Black-Scholes Model

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Pricing rules

Let us consider a market whose evolution between 0 and T is described by a probability space \((\Omega, \mathcal{F})\)
\((\mathcal{F} \text{ represents all events related to the evolution of the assets in this market between 0 and T}).

We assume that the evolution of asset prices is described by an adapted vector process :

\[
S : [0, T] \times \Omega \rightarrow \mathbb{R}^{d+1} : (t, \omega) \mapsto (S^0_t(\omega), S^1_t(\omega), \ldots, S^d_t(\omega))
\]

where we assume that the asset 0 is risk free :

\[
S^0_t = S^0_0 e^{rt}
\]

with \(r\), the risk free rate, supposed to be constant in time and for all maturities.

Filtration \((\mathcal{F}_t)\) is assumed to be the generated filtration associated to these different processes \((\mathcal{F}_t \text{ will contain the history of the market until instant } t, \text{ the process } S_t \text{ being hence adapted w.r.t. this filtration } (\mathcal{F})).\)
Problem: How to attribute a notion of “value” to any contingent asset (or derivative product) constructed from the assets of the market?

If $H$ is such a derivative, its payoff (supposed to be delivered in $T$) $H(\omega)$ is of the form: $H(\omega) = h(S_t(\omega), t \in [0, T])$.

A **pricing rule** is a methodology attributing to any contingent asset of payoff $H(T)$ in $T$ and to any instant $t$, a value $\Pi_t(H)$, with:

- $\Pi_t(H)$ is adapted;
- If the payoff is positive ($\forall \omega \in \Omega: H(\omega) \geq 0$) then we have: $\Pi_t(H) \geq 0$;
- $\Pi_t(H)$ is linear w.r.t. $H$. 
Pricing rules

For any event $A \in \mathcal{F}$, $\mathbb{I}_A$ represents the payoff of a contingent claim paying 1 at $T$ if $A$ occurs, 0 otherwise (i.e. a bet on $A$).

In particular if $A = \Omega$, we get a contract paying 1 at $T$ in all states of the world, i.e. a zero-coupon bond of maturity $T$.

Its value at $t$, $\Pi_t(1)$, represents hence the value at $t$ of 1 currency unit paid at $T$, i.e. the current value to have on the bank account at $t$ to get 1 at $T$. This should be equal to $\Pi_t(1) = e^{-r(T-t)}$, the “discount factor” (all pricing rules should give the same value of the bank account here).
We can then define $\mathbb{Q} : \mathcal{F} \to \mathbb{R}$ by:

$$\mathbb{Q}(A) = \frac{\Pi_0(\mathbb{I}_A)}{\Pi_0(1)} = e^{rT} \Pi_0(\mathbb{I}_A)$$

Then the linearity and positiveness of the pricing rule $\Pi$ implies:

- $0 \leq \mathbb{Q}(A) \leq 1$
- If $A, B$ are disjoint events, $\mathbb{I}_{A \cup B} = \mathbb{I}_A + \mathbb{I}_B$ so that $\mathbb{Q}(A \cup B) = \mathbb{Q}(A) + \mathbb{Q}(B)$

We hence see that $\mathbb{Q}$ is actually a probability measure on $(\Omega, \mathcal{F})$. 
On the other hand, $\Pi$ can be retrieved from $\mathbb{Q}$: if $H$ is of the form:

$$H = \sum c_i \mathbb{I}_{A_i},$$

by linearity of $\Pi$ we get

$$\Pi_0(H) = e^{-rT} \mathbb{E}_\mathbb{Q}[H]. \quad (*)$$

If $\Pi$ satisfies some additional continuity property, one can see that this equality also holds for any random payoff $H$. 
There is hence a \textbf{one-to-one correspondence} between linear valuation rules $\Pi$ verifying the properties above and probability measures $Q$ on scenarios. They are related by:

$$\Pi_0(H) = e^{-rT}E_Q[H] \quad \text{and} \quad Q(A) = \frac{\Pi_0(\mathbb{1}_A)}{\Pi_0(1)} = e^{rT}\Pi_0(\mathbb{1}_A)$$

**Interpretation of $Q(A)$:**

It is not the probability that $A$ happens in the real world but appears more as the \textbf{value of a bet} on $A$. 
Similarly, for each $t$, $A \mapsto Q_t(A) = e^{r(T-t)}\Pi_t(\mathbb{1}_A)$ defines a probability measure on $(\Omega, \mathcal{F}_t)$.

If we further require that the pricing rule is **time consistent**, i.e.

- the value at 0 of the payoff $H$ at $T$
- the value at 0 of the payoff $\Pi_t(H)$ at $t$

are equal, then $Q_t$ should be given by the restriction of $Q$ to $\mathcal{F}_t$, and

$$\Pi_t(H) = e^{-r(T-t)}\mathbb{E}_Q[H|\mathcal{F}_t] \quad (1)$$

So every time consistent linear pricing rule $\Pi$ can be expressed as a discounted conditional expectation with respect to some probability measure $Q$, as in (1) above.
Pricing rules - Arbitrage opportunity

We now consider such a pricing rule given by a probability measure $\mathbb{Q}$ and examine what restrictions are imposed to $\mathbb{Q}$ by the requirement of absence of arbitrage.

An **arbitrage opportunity** is a self-financing strategy $^1 \phi$ that can lead to a positive terminal gain, without any probability of intermediate loss:

\[
\mathbb{P}\left[ \forall t \in [0, T], V_t(\phi) \geq 0 \right] = 1, \quad \mathbb{P}\left[ V_T(\phi) > 0 \right] > 0
\]

\[
\mathbb{P}\left[ V_0(\phi) = 0 \right] = 1
\]

N.B.: the assumption of self-financing is important: it is trivial to find strategies which are not self-financing but verify the property above (exercise: it suffices to inject cash just before maturity...)

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1. a self-financing strategy is such that the value of the portfolio at $t$ is equal to the accumulated gains until $t$: $V_t(\phi) = G_t(\phi) = \int_0^t \phi_t \cdot dS_t = \phi_0 \cdot S_0 + \int_0^t \phi_u \cdot dS_u$
Remark that if \((\phi_t)\) and \((S_t)\) satisfy some integrability conditions, we can replace the first two conditions above by:

\[
\begin{aligned}
\mathbb{E}[V_T(\phi)] &> 0 \\
\mathbb{P} [\forall t \in [0, T], V_t(\phi) \geq 0] &= 1
\end{aligned}
\]
A first consequence of absence of arbitrage is the **law of one price**: two self-financing strategies with the same terminal payoff must have the same value at all times, otherwise the difference would generate an arbitrage.
Arbitrage-free pricing rules

Let us consider a pricing rule represented by a probability measure \( Q \), and suppose that there is no arbitrage in the market.

Let us consider an event \( A \) such that \( P[A] = 0 \) and an option that pays 1 to the holder if \( A \) occurs. Such an option should have a zero-value as it is considered to be impossible (nobody will pay anything for it...). So necessarily \( \Pi_0(1_A) = 0 \), which implies \( Q(A) = 0 \).

Conversely, if \( Q(A) = 0 \), then necessarily \( P(A) = 0 \), otherwise purchasing this option (for free) would lead to an arbitrage.

So in case of absence of arbitrage, a pricing measure \( Q \) must be equivalent to \( P \): they define the same set of possible events:

\[
P \sim Q : \quad \forall A \in \mathcal{F} \quad Q(A) = 0 \iff P(A) = 0 \quad (2)
\]

2. more formally, if the price is positive, any seller of this option would make an arbitrage.
Arbitrage-free pricing rules

Let us consider one of the assets $S_t^i$, and a zero-cost strategy consisting to borrow $S_t^i$ at the risk free rate at time $t$, invest the amount in the asset $i$, wait until $T$, sell at $T$ the asset at a price $S_T^i$, and pay back the borrowed money for an amount $e^{r(T-t)} S_t^i$.

The price at any $t$ of this strategy should be 0, unless it leads to an arbitrage, so we have:

$$\Pi_t[S_T^i - e^{r(T-t)} S_t^i] = e^{T-t} E_Q[S_T^i - e^{r(T-t)} S_t^i | \mathcal{F}_t] = 0$$

which implies that

$$E_Q[e^{-rT} S_T^i | \mathcal{F}_t] = e^{-rt} S_t^i \quad (3)$$

This means in other words that the discounted prices $e^{-rt} S_t^i$ are martingales under $Q$.

**Definition**

A probability measure $Q$ verifying (2) and (3) is called an **equivalent martingale measure**.
Arbitrage-free pricing rules

One can see that the converse is also true: any equivalent martingale measure $Q$ defines an arbitrage-free pricing rule via (1) slide 8.

It suffices to consider a piecewise constant self-financing strategy $(\phi_t)_{t \in [0, T]}$.

The value of the portfolio $V_t(\phi) = \int_0^t \phi \cdot dS$ is a martingale, and in particular $\mathbb{E}_Q[\int_0^T \phi \cdot dS] = 0$. The random variable $\int_0^T \phi \cdot dS$ must therefore take both positive and negative values:

$$Q[V_T(\phi) = \int_0^T \phi \cdot dS \geq 0] \neq 1$$

Since $\mathbb{P} \sim Q$, this implies $\mathbb{P}[V_T(\phi) \geq 0] \neq 1$ and hence $\phi$ cannot be an arbitrage strategy.
Arbitrage-free pricing rules

Hence:

Specifying an arbitrage free pricing rule on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$

\[ \uparrow \]

Specifying a probability measure $\mathbb{Q} \sim \mathbb{P}$ on market scenarios such that the discounted prices of traded assets are martingales.
Result 1 : Fundamental theorem of asset pricing

Up to now, we have assumed that such an arbitrage-free pricing rule/equivalent martingale measure exists, which is not obvious in a given arbitrary market/model. We have shown that if an equivalent martingale measure exists, then the market is arbitrage free.

One can show that the converse is also true (but this is more difficult), see [Harrison-Kreps, 1979], [Harrison-Pliska, 1983], [Delbaen, Schachermayer, 1994] for proof of this result under different assumptions on the process of asset prices \((S_t)\):

**Theorem (First fundamental theorem of asset pricing)**

A market model defined by \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) and asset prices \((S_t)_{t \in [0,T]}\) is arbitrage free if and only if there exists a probability measure \(\mathbb{Q} \sim \mathbb{P}\) such that the discounted asset prices \((e^{-rt}S_t)\) are martingales with respect to \(\mathbb{Q}\).
Result 2 : Market Completeness

A self-financing strategy \((\phi_t^0, \phi_t)\) is a replication strategy for a contingent claim delivering a stochastic payoff \(H\) at \(T\) if:

\[
H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_0^0 dS_0^0 \quad \mathbb{P} - a.s.
\]

A market is said **complete** if any contingent claim admits a replication strategy.

**Theorem (Second fundamental theorem of asset pricing)**

*An arbitrage free market defined by the assets \((S_t^0, S_t^1, ..., S_t^d)\), described as stochastic processes on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), is complete if and only if there exists a unique martingale measure \(Q \sim \mathbb{P}\).*
In summary: pricing and hedging of contingent assets

1. Equivalence between the absence of arbitrage opportunity assumption (A.O.A.) and the **existence** of a martingale measure / risk-neutral measure

2. Equivalence between completeness of the market and **uniqueness** of the martingale / risk neutral measure

3. In an arbitrage free complete market, the price of any contingent asset is equal to the expectation under the risk-neutral measure of its discounted payoff
In summary: pricing and hedging of contingent assets

**Pricing**
In a **complete and arbitrage free** market, if \( H \) is a contingent asset whose payoff is paid in \( T \), its price \( \Pi_t(H) \) at instant \( t \) is given by:

\[
\Pi_t(H) = \mathbb{E}_Q[e^{-r(T-t)}H(T)|\mathcal{F}_t]
\]

**Hedging**
In a **complete and arbitrage free** market, any contingent asset \( H \) is replicable by a self-financing strategy \( \phi^* \):

\[
V_T(\Phi^*)(\omega) = H(T, \omega) \quad \forall \omega \in \Omega
\]
Additive Brownian motion - Bachelier 1900

The price of a stock is assumed to follow a "continuous random walk" (i.e. a Brownian motion)

→ same chance that on a given period, the price increases or decreases (symmetry).

\[ S(t) = S(0) + \sigma W(t) \]

where

- \( S(0) \) is the initial value of the stock,
- \( W \) is a standard B.M.
- \( \sigma \) is the “volatility” of the stock
Properties of additive Brownian motion

- The price of the risky asset (stock) at instant $t$ is normally distributed with constant mean $S(0)$ and variance

\[ \text{var}[S(t)] = \sigma^2 t \]

- The price process is a martingale

\[ \mathbb{E}[S(t)|\mathcal{F}_u] = S(u) \]

- Independence of increments:

\[ S(t) - S(u) \text{ independent of } S(u) \quad (u < t) \]
Additive and geometric Brownian motion

**Drawbacks of the additive B.M.**
- The expected return of the stock is zero
- The stock price can take negative values

**Solution**
Assume independence between “relative increments” or arithmetic returns:

$$\frac{S(t) - S(u)}{S(u)} \text{ is independent of } S(u)$$
Additive and geometric Brownian motion

Using stochastic differential equations:

From the additive model to the geometric model

- Additive model:
  \[ \Delta S(t) = \sigma \Delta W(t) \]
  \[ dS(t) = \sigma dW(t) \]

- Geometric (or multiplicative) model:
  \[ \frac{\Delta S(t)}{S(t)} = \delta \Delta t + \sigma \Delta W(t) \]
  \[ dS(t) = \delta S(t) dt + \sigma S(t) dW(t) \]
→ geometric Brownian model:

\[ S(t) = S(0)e^{(\delta - \frac{\sigma^2}{2})t + \sigma W(t)} \]

\( S(t)/S(0) \) has hence a log-normal distribution \( LN(a, b) \) with parameters

\[ a = (\delta - \frac{\sigma^2}{2})t, \quad b = \sigma \sqrt{t} \]
Properties of the log-normal distribution:

Let $Y = e^X$ where $X$ is Gaussian $N(a, b)$. We say that $Y$ has a log-normal distribution with parameters $a$ and $b$.

- Moments of a log-normal distribution:
  \[
  \mathbb{E}[Y] = e^{a + \frac{b^2}{2}}
  \]
  \[
  var[Y] = e^{2a + b^2} (e^{b^2} - 1)
  \]

- Density of a log-normal distribution:
  \[
  f_Y(y) = \frac{1}{by\sqrt{2\pi}} e^{-\frac{(\ln(y) - a)^2}{2b^2}} \mathbb{1}_{y>0}
  \]
In consequence:

Properties of geometric Brownian motion:

- Moments:
  - $\mathbb{E}[S(t)] = S(0)e^{\delta t}$
  - $\text{var}(S(t)) = (S(0))^2 e^{2\delta t} (e^{\sigma^2 t} - 1)$

- Independence of arithmetic returns:
  \[
  \frac{S(t) - S(u)}{S(u)} = e^{(\delta - \sigma^2 / 2)(t-u)} e^{\sigma(W(t) - W(u))} - 1
  \]
  → independent of $S(u)$
Objective:
Getting the price of a European call or put on a risky asset (stock, index, ...)

This valuation will be performed in the simplest model after the Bachelier model: the Samuelson / Black-Scholes model.

We will only consider the unidimensional case, i.e. the case where there is only one underlying risky asset.
Assumptions of Black-Scholes’ model

1. The market contains one risky asset, supposed to follow a geometric Brownian motion (Samuelson model)
2. The market contains a risk free asset (saving account), with (continuously compounded) rate of return $r$
3. The market is perfect
4. No dividend is distributed on the risky asset during the considered period
5. No transaction costs, assets are infinitely divisible, no constraint on short selling

3. see later for the case with dividends
We will see three approaches to arrive to an explicit formula for the call price:

1. Direct resolution by using an A.O.A. argument
2. Resolution by using a martingale measure and general results on pricing rules
3. Limit of the binomial model
Black-Scholes’ formula

Notations:

- The stock price at instant \( t \) is denoted by \( S(t) \) and satisfies:
  \[
dS(t) = \delta S(t)dt + \sigma S(t)dW(t)
  \]

- The price of the call at instant \( t \), with strike \( K \), maturity \( T \), and on the underlying stock having a value \( S \) at \( t \) is denoted by:
  \[
  C(S, t, K, T)
  \]

In what follows, \( T \) and \( K \) will be assumed fixed, and omitted. We will hence consider \( C \) as a function of \((S, t)\) only.
We will construct a self-financing portfolio composed of assets tradable on the market, i.e. the underlying stock $S(t)$ and the risk free asset.
Let us denote by $\beta_t$ the value of 1 currency unit invested at $t = 0$:
$\beta_t = e^{rt}$.
The value of that portfolio at $t$ can hence be written:
$$V(t) = V(t, S_t) = a_t S_t + b_t \beta_t$$

The portfolio is supposed to be self-financing: no arrival of external money and no consumption.

This means that the variation of the value of that portfolio on a given period is only due to variations of the value of the assets themselves.
Denoting by $t + \Delta t$ a rebalancing instant of the portfolio, we have:

$$a(t)S(t + \Delta t) + b(t)\beta(t + \Delta t) = a(t + \Delta t)S(t + \Delta t) + b(t + \Delta t)\beta(t + \Delta t)$$

which can be written:

$$(a(t + \Delta t) - a(t))S(t + \Delta t) + (b(t + \Delta t) - b(t))\beta(t + \Delta t) = 0$$

for any value taken by the stock at $t + \Delta t$, and for any instant $t$. By assuming that rebalancing can occur at more and more frequent instants ($\Delta t \to 0$), this can be translated in:

$$S_t da_t + \beta_t db_t = 0$$
We will impose that $V$ is a replicating portfolio, i.e. such that

$$V(T, S_T) = (S_T - K)^+ \quad p.s.$$  

In that case, the A.O.A. condition implies that the value of the portfolio is equal to the value of the call at any instant: $V(t, S_t) = C(t, S_t)$.

(i) By applying Itô’s lemma to $Y_t = V(t, S_t)$:

$$dV = \left( \frac{\partial V}{\partial t} + \delta S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW(t).$$
(ii) On the other hand, as $V_t = a_t S_t + b_t \beta_t$ is self-financing, we have:

$$
\begin{align*}
    dV &= d(a_t S_t + b_t \beta_t) \\
    &= a_t dS_t + b_t d\beta_t \\
    &= (\delta a_t S_t + rb_t \beta_t)dt + a_t \sigma S_t dW(t) \\
    &= (\delta a_t S_t + r(V_t - a_t S_t))dt + a_t \sigma S_t dW(t)
\end{align*}
$$
Black-Scholes’ formula: Direct approach by A.O.A.

By using the fact that:

\[ P_t^{(1)} dt + P_t^{(2)} \, dW_t = Q_t^{(1)} \, dt + Q_t^{(2)} \, dW_t \quad \forall \, t \]

iff

\[ \begin{cases} 
P_t^{(1)} = Q_t^{(1)} \\
P_t^{(2)} = Q_t^{(2)} 
\end{cases} \quad \forall \, t \]

we get:

\[
\frac{\partial V}{\partial t} + \delta S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} = \delta a_t S_t + r(V_t - a_t S_t) \\
S_t \frac{\partial V}{\partial S} = a_t S_t
\]
which leads to the following PDE in $V(S, t)$ (the Black-Scholes PDE):

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$

with terminal condition:

$$V(T, S) = (S - K)^+ \quad \forall S \geq 0$$

The PDE only depends on the volatility $\sigma$ of the underlying and the risk free rate $r$, but not on the rate of return $\delta$ of the underlying stock.

The PDE without the terminal condition is verified for any self-financing portfolio $V$ composed of the risky asset and the risk free asset (so not necessarily replicating the call).
Solving the Black-Scholes PDE in 3 steps:

i Transformation in the heat equation

ii Solving the heat equation

iii Back to the initial variables
Etape 1 : transformation in the heat equation :

Change of variables :

\[ V(t, S) = e^{-r(T-t)}u(x, \tau) \]
\[ \tau = \frac{2r-\sigma^2}{\sigma^2}(T-t), \]
\[ x = \frac{2r-\sigma^2}{\sigma^2}(\log(S/K) + (r - \frac{\sigma^2}{2})(T-t)) \]

Rem : \( \tau \) appears as a normalized time to maturity
The PDE hence become a new equation in variables $u, x, \tau$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau} \quad (\text{heat equation}),$$

The terminal condition $V(S, T) = (S - K)^+$ becomes:

$$u(x, 0) = Ke^{x\sigma^2/(2r - \sigma^2)} - K := \phi(x)$$

i.e. an initial condition.
Etape 2 : Solving the heat equation

One can show (cf. elementary calculus course) that the solution can be obtained under the form of a Poisson integral:

\[
u(x, \tau) = \frac{1}{2 \sqrt{\pi \tau}} \int_{-\infty}^{+\infty} \phi(s) e^{-\frac{(x-s)^2}{4\tau}} ds\]

whose explicit calculation is given by:

\[
u(x, \tau) = K \Phi \left( \frac{x^*}{\sqrt{2\tau}} \right) e^{-\frac{x^*^2-x^2}{4\tau}} - K \Phi \left( \frac{x}{\sqrt{2\tau}} \right).
\]

with \(x^* = x + \frac{\tau \sigma^2}{2 (r - \frac{\sigma^2}{2})}\) and where \(\Phi\) is the cumulative distribution function of the standard normal distribution:

\[
\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{s^2}{2}} ds
\]
Etape 3 : Back to the initial variables

\[ u(x, \tau) \rightarrow V(S, t) \]

Returning to the initial variables, we obtain the price of the replicating portfolio \( V(t) \), and hence to the call price \( C(S, t) \) at instant \( t \) :

\[
C(S, t) = V(S, t) = S(t) \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2),
\]

with

\[
d_1 = \frac{\log(S/K) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = \frac{\log(S/K) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}
\]

which is the so-called Black-Scholes’ formula.
The price of the call hence depends on:

- The volatility $\sigma$
- The risk free rate $r$

but NOT on the average rate of return of the underlying $\delta$. 
Rem : Price of a European put :

We can immediately deduce from that formula the price of a European put by using the put-call parity relationship :

\[ P(S, t) + S = C(S, t) + e^{-r(T-t)}K \]

which leads here to :

\[ P(S, t) = Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1) \]
We will apply the general result on pricing in a complete market without arbitrage:

\[ C(S, t) = \mathbb{E}_Q [e^{-r(T-t)}(S(T) - K)^+] | \mathcal{F}_t ] \]

Problem:
does a martingale measure/risk-neutral measure exist in this model? if yes, what is the dynamics of \((S_t)\) under the risk-neutral measure \(Q\)?

We are looking for a probability measure \(Q\) such that \(e^{-rt} S_t\) is a martingale under \(Q\).
\[ \rightarrow \text{Girsanov Theorem} \]
Theorem

Let \( \{W(t, \omega); t \in [0, T], \omega \in \Omega\} \) be a B.M. defined on a probability space \((\Omega, \mathcal{F}, P)\).

Let \( W^* \) be a process be defined by:

\[
W^*(t) = W(t) - \theta t \quad (\theta \in \mathbb{R})
\]

The the process \( W^* \) is a B.M. under measure \( Q \) equivalent to \( P \) whose density is defined by:

\[
\rho_T = Y(T) = \exp \left( \theta W(T) - \frac{\theta^2}{2} T \right)
\]

We will apply this result with \( \theta = -\frac{\delta - r}{\sigma} \).
Indeed, we can write:

\[
    dS(t) = \delta S(t) dt + \sigma S(t) dW(t) = rS(t) dt + \sigma S(t) dW^*(t)
\]

with \( W^*(t) = W(t) + \frac{\delta - r}{\sigma} t \) (Brownian motion with drift under \( \mathbb{P} \)).

By applying the THM with \( \theta = -\frac{\delta - r}{\sigma} \), we know that there exists a measure \( \mathbb{Q} \sim \mathbb{P} \) such that \( W^*(t) \) is a standard B.M. under \( \mathbb{Q} \).
Black-Scholes’ formula: Approach by martingale measure

Under this new measure, $X(t) = e^{-rt}S(t)$ satisfies (by Itô Lemma):

$$dX(t) = -rX(t)dt + e^{-rt}dS(t)$$

$$= -rX(t)dt + e^{-rt}(rS(t)dt + \sigma S(t)dW^*(t))$$

$$= -rX(t)dt + rX(t)dt + \sigma X(t)dW^*(t))$$

$$= \sigma X(t)dW^*(t))$$

i.e.:

$$X(t) = X(0) + \sigma \int_0^t X(t)dW^*(t)$$

which implies $X(t) = e^{-rt}S(t)$ is a martingale under $Q$.

This measure $Q$ is hence well a martingale measure, and it is equivalent to $\mathbb{P}$ by Girsanov thm

$\rightarrow$ existence of an equivalent martingale measure

$\rightarrow$ there is no arbitrage opportunity in that model
On the other hand, we can show that the market is complete in this model.

**Theorem (cf. Karatzas-Shreve Th 6.6)**

Let a financial market composed of $d$ risky assets whose prices $S_i$ are Itô processes:

$$dS_i(t) = \mu_i(t)dt + \sigma_i(t).dW_t$$

with $\sigma_i = (\sigma^{i,1}, ..., \sigma^{i,k})$ and $W = (W^1, ..., W^k)$ a B.M. $k$ dimensional. Then, this market is complete iff the number $d$ of risky assets is equal to the dimension $k$ of the Brownian motion $B$ and if its volatility matrix is non singular a.s. for $t \in [0, T]$.

We have hence uniqueness of the martingale measure (and hence of option prices...).
We can hence write the price of the call $C(S, t)$ as:

$$C(S, t) = \mathbb{E}_Q[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}_t]$$

with under $\mathbb{Q}$: $dS(t) = rS(t)dt + \sigma S(t)dW^*(t)$ and $W^*$ is a standard B.M., where $\mathbb{Q} \sim \mathbb{P}$ has a RN derivative given by the Girsanov theorem (by replacing $\theta$ by $-\frac{\delta - r}{\sigma}$).

We just need now to calculate that price...
Black-Scholes’ formula : Approach by martingale measure

Dynamics of $S(t)$ under $\mathbb{Q}$:

$$dS(t) = rS(t)dt + \sigma S(t)dW^*(t)$$

with $W^*$ a standard B.M. under $\mathbb{Q}$.

In particular, we can write (cf. solving the EDS of the geometric Brownian motion):

$$S(T) = S(t)e^{(r-\frac{\sigma^2}{2})(T-t)}e^{\sigma(W^*(T)-W^*(t))}$$

$$\Rightarrow \frac{S(T)}{S(t)} \sim LN\left((r-\frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}\right)$$
Black-Scholes’ formula: Approach by martingale measure

The expectation $E_Q[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}_t]$ hence becomes:

$$e^{-r(T-t)} \int_{K}^{+\infty} (x - K) dF^*(x)$$

where

$$F^*(x) = Q[S(T) \leq x | S(t) = S]$$

$$= \Phi \left( \frac{ln(x/S) - (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right)$$

(cf. CDF of the log-normal distribution)
Black-Scholes’ formula: Approach by martingale measure

\[ C(S, t) = e^{-r(T-t)} \int_{K}^{+\infty} x dF^*(x) - e^{-r(T-t)} \int_{K}^{+\infty} K dF^*(x) \]

\[ = e^{-r(T-t)} \int_{K}^{+\infty} x dF^*(x) - Ke^{-r(T-t)} \Phi(d_2) \]

\[ = S\Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2) \]

→ Black-Scholes’ formula
We can also directly deduce the formula

\[ C(S, t) = \mathbb{E}[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}_t] \]

by using the Feynman-Kac lemma from the Black-Scholes PDE with terminal condition:

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0,
\]

\[ C(T, S) = (S - K)^+ \quad \forall S \geq 0 \]
The variant of Feynman-Kac lemma seen with the additional cooling term (cf. preceding course, slide 81) shows that

\[ F(t, S) = \mathbb{E}[e^{-r(T-t)}(S(T) - K)^+|S(t) = S] \]

is a solution of the PDE with terminal condition, where \( S(t) \) is a process satisfying the SDE:

\[ dS(t) = rS(t)dt + \sigma S(t)dW(t) \]

where \( W(t) \) is a standard B.M.

(Rem : the risk-neutral measure is not explicitly mentioned here : the solution appears as the expectation of a discounted payoff, with a process whose dynamics is known. Here this known dynamics is the risk-neutral dynamics.)
We have already seen that the (discrete) random walk converges to the Brownian motion.

In the binomial model with $n$ periods, we have seen that the price of a European call of maturity $n$ and strike $K$ on an underlying asset $S = S(i)$ evolving as:

$$S(j) = \begin{cases} uS(j - 1) & p \\ dS(j - 1) & 1 - p \end{cases}$$

is given by:

$$C = C(S, 0) = \frac{1}{(1 + r)^n} E_Q[(S_n - K)^+]$$

where $Q$ is the risk-neutral measure, under which $p$ is replaced by

$$\pi = \frac{1+r-d}{u-d}.$$
We hence have:

\[
C(S, 0) = \frac{1}{(1 + r)^n} \sum_{j=0}^{n} \binom{n}{j} \pi^j (1 - \pi)^{n-j} (u^j d^{n-j} S - K)^+
\]

Many terms in the preceding sum are equal to zero. This is the case as soon as \(u^j d^{n-j} (S - K) < 0\) → Let \(\eta\) be defined by:

\[
\eta = \inf \{ j \in \mathbb{N} | u^j d^{n-j} S - K > 0 \}
\]

hence, the formula can be rewritten as:

\[
C(S, 0) = \frac{1}{(1 + r)^n} \sum_{j=\eta}^{n} \binom{n}{j} \pi^j (1 - \pi)^{n-j} (u^j d^{n-j} S - K)
\]
By denoting

\[ D(n, \eta, \pi) = \sum_{j=\eta}^{n} \binom{n}{j} \pi^j (1 - \pi)^{n-j} \]

\( D(n, \eta, \pi) \) appears as the survival function of a binomial distribution \( \text{Bin}(n, \pi) \).

By using

\[ 1 - \frac{\pi u}{1+r} = \frac{d(1 - \pi)}{1+r} \]

we get :

\[ C(S, 0) = SD \left( n, \eta, \frac{\pi u}{1+r} \right) - \frac{K}{1+r} D(n, \eta, \pi) \]
Convergence of the binomial model to the BS model:

Parameterization:

- Time intervals of length $\Delta_n$
- Risk free rate: $1 + r_n = e^{r\Delta_n}$
- Risky asset:

$$u_n = e^{\sigma \sqrt{\Delta_n}}$$

$$d_n = \frac{1}{u_n} = e^{-\sigma \sqrt{\Delta_n}}$$

In this case, we can show that the price of the call converges to the BS formula.

Idea of the proof: ...
Black-Scholes’ model - The Greeks

The “Greeks” allow to measure the sensitivity of an option price (here, the call) in function of its parameters.

They allow a quantification of the risks of a position in options, and are practical management tools for traders.

\[ \Delta = \text{delta} = \frac{\partial C}{\partial S} \]

\[ \Gamma = \text{gamma} = \frac{\partial^2 C}{\partial S^2} \]

\[ \nu = \text{vega} = \frac{\partial C}{\partial \sigma} \]

\[ \theta = \text{theta} = \frac{\partial C}{\partial t} \]

\[ \rho = \text{rho} = \frac{\partial C}{\partial r} \]

The Greeks depend on the model that we use!
Black-Scholes’ model - The delta

The delta, denoted by $\Delta$, measures the sensitivity the call price to a variation of the underlying asset: when the underlying moves by 1 currency unit, the call price moves (approximately) by an amount of $\Delta$ currency units.

This is also the quantity of risky asset composing the replicating portfolio constructed on the underlying (cf. slide 36).

We can get by direct calculation:

$$\frac{\partial C}{\partial S} = \Phi(d_1)$$

In particular $0 < \Delta < 1$

We can also show this by departing from the integral representation:

$$C(0, S(0)) = \mathbb{E}_Q[e^{-rT}(S(0)e^{(r-\sigma^2/2)T + \sigma W_T} - K)^+]$$
The Gamma measures the convexity of the price of the call with respect to the underlying:

\[
\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S}
\]

Departing from the expression of the Delta, we directly get:

\[
\Gamma = \frac{e^{-\sigma^2 t / 2}}{S \sigma \sqrt{T - t} \sqrt{2\pi}} > 0
\]

The call price is hence a convex function of the underlying price.
Illustration of the call price in B-S model in function of the underlying price \((\sigma = 20\% \text{ vs } \sigma = 10\%)\)
Black-Scholes’ formula - The Vega

The Vega: sensitivity of the call price w.r.t. the volatility of the underlying:

\[ \nu = \frac{\partial C}{\partial \sigma} = \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} S \sqrt{T - t} > 0 \]

Hence, the call price increases with the volatility of the underlying.
Sensitivity of the call price w.r.t. the time to maturity: \( \frac{\partial C}{\partial \tau} \) where \( \tau = T - t \) is the time to maturity.

We can show that:

\[
\frac{\partial C}{\partial \tau} = \frac{S\sigma}{2\sqrt{\tau}} \Phi'(d_1) + rKe^{-r\tau} \Phi(d_2) > 0
\]

The price of the call is hence an increasing function of the time to maturity.
The Theta is defined as the sensitivity of the call price w.r.t. to time $t$:

$$ \Theta = \frac{\partial C}{\partial t}.$$

We can show from the preceding that:

$$\Theta = -\frac{\partial C}{\partial \tau} = -\frac{S \sigma}{2\sqrt{T-t}} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} - rK e^{-r(T-t)} \Phi(d_2) < 0$$

The call price decreases when time passes.
Illustration of the call price within the B-S model in function of the underlying price for different remaining maturities ($\sigma = 20\%$)
On peut montrer que :

\[ \rho = \frac{\partial C}{\partial r} = K(T - t)e^{-r(T-t)}\Phi(d_2) > 0 \]

Exercice :
Montrer que \( \frac{\partial C}{\partial K} \leq 0 \), et que \( \frac{\partial^2 C}{\partial K^2} \geq 0 \).

The call price is hence a convex decreasing function of the strike.
We can also be interested to the “volatility” of the call. For that purpose, we apply the Itô formula to the call seen as a function of \((t, S)\):

\[
dC_t = \left( \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} \delta S_t + \frac{1}{2} \left( \sigma S_t \right)^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} \sigma S_t dW_t
\]

that we rewrite under the form:

\[
dC_t = \delta_{\text{call}} C_t dt + \sigma_{\text{call}} C_t dW(t).
\]

We then arrive to:

\[
\text{volatility of the call} = \sigma_{\text{call}} = \frac{\partial C}{\partial S} \frac{S}{C} \sigma = \Delta \frac{S}{C} \sigma_S
\]

We remark that the ratio between \(\sigma_{\text{call}}\) and \(\sigma_S\) is equal to:

\[
\frac{\Delta \cdot S}{C} = \frac{S \Phi(d_1)}{\Phi(d_2) - K e^{-r(T-t)} \Phi(d_2)} \geq 1
\]

The risk of the call is hence superior to the risk of the underlying.
Moreover, by departing from the application of the Itô formula to $C(S, t)$, we get that the instantaneous return of the call, denoted by $\delta_{\text{call}}$, is equal to:

\[
\delta_{\text{call}} = \frac{1}{C_t} \left( \frac{\partial C}{\partial t} + \frac{1}{2} (\sigma S_t)^2 \cdot \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial S} \delta S_t \right) \\
= \frac{1}{C_t} \left( rC_t - rS_t \frac{\partial C}{\partial S} \right) + \frac{1}{C_t} \frac{\partial C}{\partial S} \delta S_t
\]

which implies:

\[
\delta_{\text{call}} - r = \frac{S_t}{C_t} \frac{\partial C}{\partial S} (\delta S - r) = \frac{\Delta S}{C} (\delta S - r)
\]

which generalizes the discrete case.
Black-Scholes’ formula avec dividendes

Let us assume that the market is constituted of a risk free asset (risk free rate $r$) and of a risky asset paying a (continuous) dividend yield $d$:

$$dS(t) = (\mu - d)S(t)dt + \sigma dW(t)$$

We can reuse the reasoning involving a replicating portfolio $V_t = a_t S_t + b_t \beta_t$. The argument is similar, but the self-financing condition for $V(t)$ becomes:

$$d(a_t S_t + b_t d\beta_t) = a_t(dS(t) + d.S(t)dt) + b_t d\beta_t$$

because a dividend amount $d.S(t)dt$ is paid during the time interval $[t, t + dt]$, and hence:

$$dV = d(a_t S_t + b_t \beta_t) = a_t dS_t + a_t S_t dt + b_t d\beta_t$$

$$= ((\mu - d)a_t S_t + d a_t S_t + rb_t \beta_t)dt + a_t \sigma S_t dW(t)$$

$$= (\mu a_t S_t + r(V_t - a_t S_t))dt + a_t \sigma S_t dW(t)$$
Black-Scholes’ formula with dividends

The application of the Itô lemma and the self-financing condition lead to:

\[
\frac{\partial V}{\partial t} + (\mu - d)S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} = \mu a_t S_t + r(V_t - a_t S_t)
\]

\[
S_t \frac{\partial V}{\partial S} = a_t S_t
\]

which leads to the Black-Scholes PDE with dividends in \( V(S, t) \):

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - d)S \frac{\partial V}{\partial S} = rV,
\]

with terminal condition:

\[
V(T, S) = (S - K)^+ \quad \forall S \geq 0
\]
Black-Scholes’ formula with dividends

By the Feynman-Kac lemma with \( \mu'' = (r - d)S, \sigma'' = \sigma S \) and \( r'' = r \) (cf. slide 82), the solution is given by:

\[
E[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}_t] \quad (1)
\]

where the process \( S(t) \) satisfies the SDE:

\[
dS(t) = (r - d)S(t)dt + \sigma S(t)dW(t)
\]

As (1) can be rewritten as

\[
e^{-d(T-t)}E[e^{-(r-d)(T-t)}(S(T) - K)^+|\mathcal{F}_t],
\]

we can apply the Black-Scholes’ formula by replacing \( r \) by \( r - d \) and multiplying the whole by \( e^{-d(T-t)} \).
We get then the Black-Scholes’ formula with dividends:

\[ C(t, S) = e^{-d(T-t)} S(t) \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2) \]

\[
d_1 = \frac{\ln(S_t/K) + (r - d + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t}
\]
In fact, the risk-neutral measure $\mathbb{Q}$ is always characterized by a (total) return of the risky asset equal to $r$:

$$dS(t) = (r - d)S(t)dt + \sigma S(t)dW^*(t)$$

This actually means that the return ex dividend is $r - d$ (return on the value only), which leads to a total return of $r$. 
Black-Scholes’ formula