi^2}{u_\varepsilon^4} \\ \overrightarrow{\triangle}_g T_\varepsilon = u_\varepsilon^3 X_\varepsilon + Y, \end{cases} \tag{2.10}$$

for ε small enough, where we have let:

$$h_\varepsilon = \frac{1}{5} S_g + 2 f_0 u_\varepsilon - \mu_\varepsilon H\left(\frac{1}{\beta_\varepsilon}\left(\exp_{\xi_0}^{g_{\xi_0}}\right)^{-1}(x)\right). \tag{2.11}$$

Such a h_ε is smooth by (2.6). We refer to [24], Section 2, for the details. Since (M, g) is of positive Yamabe type we also have:

$$\int_M |\nabla \phi|_g^2 + \left[h_\varepsilon - 2 f_0 u_\varepsilon + 4 \frac{|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 + \pi^2}{u_\varepsilon^5}\right] \phi^2 dv_g \ge \frac{1}{C} \|\phi\|_{H^1}^2, \tag{2.12}$$

for all $\phi \in H^1(M)$ and for $C > 0$ independent of η and ε. Also, by (2.5), $(\mathcal{L}_g T_\varepsilon + \sigma)(\xi_0)$ is again invertible, uniformly in η and ε small enough. Standard elliptic theory shows that $(u_\varepsilon, T_\varepsilon) \to (u_0, T_0)$ in $C^r(M)$ for all $r \in \mathbb{N}$ as $\varepsilon \to 0$, where $u_0 > 0$. As a consequence, $h_\varepsilon \to \frac{1}{5} S_g + 2 f_0 u_0$ in $C^r(M)$ for any $r \in \mathbb{N}$. Note that our proof still works if we allow small perturbations of f_0, see again [24].

2.2. **Blow-up profiles.** We denote by $H^1(M)$ the usual Sobolev space of functions, endowed with the following scalar product:

$$\langle u, v \rangle_\varepsilon = \int_M \left(\langle \nabla u, \nabla v \rangle_g + h_\varepsilon u v\right) dv_g, \text{ for any } u, v \in H^1(M), \tag{2.13}$$

where h_ε is given by (2.11). For $t > 0$ we define:

$$\delta_\varepsilon(t) = \mu_\varepsilon t. \tag{2.14}$$

We let $r_0 > 0$ be such that $2r_0 < i_{g_\xi}(M)$ for all $\xi \in M$, where i_{g_ξ} is the injectivity radius of the metric g_ξ given by (2.2). We let $\chi \in C^\infty(\mathbb{R})$ be a nonnegative, smooth, compactly supported function such that $\chi \equiv 1$ in $[-r_0, r_0]$ and $\chi \equiv 0$ outside of $[-2r_0, 2r_0]$. For $t > 0$ and $\xi \in M$, and for any $x \in M$, let:

$$W_{\varepsilon,t,\xi}(x) = \Lambda_\xi(x)\chi\left(d_{g_\xi}(\xi,x)\right)\delta_\varepsilon^2\left(\delta_\varepsilon^2 + \frac{f_0}{24}d_{g_\xi}(\xi,x)^2\right)^{-2}, \tag{2.15}$$

where Λ_ξ is as in (2.3) and δ_ε is given by (2.14). We define $V_0, \cdots, V_6 : \mathbb{R}^6 \to \mathbb{R}$ by:

$$V_0(x) = \left(\frac{f_0}{24}|x|^2 - 1\right)\left(1 + \frac{f_0}{24}|x|^2\right)^{-3} \text{ and } V_i(y) \ = f_0 x_i\left(1 + \frac{f_0}{24}|x|^2\right)^{-3}. \tag{2.16}$$

The $(V_i)_{0 \le i \le 6}$ span the set of solutions in $H^1(\mathbb{R}^6)$ of the linearized equation (see [4]):

$$\triangle_{\text{eucl}} V_i = 2 f_0 U V_i, \tag{2.17}$$

where:

$$U(y) = \left(1 + \frac{f_0}{24}|y|^2\right)^{-2}, \text{ for any } y \in \mathbb{R}^6. \tag{2.18}$$

We also define, for $x \in M$, $1 \le i \le 6$ and $\xi \in M$:

$$\begin{aligned} Z_{0,\varepsilon,t,\xi}(x) &= \Lambda_\xi(x)\chi\left(d_{g_\xi}(\xi,x)\right)\delta_\varepsilon^2\left(\delta_\varepsilon^2 + \frac{f_0}{24}d_{g_\xi}(\xi,x)^2\right)^{-3} \times \left(\frac{f_0}{24}d_{g_\xi}(\xi,x)^2 - \delta_k^2\right) \\ Z_{i,\varepsilon,t,\xi}(x) &= \Lambda_\xi(x)\chi\left(d_{g_\xi}(\xi,x)\right)\delta_\varepsilon^3\left(\delta_\varepsilon^2 + \frac{f_0}{24}d_{g_\xi}(\xi,x)^2\right)^{-3} \times f_0\left\langle\left(\exp_\xi^{g_\xi}\right)^{-1}(x), e_i(\xi)\right\rangle_{g_\xi(\xi)}. \end{aligned} \tag{2.19}$$

In (2.19), the $(e_i)_i$ denote an orthonormal basis of g_ξ around ξ_0. Finally, we let

$$K_{\varepsilon,t,\xi} = \text{Span}\left\{Z_{i,\varepsilon,t,\xi}, i = 0 \ldots 6\right\}. \tag{2.20}$$

Since $(V_0, \cdots, V_6)$ forms an orthonormal family for the scalar product $(u,v) = \int_{\mathbb{R}^6}\langle\nabla u, \nabla v\rangle dx$ in $\mathbb{R}^6$, $K_{\varepsilon,t,\xi}$ is 7-dimensional for ε small enough. We denote by $K_{\varepsilon,t,\xi}^\perp$ its orthogonal in $H^1(M)$ for the scalar product (2.13).

2.3. **The pointwise finite-dimensional reduction.**

The starting point of our analysis is the following Theorem which was proved in [24].

Theorem 2.1 ([24], Prop. 6.1)**.** *Let (M,g) be a 6-dimensional Riemannian manifold of positive Yamabe type and such that $\overrightarrow{\triangle}_g$ has no kernel. Let $D > 0$, and let $h_\varepsilon, f_0, \pi, \sigma, X_\varepsilon$ and Y be as in Section 2. Assume that η and α in (2.8) and (2.9) are small enough. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$ and any sequence $(t_\varepsilon, \xi_\varepsilon)_\varepsilon$ in $[1/D, D] \times M$, there exists $\phi_\varepsilon = \phi_\varepsilon\big(t_\varepsilon, \xi_\varepsilon\big) \in K_{\varepsilon,t_\varepsilon,\xi_\varepsilon}^\perp$ that satisfies*

$$\begin{cases} \Pi_{K_{\varepsilon,t_\varepsilon,\xi_\varepsilon}^\perp}\left[u_{\varepsilon,t_\varepsilon,\xi_\varepsilon} - (\triangle_g + h_\varepsilon)^{-1}\left(f_0 u_{\varepsilon,t_\varepsilon,\xi_\varepsilon}^2 + \dfrac{|\mathcal{L}_g T_{\varepsilon,t_\varepsilon,\xi_\varepsilon} + \sigma|_g^2 + \pi^2}{u_{\varepsilon,t_\varepsilon,\xi_\varepsilon}^4}\right)\right] = 0, \\ \overrightarrow{\triangle}_g T_{\varepsilon,t_\varepsilon,\xi_\varepsilon} = u_{\varepsilon,t_\varepsilon,\xi_\varepsilon}^3 X_\varepsilon + Y, \end{cases} \tag{2.21}$$

where we have let

$$u_{\varepsilon,t_\varepsilon,\xi_\varepsilon} = u_\varepsilon + W_{\varepsilon,t_\varepsilon,\xi_\varepsilon} + \phi_\varepsilon(t_\varepsilon,\xi_\varepsilon). \tag{2.22}$$

In addition there exists a positive constant C, independent of $(t_\varepsilon,\xi_\varepsilon)_\varepsilon$, such that

$$\|\phi_\varepsilon(t_\varepsilon,\xi_\varepsilon)\|_{H^1(M)} \le C\delta_\varepsilon, \tag{2.23}$$

and that, for any $x \in M$:

$$\begin{aligned} |\phi_\varepsilon(t_\varepsilon,\xi_k)(x)| &\le C\delta_\varepsilon\Big(u_\varepsilon + W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}\Big)(x), \\ |\nabla\phi_\varepsilon(t_\varepsilon,\xi_\varepsilon)(x)| &\le C\delta_\varepsilon\Big(1 + \big|\nabla W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}(x)\big|\Big). \end{aligned} \tag{2.24}$$

Also, $\phi_\varepsilon(t_\varepsilon,\xi_\varepsilon)$ is the only solution of (2.21)–(2.22) *which belongs to $K^\perp_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ and satisfies* (2.23) *and* (2.24)*, and for ε fixed, the mapping $(t,\xi) \mapsto \phi_\varepsilon(t,\xi) \in C^1(M)$ is continuous.*

In the statement of Theorem 2.1, δ_ε and $K_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ are defined in (2.14) and (2.20), and $\Pi_{K^\perp_{\varepsilon,t_\varepsilon,\xi_\varepsilon}}$ denotes the orthogonal projection on $K^\perp_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ for the scalar product (2.13), u_ε is given by (2.10) and $W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ is as in (2.15).

Since we assumed $X_0 \not\equiv 0$, (2.21) is strongly coupled and ill-posed in $H^1(M)$. Theorem 2.1 is therefore not obtained by a standard finite-dimensional reduction approach. Its proof (which is in [24]) follows from an involved "ping-pong" method: the function $\phi_\varepsilon(t_\varepsilon,\xi_\varepsilon)$ is constructed as a fixed-point of the remainders in the decomposition of the ping-pong mapping. This approach combines asymptotic analysis techniques – in the spirit of those developed for the C^0 theory of blow-up in [8, 11, 15] – with a Lyapunov-Schmidt reduction – as developed in [3, 5, 7, 13, 21, 27, 28, 29, 30, 31]. As a consequence of (2.24) we also obtained in [24] refined pointwise estimates on ϕ_ε. The first one improves (2.24) when the bubbling profile is dominant in (2.22):

Proposition 2.2 ([24], Prop. 5.1)**.** *There exists $C > 0$ such that for any family $(t_\varepsilon,\xi_\varepsilon)_\varepsilon$ in $[1/D, D] \times M$ and for any family $(x_\varepsilon)_\varepsilon \in B_{\xi_\varepsilon}(2\sqrt{\delta_\varepsilon})$ there holds, for small ε:*

$$\begin{aligned} &\Big(\delta_\varepsilon + d_g(\xi_\varepsilon,x_\varepsilon)\Big)\,|\nabla\phi_\varepsilon(x_\varepsilon)| + |\phi_\varepsilon(x_\varepsilon)| \\ &\qquad \le C\left(\delta_\varepsilon + \left[\delta_\varepsilon^3\left|\ln\left(1 + \frac{d_g(\xi_\varepsilon,x_\varepsilon)}{\delta_\varepsilon}\right)\right| + \delta_\varepsilon\Big(\delta_\varepsilon + d_g(\xi_\varepsilon,x_\varepsilon)\Big)^2\right] W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}(x_\varepsilon)\right), \end{aligned} \tag{2.25}$$

where $\phi_\varepsilon = \phi_\varepsilon\big(t_\varepsilon,\xi_\varepsilon\big)$ is given by Theorem 2.1.

The second one improves (2.24) when the approximate weak limit u_ε dominates:

Proposition 2.3 ([24], Prop. 6.3)**.** *There exists $C > 0$ such that for any family $(t_\varepsilon,\xi_\varepsilon)_\varepsilon$ in $[1/D, D] \times M$ and for any family $(R_\varepsilon)_\varepsilon$, $R_\varepsilon \ge 1$, there holds:*

$$\|\phi_\varepsilon\|_{L^\infty(M\setminus B_{\xi_\varepsilon}(R_\varepsilon\sqrt{\delta_\varepsilon}))} \le C\left(\frac{\delta_\varepsilon}{R_\varepsilon^2} + R_\varepsilon^2\delta_\varepsilon^2 + \delta_\varepsilon^2\right), \tag{2.26}$$

where $\phi_\varepsilon = \phi_\varepsilon\big(t_\varepsilon,\xi_\varepsilon\big)$ is given by Theorem 2.1.

An important feature of the construction in [24] is that the precision of (2.23) is limited by the coupling of the system, and does not depend on the choice of the approximate solution $u_\varepsilon + W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$. The analysis in Section 3 will therefore crucially rely on (2.25) and (2.26) and not on (2.23).

3. Uniform asymptotic expansions.

Let $D > 0$. For any $(t,p) \in [1/D, D] \times \overline{B_0(1)}$, where $B_0(1)$ is the unit ball in $\mathbb{R}^6$, we let $\phi_\varepsilon(t,p) \in K^{\perp}_{\varepsilon,t,\xi_\varepsilon}$ be given by Theorem 2.1, where $(\delta_\varepsilon)_\varepsilon = (\delta_\varepsilon(t))_\varepsilon$ and $(\xi_\varepsilon)_\varepsilon = (\xi_\varepsilon(p))_\varepsilon$ are given by (2.14) and by:

$$\xi_\varepsilon(p) = \exp_{\xi_0}^{g_{\xi_0}}(\beta_\varepsilon p), \tag{3.1}$$

where β_ε is given by (2.6) and ξ_0 is as in (2.5). By (2.21), there exist $(\lambda^i_\varepsilon(t,p))_{0\le i\le 6}$ such that $u_{\varepsilon,t,p}$ given by (2.22) satisfies:

$$\begin{cases} \left(\triangle_g + h_\varepsilon\right) u_{\varepsilon,t,p} = f_0 u^2_{\varepsilon,t,p} + \dfrac{|\mathcal{L}_g T_{\varepsilon,t,p} + \sigma|^2_g + \pi^2}{u^4_{\varepsilon,t,p}} + \displaystyle\sum_{i=0}^{6} \lambda^i_\varepsilon(t,p) \left(\triangle_g + h_\varepsilon\right) Z_{i,\varepsilon,t,p}, \\ \overrightarrow{\triangle}_g T_{\varepsilon,t,p} = u^3_{\varepsilon,t,p} X_\varepsilon + Y, \end{cases} \tag{3.2}$$

where $Z_{i,\varepsilon,t,p}$ is given by (2.19) and h_ε and X_ε are given by (2.11) and (2.7). In this section we obtain an asymptotic expansion of the $\lambda^i_\varepsilon(t,p)$ in $C^0([1/D,D] \times \overline{B_0(1)})$.

Proposition 3.1. *For $1 \le i \le 6$ there holds in $C^0\left([1/D,D] \times \overline{B_0(1)}\right)$, as $\varepsilon \to 0$:*

$$\begin{aligned} \lambda^0_\varepsilon(t,p) &= -C_0 H(p) t^2 \mu^3_\varepsilon + \kappa_1 \delta^3_\varepsilon + o(\delta^3_\varepsilon), \\ \lambda^i_\varepsilon(t,p) &= -\kappa_2 \left(\mathcal{L}_g T_0 + \sigma\right)_{iq} (\xi_0) Z^q(p) \cdot t^{\frac{7}{2}} \mu_\varepsilon^{\frac{7}{2}} + o(\delta_\varepsilon^{\frac{7}{2}}), \end{aligned}$$

for some positive numerical constant C_0, where κ_1 and κ_2 are positive constants given by (3.18) *and* (3.25) *below and δ_ε is as in* (2.14).

Proof. Since $(u_\varepsilon, T_\varepsilon)$ solve (2.10) we can rewrite system (3.2) in the following form:

$$\begin{aligned} \sum_{i=0}^{6} \lambda^i_\varepsilon(t,p) \left(\triangle_g + h_\varepsilon\right) Z_{i,\varepsilon,t,p} = &\left(\triangle_g + h_\varepsilon - 2f_0 u_\varepsilon\right) W_{\varepsilon,t,p} - f_0 W^2_{\varepsilon,t,p} - f_0 \phi_\varepsilon(t,p)^2 \\ &+ \left(\triangle_g + h_\varepsilon - 2f_0 u_\varepsilon\right) \phi_\varepsilon(t,p) - 2 f_0 W_{\varepsilon,t,p} \phi_\varepsilon(t,p) \\ &+ \frac{|\mathcal{L}_g T_\varepsilon + \sigma|^2_g + \pi^2}{u^4_\varepsilon} - \frac{|\mathcal{L}_g T_{\varepsilon,t,p} + \sigma|^2_g + \pi^2}{\left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^4}. \end{aligned} \tag{3.3}$$

For any $0 \le i \le 6$ we now let:

$$\begin{aligned} I_{1,i} &= \int_M \left[\left(\triangle_g + h_\varepsilon - 2f_0 u_\varepsilon\right) W_{\varepsilon,t,p} - f_0 W_{\varepsilon,t,p}^2 \right] Z_{i,\varepsilon,t,p} dv_g, \\ I_{2,i} &= -\int_M f_0 \phi_\varepsilon(t,p)^2 Z_{i,\varepsilon,t,p} dv_g, \\ I_{3,i} &= \int_M \left[\left(\triangle_g + h_\varepsilon - 2f_0 u_\varepsilon\right) \phi_\varepsilon(t,p) - 2 f_0 W_{\varepsilon,t,p} \phi_\varepsilon(t,p) \right] Z_{i,\varepsilon,t,p} dv_g, \\ I_{4,i} &= \int_M \left(\frac{|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 + \pi^2}{u_\varepsilon^4} - \frac{|\mathcal{L}_g T_{\varepsilon,t,p} + \sigma|_g^2 + \pi^2}{\left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^4} \right) Z_{i,\varepsilon,t,p} dv_g. \end{aligned} \tag{3.4}$$

These integrals are now estimated using (2.24), (2.25) and (2.26). Integrals $I_{1,i}, I_{2,i}$ and $I_{3,i}$ are computed using the same methods as in [24], while the expansion of $I_{4,i}$ is obtained differently. Until the end of this section the notations $O(\cdot)$ and $o(\cdot)$ will denote quantities which are uniform in the choice of $(t,p) \in [1/D, D] \times \overline{B_0(1)}$. We first compute $I_{1,i}, I_{2,i}$ and $I_{3,i}$. Explicit computations using (2.11), (2.15) and (2.19) show that there holds:

$$I_{1,0} = -C_0 H(p) t^2 \mu_\varepsilon^3 + o(\delta_\varepsilon^3) \tag{3.5}$$

for some positive C_0 (which depends on f_0) and, for $1 \le i \le 6$, using (2.6), that:

$$I_{1,i} = o(\delta_\varepsilon^{\frac{7}{2}}). \tag{3.6}$$

We now compute $I_{2,i}$ in (3.4). Using (2.24) and the definition of the $Z_{i,\varepsilon,t,p}$ in (2.19) we write that:

$$\begin{aligned} &\int_{M \backslash B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} f_0 \phi_\varepsilon(t,p)^2 |Z_{0,\varepsilon,t,p}| dv_g \\ &\qquad = O\left(\delta_\varepsilon^2 \int_{B_{\xi_\varepsilon}(2r_0) \backslash B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} |Z_{0,\varepsilon,t,p}| dv_g \right) + O\left(\delta_\varepsilon^2 \int_{B_{\xi_\varepsilon}(2r_0) \backslash B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} W_{\varepsilon,t,p}^3 dv_g \right) \\ &\qquad = o(\delta_\varepsilon^3), \end{aligned}$$

while Proposition 2.2 shows that there holds:

$$\int_{B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} f_0 \phi_\varepsilon(t,p)^2 |Z_{0,\varepsilon,t,p}| dv_g = o(\delta_\varepsilon^3).$$

Let now $1 \le i \le 6$ be fixed. On one side, using again (2.24) and (2.19) and since $|Z_{i,\varepsilon,t,p}| = O(W_{\varepsilon,t,p})$ we have that

$$\int_{M \backslash B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} f_0 \phi_\varepsilon(t,p)^2 |Z_{i,\varepsilon,t,p}| dv_g = o(\delta_\varepsilon^4),$$

while on the other side Proposition 2.2 and (2.11) show that

$$\int_{B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} f_0 \phi_\varepsilon(t,p)^2 Z_{i,\varepsilon,t,p} dv_g = o(\delta_\varepsilon^4).$$

Combining the latter estimates we get:

$$I_{2,i} = \begin{cases} o(\delta_\varepsilon^3), & \text{if } i = 0, \\ o(\delta_\varepsilon^4), & \text{if } 1 \le i \le 6. \end{cases} \tag{3.7}$$

We now rewrite $I_{3,i}$ as:

$$I_{3,i} = \int_M \Big(\left[\triangle_g + (h_\varepsilon - 2f_0 u_\varepsilon) \right] Z_{i,\varepsilon,t,p} - 2f_0 W_{\varepsilon,t,p} Z_{i,\varepsilon,t,p} \Big) \phi_\varepsilon(t,p) dv_g, \tag{3.8}$$

and use (5.1) below. Let $(R_\varepsilon)_\varepsilon$ be a family of positive numbers such that $R_\varepsilon \sqrt{\delta_\varepsilon} = o(1)$. Using (2.24) and (2.26) yields:

$$\begin{aligned} \int_M \left(\delta_\varepsilon^2 \mathbb{1}_{r_0 \le d_{g_{\xi_\varepsilon}} \le 2r_0} + \delta_\varepsilon^2 \mathbb{1}_{nlcf} \right) \phi_\varepsilon(t,p) dv_g &= \int_{M \backslash B_{\xi_\varepsilon}(R_\varepsilon \sqrt{\delta_\varepsilon})} \left(\delta_\varepsilon^2 \mathbb{1}_{r_0 \le d_{g_{\xi_\varepsilon}} \le 2r_0} + \delta_\varepsilon^2 \mathbb{1}_{nlcf} \right) \phi_\varepsilon(t,p) dv_g \\ &\quad + \int_{B_{\xi_\varepsilon}(R_\varepsilon \sqrt{\delta_\varepsilon})} \left(\delta_\varepsilon^2 \mathbb{1}_{r_0 \le d_{g_{\xi_\varepsilon}} \le 2r_0} + \delta_\varepsilon^2 \mathbb{1}_{nlcf} \right) \phi_\varepsilon(t,p) dv_g \\ &= o(\delta_\varepsilon^3), \end{aligned}$$

while using (2.11) and (2.24) gives:

$$\int_M \left(h_\varepsilon - \frac{1}{5} S_g - 2f_0 u_\varepsilon \right) Z_{0,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g = o(\delta_\varepsilon^3).$$

If (M,g) is not locally conformally flat there holds, for some R_ε satisfying $R_\varepsilon \sqrt{\delta_\varepsilon} = o(1)$:

$$\begin{aligned} &\int_M \frac{1}{5} \Lambda_{\xi_\varepsilon} S_{g_{\xi_\varepsilon}} Z_{0,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g \\ &= \int_{M \backslash B_{\xi_\varepsilon}(R_\varepsilon \sqrt{\delta_\varepsilon})} \frac{1}{5} \Lambda_{\xi_\varepsilon} S_{g_{\xi_\varepsilon}} Z_{0,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g + \int_{B_{\xi_\varepsilon}(R_\varepsilon \sqrt{\delta_\varepsilon}) \backslash B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} \frac{1}{5} \Lambda_{\xi_\varepsilon} S_{g_{\xi_\varepsilon}} Z_{0,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g \\ &+ \int_{B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} \frac{1}{5} \Lambda_{\xi_\varepsilon} S_{g_{\xi_\varepsilon}} Z_{0,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g \\ &= o(\delta_\varepsilon^3), \end{aligned} \tag{3.9}$$

where we used (2.4) and we estimated the first integral using (2.26), the second one using (2.24) and the third one using (2.25). With (3.8) and (5.1) below we therefore get:

$$I_{3,0} = o(\delta_\varepsilon^3). \tag{3.10}$$

Let now $1 \leq i \leq 6$. Similarly, using (2.24) and (2.26) for a radius $R_\varepsilon = o(\delta_\varepsilon^{-\frac{1}{2}})$ yields:

$$\int_M \delta_\varepsilon^3 |\phi_\varepsilon(t,p)| dv_g = o(\delta_\varepsilon^4),$$

and, using (2.24), (2.25), (2.26) and (2.4) and mimicking the proof of (3.9) shows that, if (M,g) is not locally conformally flat, then:

$$\int_M \frac{1}{5} \Lambda_{\xi_\varepsilon} S_{g_{\xi_\varepsilon}} Z_{i,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g = o(\delta_\varepsilon^4).$$

Independently, there holds that:

$$\begin{aligned} &\int_M \big(h_\varepsilon - \frac{1}{5} S_g - 2 f_0 u_\varepsilon\big) Z_{i,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g \\ &\qquad = \int_{B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} \big(h_\varepsilon - \frac{1}{5} S_g - 2 f_0 u_\varepsilon\big) Z_{i,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g \\ &\qquad + \int_{M \backslash B_{\xi_\varepsilon}(\sqrt{\delta_\varepsilon})} \big(h_\varepsilon - \frac{1}{5} S_g - 2 f_0 u_\varepsilon\big) Z_{i,\varepsilon,t,p} \phi_\varepsilon(t,p) dv_g \\ &\qquad = o(\delta_\varepsilon^4), \end{aligned}$$

where we used (2.11) and (2.25) to estimate the first integral and (2.11) and (2.24) to estimate the second one. Finally, in case (M,g) is not locally conformally flat, mimicking the computations that led to (3.9) we get that:

$$\int_M \delta_\varepsilon^3 \Big(\delta_\varepsilon + d_g(\xi_\varepsilon, \cdot)\Big)^{-4} |\phi_\varepsilon(t,p)| dv_g = o(\delta_\varepsilon^4).$$

With (3.8) and (5.1) below the previous estimates give in the end, for $1 \leq i \leq 6$:

$$I_{3,i} = o(\delta_\varepsilon^4). \tag{3.11}$$

We now turn to the estimates of $I_{4,i}$. These are the core of the proof of Theorem 1.1 and the main novelty of the 6-dimensional case and are the subject of the following lemma:

Lemma 3.2. *For $1 \leq i \leq 6$ there holds in $C^0\Big([1/D, D] \times \overline{B_0(1)}\Big)$, as $\varepsilon \to 0$:*

$$\begin{aligned} I_{4,0} &= \kappa_1 \delta_\varepsilon^3 + o(\delta_\varepsilon^3), \\ I_{4,i} &= -\kappa_2 \left(\mathcal{L}_g T_0 + \sigma\right)_{iq} (\xi_0) Z^q(p) \cdot \delta_\varepsilon^{\frac{7}{2}} + o(\delta_\varepsilon^{\frac{7}{2}}), \end{aligned}$$

where κ_1 and κ_2 are positive constants given by (3.18) *and* (3.25) *below.*

Proof. For any $0 \le i \le 6$ we write that:

$$\begin{aligned} I_{4,i} &= \int_M \left(|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 + \pi^2\right)\left(u_\varepsilon^{-4} - \left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^{-4}\right) Z_{i,\varepsilon,t,p} dv_g \\ &+ \int_M \left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^{-4}\left(|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 - |\mathcal{L}_g T_{\varepsilon,t,p} + \sigma|_g^2\right) Z_{i,\varepsilon,t,p} dv_g \\ &:= I_{4,i}^1 + I_{4,i}^2. \end{aligned} \tag{3.12}$$

Let first $i = 0$. Let $R \ge 1$. Since there holds

$$\left| u^{-4} - \left(u + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^{-4}\right| = O\big(W_{\varepsilon,t,p} + |\phi_\varepsilon(t,p)|\big) \text{ in } M \backslash B_{\xi_\varepsilon}(R\sqrt{\delta_\varepsilon}),$$

we get with (2.26) that:

$$\left| \int_{M \backslash B_{\xi_\varepsilon}(R\sqrt{\delta_\varepsilon})} \left(|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 + \pi^2\right)\left(u_\varepsilon^{-4} - \left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^{-4}\right) Z_{0,\varepsilon,t,p} dv_g \right| = O\left(\frac{\delta_\varepsilon^3}{R^2}\right) + o(\delta_\varepsilon^3). \tag{3.13}$$

Independently, by (2.24), Lebesgue's dominated convergence theorem shows that

$$\begin{aligned} &\delta_\varepsilon^{-3} \int_{B_{\xi_\varepsilon}(R\sqrt{\delta_\varepsilon})} \left(|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 + \pi^2\right)\left(u_\varepsilon^{-4} - \left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^{-4}\right) Z_{0,\varepsilon,t,p} dv_g \\ &= \frac{(24)^2}{f_0^2}\big(|\mathcal{L}_g T_0 + \sigma|_g^2 + \pi^2\big)(\xi_0) \int_{B_0(R)} \left[u_0(\xi_0)^{-4} - \left(u_0(\xi_0) + \left(\frac{24}{f0}\right)^2 |y|^{-4} \right)^{-4} \right] |y|^{-4} dy + o(1) \end{aligned} \tag{3.14}$$

as $\varepsilon \to 0$, so that (3.13) and (3.14) together show that there holds

$$\begin{aligned} I_{4,0}^1 = &\frac{(24)^2}{f_0^2}\big(|\mathcal{L}_g T_0 + \sigma|_g^2 + \pi^2\big)(\xi_0) \\ &\times \int_{\mathbb{R}^6} \left[u_0(\xi_0)^{-4} - \left(u_0(\xi_0) + \left(\frac{24}{f_0}\right)^2 |y|^{-4} \right)^{-4} \right] |y|^{-4} dy \cdot \delta_\varepsilon^3 + o(\delta_\varepsilon^3), \end{aligned} \tag{3.15}$$

where $(u_0, T_0) = \lim_{\varepsilon \to 0}(u_\varepsilon, T_\varepsilon)$ in $C^2(M)$. Using (5.4) below with (2.7) one now gets that:

$$|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 - |\mathcal{L}_g T_{\varepsilon,t,p} + \sigma|_g^2 = -|\mathcal{L}_g \Theta_{\varepsilon,t,p}|_g^2 + O\big(|\mathcal{L}_g \Theta_{\varepsilon,t,p}|_g\big) + O(\delta_\varepsilon),$$

where $\Theta_{\varepsilon,t,p}$ is defined in (5.3) below. Using the asymptotic (5.7) together with (2.7) and the dominated convergence theorem shows that:

$$\begin{aligned} &I_{4,0}^2 = \\ &-C_1 \alpha^2 \cdot \int_{\mathbb{R}^6} \left(u_0(\xi_0) + \left(\frac{f_0}{24}\right)^{-2} |y|^{-4} \right)^{-4} \left(2 + 28 \left| \left\langle \frac{Z(0)}{|Z(0)|}, \frac{y}{|y|} \right\rangle \right|^2 \right) |y|^{-14} dy \cdot \delta_\varepsilon^3 + o(\delta_\varepsilon^3), \end{aligned} \tag{3.16}$$

where C_1 denotes a positive constant (depending on f_0) and where α is as in (2.9). Combining (3.15) and (3.16) in (3.12) one obtains in the end that:

$$I_{4,0} = \kappa_1 \delta_\varepsilon^3 + o(\delta_\varepsilon^3), \tag{3.17}$$

where κ_1 is explicitly given by:

$$\begin{aligned}\kappa_1 = \frac{(24)^2}{f_0^2}\big(|\mathcal{L}_g T_0 + \sigma|_g^2 + \pi^2\big)(\xi_0) \cdot \int_{\mathbb{R}^6} \left[u_0(\xi_0)^{-4} - \left(u_0(\xi_0) + \left(\frac{24}{f_0}\right)^2 |y|^{-4}\right)^{-4}\right] |y|^{-4} dy \\ - C_1 \alpha^2 \times \int_{\mathbb{R}^6} \left(u_0(\xi_0) + \left(\frac{f_0}{24}\right)^{-2} |y|^{-4}\right)^{-4} \left(2 + 28 \left|\left\langle \frac{Z(0)}{|Z(0)|}, \frac{y}{|y|}\right\rangle\right|^2\right) |y|^{-14} dy.\end{aligned} \tag{3.18}$$

In particular, up to choosing α as in (2.9) small enough, we have $\kappa_1 > 0$.

Let now $1 \le i \le 6$. Mimicking the proof of (3.13) and (3.14) and using (2.26), the dominated convergence theorem and an antisymmetry argument shows that

$$I_{4,i}^1 = o(\delta_\varepsilon^{\frac{7}{2}}). \tag{3.19}$$

We now use (5.4) below to write that there holds:

$$\begin{aligned}|\mathcal{L}_g T_\varepsilon + \sigma|_g^2 - |\mathcal{L}_g T_{\varepsilon,t,p} + \sigma|_g^2 = -2\left\langle \mathcal{L}_g T_\varepsilon + \sigma, \mathcal{L}_g \Theta_{\varepsilon,t,p}\right\rangle_g - |\mathcal{L}_g \Theta_{\varepsilon,t,p}|_g^2 \\ + O\big(\delta_\varepsilon |\mathcal{L}_g \Theta_{\varepsilon,t,p}|_g\big) + O(\delta_\varepsilon),\end{aligned} \tag{3.20}$$

where $\Theta_{\varepsilon,t,p}$ is defined in (5.3) and an expansion of $\mathcal{L}_g \Theta_{\varepsilon,t,p}$ is given by (5.7) (using also (2.7)). By (2.24) there always holds:

$$u_\varepsilon + W_{\varepsilon,t,p} \le C\left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)$$

for some positive constant C independent on ε, so straightforward computations using (5.7) below give that:

$$\int_M \delta_\varepsilon \big(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\big)^{-4} |Z_{i,\varepsilon,t,p}| dv_g = o(\delta_\varepsilon^{\frac{7}{2}}),$$

and that

$$\int_M \delta_\varepsilon |\mathcal{L}_g \Theta_{\varepsilon,t,p}| \big(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\big)^{-4} |Z_{i,\varepsilon,t,p}| dv_g = o(\delta_\varepsilon^{\frac{7}{2}}).$$

An antisymmetry argument with (5.7) also shows that there holds:

$$\int_M |\mathcal{L}_g \Theta_{\varepsilon,t,p}|^2 \big(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\big)^{-4} Z_{i,\varepsilon,t,p} dv_g = o(\delta_\varepsilon^{\frac{7}{2}}).$$

Let now $(R_\varepsilon)_\varepsilon$ be a sequence of positive numbers satisfying $R_\varepsilon \sqrt{\delta_\varepsilon} = o(1)$. There holds:

$$\int_{M \backslash B_{\xi_\varepsilon}(R_\varepsilon \sqrt{\delta_\varepsilon})} |\mathcal{L}_g \Theta_{\varepsilon,t,p}| \big(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\big)^{-4} |Z_{i,\varepsilon,t,p}| dv_g = O\left(\delta_\varepsilon^{\frac{7}{2}} R_\varepsilon^{-4}\right),$$

so that the (3.12) and the previous computations show that there holds:

$$I_{4,i}^2 = -2f_0 \int_{B_0(R_\varepsilon)} \frac{\left\langle \mathcal{L}_g T_\varepsilon + \sigma, \mathcal{L}_g \Theta_{\varepsilon,t,p} \right\rangle_g (z_\varepsilon)}{\left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^4 (z_\varepsilon)} \frac{y_i}{\left(\delta_\varepsilon + \frac{f_0}{24}|y|^2\right)^3} \Big(1+O(\delta_\varepsilon |y|^2)\Big) dy \cdot \delta_\varepsilon^{\frac{7}{2}} + o(\delta_\varepsilon^{\frac{7}{2}}), \tag{3.21}$$

where we have let, for any $y \in B_0(R_\varepsilon)$:

$$z_\varepsilon = \exp_{\xi_\varepsilon}^{g_{\xi_\varepsilon}}(\sqrt{\delta_\varepsilon} y). \tag{3.22}$$

Define, for any $y \in \mathbb{R}^6$:

$$\tilde{W}_\varepsilon(y) = \left(\delta_\varepsilon + \frac{f_0}{24}|y|^2\right)^{-2}.$$

Using (2.24) it is easily seen that there holds, for any $y \in B_0(R_\varepsilon)$:

$$\Big| \left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^4 (z_\varepsilon) - \left(u_\varepsilon(\xi_\varepsilon) + \tilde{W}_\varepsilon(y)\right)^{-4} \Big| \\ \le C\sqrt{\delta_\varepsilon}|y| \left(u_\varepsilon + W_{\varepsilon,t,p} + \phi_\varepsilon(t,p)\right)^{-4}(z_\varepsilon)$$

for some positive constant C that does not depend on ε. As a consequence we obtain with (3.21) that:

$$I_{4,i}^2 = -2f_0 \left(\mathcal{L}_g T_\varepsilon + \sigma\right)^{pq}(\xi_0) \int_{B_0(R_\varepsilon)} \frac{\left(\mathcal{L}_g \Theta_{\varepsilon,t,p}\right)_{pq}(z_\varepsilon)}{\left(u_\varepsilon(\xi_\varepsilon) + \tilde{W}_\varepsilon(y)\right)^4} \frac{y_i}{\left(\delta_\varepsilon + \frac{f_0}{24}|y|^2\right)^3} dy \cdot \delta_\varepsilon^{\frac{7}{2}} + o(\delta_\varepsilon^{\frac{7}{2}}). \tag{3.23}$$

By using (2.7) in the expansion (5.7) below we obtain that:

$$|\mathcal{L}_g \Theta_{\varepsilon,t,p}(z_\varepsilon)| = O(|y|^{-5})$$

for any $y \in B_0(R_\varepsilon)$, where z_ε is given by (3.22), so that there holds:

$$\left| \frac{\left(\mathcal{L}_g \Theta_{\varepsilon,t,p}\right)_{pq}(z_\varepsilon)}{\left(u_\varepsilon(\xi_\varepsilon) + \tilde{W}_\varepsilon(y)\right)^4} \frac{y_i}{\left(\delta_\varepsilon + \frac{f_0}{24}|y|^2\right)^3} \right| \le C \cdot \begin{cases} |y|^{-4} \text{ if } |y| \le 1, \\ |y|^{-10} \text{ if } |y| > 1, \end{cases}$$

for some positive contant C independent of ε. In particular, with (5.7) below, Lebesgue's dominated convergence theorem applies. Since $\mathcal{L}_g T_\varepsilon + \sigma$ is a symmetric and traceless field of bilinear forms, straightforward computations from (3.23) give in the end, with (3.19), that:

$$I_{4,i} = -\kappa_2 \left(\mathcal{L}_g T_0 + \sigma\right)_{iq}(\xi_0) Z^q(p) \cdot \delta_\varepsilon^{\frac{7}{2}} + o(\delta_\varepsilon^{\frac{7}{2}}), \tag{3.24}$$

where we have let

$$\kappa_2 = \left(C_2 \int_{\mathbb{R}^6} \left[u_0(\xi_0)^{-4} - \left(u_0(\xi_0) + \left(\frac{24}{f_0}\right)^2 |y|^{-4} \right)^{-4} \right] |y|^{-10} dy \right) \tag{3.25}$$

for some positive positive constant C_2. □

The proof of Proposition 3.1 now follows from (3.3), (3.4), (3.5), (3.6), (3.7), (3.10), (3.11), (3.17) and (3.24).

□

4. Conclusive argument

In this section we find, for any ε small enough, a family $(t_\varepsilon, p_\varepsilon)_\varepsilon \in [1/D, D] \times \overline{B_0(1)}$ that annihilates all the $\lambda_\varepsilon^i(t,p)$ in (3.2). Lemma 3.1 shows that there holds:

$$\begin{pmatrix} \lambda_\varepsilon^0(t,p) \\ \vdots \\ \lambda_\varepsilon^6(t,p) \end{pmatrix} = \begin{pmatrix} \mu_\varepsilon^3 \Big(- C_0 H(p) t^2 + \kappa_1 t^3 + R_\varepsilon^0(t,p) \Big) \\ \\ \mu_\varepsilon^{\frac{7}{2}} \Big(- \kappa_2 \left(\mathcal{L}_g T_0 + \sigma \right)_{iq} (\xi_0) Z^q(p) t^{\frac{7}{2}} + R_\varepsilon^i(t,p) \Big)_{1 \le i \le 6} \end{pmatrix}, \tag{4.1}$$

where $R_\varepsilon^i(t,p)$, $0 \le i \le 6$, goes to zero in $C^0([1/D, D] \times \overline{B_0(1)})$ as $\varepsilon \to +\infty$ and where $\kappa_1, \kappa_2 > 0$ are given by (3.18) and (3.25). Define now $F : [1/D, D] \times \overline{B_0(1)} \to \mathbb{R}^7$ by:

$$F(t,p) = \begin{pmatrix} - C_0 H(p) t^2 + \kappa_1 t^3 \\ \Big(-\kappa_2 \left(\mathcal{L}_g T_0 + \sigma \right)_{iq} (\xi_0) Z^q(p) t^{\frac{7}{2}} \Big)_{1 \le i \le 6} \end{pmatrix}.$$

Remember that by assumption there exists $|p_0| = \frac{1}{2}$ such that $Z(p_0) = 0$ and $\nabla Z(p_0)$ is an invertible matrix. Let $t_0 = \frac{C_0}{\kappa_1}$ (note that, for D large enough, $t_0 \in [2/D, D/2]$). Since $H \equiv 1$ on $B_0(\frac{3}{4})$ and since by construction $(\mathcal{L}_g T_0 + \sigma)(\xi_0)$ is an invertible matrix, it follows that (t_0, p_0) is a non-degenerate zero of F. Standard degree-theoretic arguments ensure then that there exists a family $(t_\varepsilon, p_\varepsilon)_\varepsilon$ converging towards (t_0, p_0) such that

$$F(t_\varepsilon, p_\varepsilon) + \begin{pmatrix} R_\varepsilon^0(t_\varepsilon, p_\varepsilon) \\ \Big(R_\varepsilon^i(t_\varepsilon, p_\varepsilon) \Big)_{1 \le i \le 6} \end{pmatrix} = 0.$$

This implies, with (4.1), that $\lambda_k^i(t_\varepsilon, p_\varepsilon) = 0$ for all $0 \le i \le 6$ and hence that $(u_{\varepsilon, t_\varepsilon, p_\varepsilon}, T_{\varepsilon, t_\varepsilon, p_\varepsilon})$ solves (1.5). By (2.24) this concludes the proof of Theorem 1.1.

5. Miscellaneous technical results

5.1. **Conformal laplacian of the $Z_{i,\varepsilon,t,\xi}$, $0 \le i \le 6$.** Let $D > 0$ and let $(t_\varepsilon, \xi_\varepsilon)_\varepsilon$ be a family in $[1/D, D] \times M$. We recall the laplacian of the $Z_{i,\varepsilon,t_\varepsilon,\xi_\varepsilon}$.

Claim 5.1. *For any family of functions $(h_\varepsilon)_\varepsilon$, $h_\varepsilon \in C^\infty(M)$, and for any $1 \le i \le 6$ there holds:*

$$
\begin{aligned}
(\triangle_g + h_\varepsilon) Z_{0,\varepsilon,t_\varepsilon,\xi_\varepsilon} &= 2 f_0 W_{\varepsilon,t_\varepsilon,\xi_\varepsilon} Z_{0,\varepsilon,t_\varepsilon,\xi_\varepsilon} + (h_\varepsilon - \frac{1}{5} S_g) Z_{0,\varepsilon,t_\varepsilon,\xi_\varepsilon} \\
&\quad + \frac{1}{5} \Lambda_{\xi_\varepsilon} S_{g_{\xi_\varepsilon}} Z_{0,\varepsilon,t_\varepsilon,\xi_\varepsilon} + O\big(\delta_\varepsilon^2 \mathbb{1}_{r_0 \le d_{g_{\xi_\varepsilon}} \le 2r_0}\big) + O(\delta_\varepsilon^2 \mathbb{1}_{nlcf, d_{g_{\xi_\varepsilon}} \le 2r_0}), \\
(\triangle_g + h_\varepsilon) Z_{i,\varepsilon,t_\varepsilon,\xi_\varepsilon} &= 2 f_0 W_{\varepsilon,t_\varepsilon,\xi_\varepsilon} Z_{i,\varepsilon,t_\varepsilon,\xi_\varepsilon} + (h_\varepsilon - \frac{1}{5} S_g) Z_{i,\varepsilon,t_\varepsilon,\xi_\varepsilon} \\
&\quad + \frac{1}{5} \Lambda_{\xi_\varepsilon} S_{g_{\xi_\varepsilon}} Z_{i,\varepsilon,t_\varepsilon,\xi_\varepsilon} + O\big(\delta_\varepsilon^3 \mathbb{1}_{r_0 \le d_{g_{\xi_\varepsilon}} \le 2r_0}\big) + O\left(\delta_\varepsilon^3 \Big(\delta_k + d_g(\xi_\varepsilon, \cdot)\Big)^{-4} \mathbb{1}_{nlcf, d_{g_{\xi_\varepsilon}} \le 2r_0}\right).
\end{aligned}
\tag{5.1}
$$

In (5.1) $\Lambda_{\xi_\varepsilon}$ is as in (2.3) and $d_\varepsilon = d_{g_{\xi_\varepsilon}}(\xi_\varepsilon, \cdot)$. The notational shorthand $\mathbb{1}_{nlcf}$ indicates that the associated term vanishes if (M, g) is locally conformally flat around ξ_ε. The notations $\mathbb{1}_{r_0 \le d_{g_{\xi_\varepsilon}}} \le 2r_0$ and $\mathbb{1}_{nlcf, d_{g_{\xi_\varepsilon}} \le 2r_0}$ are defined in the same way.

5.2. **Asymptotic estimates for solutions of the 1-form equation.** Let (M, g) be a closed Riemannian manifold such that $\overrightarrow{\triangle}_g$ has no kernel. Let $(u_\varepsilon)_\varepsilon$ be a family of smooth positive functions in M, converging in $C^2(M)$ as $\varepsilon \to 0$ towards some smooth positive function u_0 in M. Let $D > 0$, let $(t_\varepsilon, \xi_\varepsilon)_\varepsilon \in [\frac{1}{D}, D] \times M$ and consider the function $W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ given by (2.15), where δ_ε is given by (2.14) and $(\mu_\varepsilon)_\varepsilon$ denotes any family of positive numbers with $\mu_\varepsilon \to$ as $\varepsilon \to 0$. Let v_ε be a family of continuous functions satisfying

$$
\left\| \frac{v_\varepsilon}{u_\varepsilon + W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}} \right\|_{C^0(M)} \le \mu_\varepsilon. \tag{5.2}
$$

Let $(X_\varepsilon)_\varepsilon$ be a C^1-bounded family of 1-forms and Y be a given 1-form and, for any ε, let $T_{\varepsilon,t_\varepsilon,\xi_\varepsilon}, \Theta_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ and T_ε be the unique solutions of the following equations in M:

$$
\begin{aligned}
\overrightarrow{\triangle}_g T_{\varepsilon,t_\varepsilon,\xi_\varepsilon} &= (u_\varepsilon + W_{\varepsilon,t_\varepsilon,\xi_\varepsilon} + v_\varepsilon)^{2^*} X_\varepsilon + Y, \\
\overrightarrow{\triangle}_g \Theta_{\varepsilon,t_\varepsilon,\xi_\varepsilon} &= W_{\varepsilon,t_\varepsilon,\xi_\varepsilon}^{2^*} X_\varepsilon, \\
\overrightarrow{\triangle}_g T_\varepsilon &= u_\varepsilon^{2^*} X_\varepsilon + Y.
\end{aligned}
\tag{5.3}
$$

We recall the following pointwise asymptotic estimate on $\mathcal{L}_g T_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$, obtained in [24]:

Proposition 5.2. *For any family $(x_\varepsilon)_\varepsilon$ of points in M, there holds:*

$$
\mathcal{L}_g T_{\varepsilon,t_\varepsilon,\xi_\varepsilon}(x_\varepsilon) = \Big(1 + O(\delta_\varepsilon)\Big) \mathcal{L}_g \Theta_{\varepsilon,t_\varepsilon,\xi_\varepsilon}(x_\varepsilon) + \mathcal{L}_g T_\varepsilon(x_\varepsilon) + O\Big(\mu_\varepsilon + |X_\varepsilon(\xi_\varepsilon)|_g + \delta_\varepsilon \|\nabla X_\varepsilon\|_{L^\infty(2r_0)}\Big), \tag{5.4}
$$

uniformly as $\varepsilon \to 0$. As a consequence, there holds, for any $x \in M$:

$$
|\mathcal{L}_g T_{\varepsilon,t_\varepsilon,\xi_\varepsilon} - \mathcal{L}_g T_\varepsilon|_g(x) \le C \left(\Big[|X_\varepsilon(\xi_\varepsilon)|_g + \delta_\varepsilon \|\nabla X_\varepsilon\|_{L^\infty(2r_0)} \Big] \Big(\delta_\varepsilon + d_g(\xi_\varepsilon, x)\Big)^{-5} + \mu_\varepsilon \right), \tag{5.5}
$$

where C is a positive constant independent of ε.

We refer to [24] (Section 9, Prop. 9.2) for a proof of this statement. In the course of the proof of this result it is also shown that there holds:

$$|\mathcal{L}_g \Theta_{\varepsilon,t_\varepsilon,\xi_\varepsilon}| \le C\left(\Big(|X_\varepsilon(\xi_\varepsilon)|_g + \delta_\varepsilon \|\nabla X_\varepsilon\|_{L^\infty(2r_0)}\Big)\Big(\delta_\varepsilon + d_g(\xi_\varepsilon, x)\Big)^{-5}\right), \tag{5.6}$$

for some positive constant C that does not depend on ε, and where $\Theta_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ is defined in (5.3). The following asymptotic of $\mathcal{L}_g \Theta_{\varepsilon,t_\varepsilon,\xi_\varepsilon}$ is also given in [24]: for any family $(x_\varepsilon)_\varepsilon$ of points in M satisfying $\delta_\varepsilon << d_g(x_\varepsilon, \xi_\varepsilon) \le r_0$[1] there holds:

$$\begin{aligned}\mathcal{L}_g \Theta_{\varepsilon,t_\varepsilon,\xi_\varepsilon}(x_\varepsilon)_{ij} =& \gamma \cdot f_0^{-3}\Big[\delta_{ij}\zeta^p \check{x}_p - \zeta_i \check{x}_j - \zeta_j \check{x}_i - 4\zeta^p \check{x}_p \check{x}_i \check{x}_j\Big] \\ & \times \Big(|X_\varepsilon(\xi_\varepsilon)|_g + O(\delta_\varepsilon \|\nabla X_\varepsilon\|_{L^\infty(2r_0)})\Big) d_g(\xi_\varepsilon, x_\varepsilon)^{-5}\big(1 + o(1)\big),\end{aligned} \tag{5.7}$$

where we have let, up to a subsequence:

$$\check{x} = \lim_{\varepsilon\to 0} \frac{1}{d_g(\xi_\varepsilon, x_\varepsilon)} \exp_{\xi_\varepsilon}^{-1}(x_\varepsilon), \quad \zeta = \lim_{\varepsilon \to 0} \frac{X_\varepsilon(\xi_\varepsilon)}{|X_\varepsilon(\xi_\varepsilon)|_g}$$

and $\gamma(n)$ is a positive numerical constant.

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[1] here $2r_0$ is the width of the support of the cut-off function appearing in (2.15).

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Bruno Premoselli, Université Libre de Bruxelles, Service de Géométrie Différentielle, CP 218, Boulevard du Triomphe, B-1050 Bruxelles, Belgique.

E-mail address: bruno.premoselli@ulb.ac.be