Applications of Black-Scholes model

Black-Scholes formula: applications

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Forward contract

Let us consider a financial product with payoff at $T : \chi = f (S_T)$ $\mathcal{F}_T$–measurable, and suppose that we are in the framework of BS model. Let us suppose that we are now at instant $t < T$.

Definition

A forward contract with maturity $T$ on $\chi$, concluded at instant $t < T$, is a contract in which the buyer will pay a price $K$ (forward price) at date $T$ and will receive $\chi$ at this date.

Nothing is paid nor received at date $t$, but the forward price $K$ that will be paid at $T$ is determined at date $t$ (at the signature of the forward contract).

Notation

The forward price $K$ determined at instant $t$ will be denoted by $f (t; T, \chi)$.
We will compute it now.
Clearly, the amount received at $T$ by the buyer is

$$Y = \chi - K$$

and the amount received at date $t$ is equal to 0 (no cash-flow at $t=0$ by definition of the forward contract).

By using the fundamental thm on pricing in an arbitrage free and complete market, we directly get:

$$E_Q \left[ e^{-r(T-t)} (\chi - K) \mid \mathcal{F}_t \right] = 0$$

which implies

$$E_Q [\chi \mid \mathcal{F}_t] = E_Q [K \mid \mathcal{F}_t] = f(t; T, \chi)$$

since $K$ is $\mathcal{F}_t$–measurable ($K$ is fixed at $t$)
We hence have the following result:

**Proposition**

The forward price $f(t; T, \chi)$ determined at instant $t$, on the financial product with payoff $\chi \mathcal{F}_T$-measurable is:

$$f(t; T, \chi) = E_Q [\chi | \mathcal{F}_t]$$

In particular, if $\chi = S_T$, then

$$f(t; T, S_T) = e^{r(T-t)} S_t.$$

This last inequality simply comes from the fact that $(e^{-rt}S_t)$ is a martingale under $Q$. 
Call on forward

We now consider a European call option with maturity $T$ and strike $K$, on a forward contract with underlying $S_t$ and maturity $T_1 > T$.

The call holds more specifically on the forward price with maturity $T_1$, which is worth at $t$:

$$F(t, T_1; S) = F(t, T_1) = e^{r(T_1-t)} S_t$$

The underlying is hence a forward price associated to a given maturity date $T_1$.

At date $T$, the payoff of the call is:

$$(F(T, T_1) - K)_+ = \left( e^{r(T_1-T)} S_T - K \right)_+$$

$$= e^{-r(T-t)} \left( e^{r(T_1-t)} S_T - e^{r(T-t)} K \right)_+$$
Call on forward

The call price is hence equal to

\[ e^{-r(T-t)} C(t, e^{r(T_1-t)} S_t, K') = e^{r(T-t)} K, T) \]

i.e.:

\[ e^{-r(T-t)} \left[ e^{r(T_1-t)} S_t \Phi (d_1) - K \Phi (d_2) \right] \]

\[ = e^{-r(T-t)} \left[ F_t \Phi (d_1) - K \Phi (d_2) \right] \]

\[ d_1 = \frac{\ln \frac{F_t}{e^{r(T-t)} K} + (r + \sigma^2/2) (T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = d_1 - \sigma \sqrt{T-t}. \]
Proposition (Black’s formula)

The price at \( t \) of a European call with maturity \( T \) and strike \( K \), on the forward price of maturity \( T_1 > T \) on the underlying \( S_t \), is given by:

\[
e^{-r(T-t)} \left[ F(t, T_1) \phi(d_1) - K \phi(d_2) \right]
\]

with

\[
d_1 = \frac{\ln(F/K) + \sigma^2(T-t)/2}{\sigma \sqrt{T-t}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t}.
\]
Let us consider a market composed of a risk free asset (risk free rate $r$) and of two risky assets of price $S_1(t)$ and $S_2(t)$ satisfying the assumptions of Black-Scholes’ model:

$$d\beta_t = r\beta_t dt$$

$$dS_1(t) = \alpha_1 S_1(t) dt + \sigma_1 S_1(t) dW^1_t$$

$$dS_2(t) = \alpha_2 S_2(t) dt + \sigma_2 S_2(t) dW^2_t$$

with corr ($W^1_t, W^2_t$) = 0 (independence between both stocks $S_1$ et $S_2$).

We consider an option allowing to exchange stock 2 against stock 1 at instant $T$ (“exchange option”):

payoff of the option:

$$(S_1(T) - S_2(T))_+$$
Exchange option: pricing by replication

A first method consists to build a self-financing replicating portfolio, of value

\[ V(t) = V(t, S_1, S_2) = a_1(t)S_1(t) + a_2(t)S_2(t) + b(t)\beta_t \]

like in the case of the European call, and apply the multidimensional Itô lemma.

We can then see than we arrive to the following PDE:

\[ \partial_t V + r s_1 \partial_1 V + r s_2 \partial_2 V + \frac{1}{2} s_1^2 \sigma_1^2 \partial_{11} V + \frac{1}{2} s_2^2 \sigma_2^2 \partial_{22} V = rV \]

with terminal condition \( V(T, s_1, s_2) = (s_1 - s_2)^+ \) for all \((s_1, s_2) \in \mathbb{R}^+ \times \mathbb{R}^+\).
The key of the remaining part is to see that the terminal condition is homogeneous with degree 1 in the following sense:

\[ H(s_1, s_2) = (s_1 - s_2)^+ = s_2 \left( \frac{s_1}{s_2} - 1 \right)^+ = s_2 H \left( \frac{s_1}{s_2}, 1 \right). \]

One can show that this property is kept in the solution of the PDE: by introducing the change of variable

\[ V(t, s_1, s_2) = s_2 F \left( t, \frac{s_1}{s_2} \right) \]

we get a new PDE on the unknown function \( F(t, z) \), more simple than the preceding PDE (one dimension less), and which has a solution.
Exchange option: pricing by replication

Indeed, the PDE for the new unknown function $F(t, z)$ becomes:

\[
\begin{aligned}
\frac{\partial F}{\partial t} + \frac{1}{2} z^2 \sigma_1^2 \frac{\partial^2 F}{\partial z^2} + \sigma_1^2 \frac{\partial F}{\partial z} &= 0 \\
F(T, z) &= (z - 1)_+
\end{aligned}
\]

which is exactly the Black-Scholes PDE:

\[
\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS \frac{\partial F}{\partial S} - rF = 0, \quad F(T, S) = (S - K)_+
\]

avec:

\[
''r'' = 0 \\
''K'' = 1 \\
''\sigma'' = \sqrt{\sigma_1^2 + \sigma_2^2}
\]
The solution $F$ is hence given by the following formula:

$$F(t, z) = z\Phi(d_1(z)) - \Phi(d_2(z))$$

$$d_1 = \ln z + \frac{\sigma_1^2 + \sigma_2^2}{2}(T-t) \frac{1}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}}$$

$$d_2 = d_1 - \sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{T-t}$$

The price at $t$ of the exchange option is hence:

$$V(t, s_1, s_2) = s_2 F\left(t, \frac{s_1}{s_2}\right) = s_1\Phi\left(d_1\left(\frac{s_1}{s_2}\right)\right) - s_2\Phi\left(d_2\left(\frac{s_1}{s_2}\right)\right)$$

This formula is known as the so-called “Margrabe formula”.

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Exchange option

Theorem (Margrabe formula)

Let two risky assets $S_1(t)$ and $S_2(t)$ satisfying the assumptions of the Black-Scholes model:

$$dS_1(t) = \alpha_1 S_1(t)dt + S_1(t)\sigma_1 dW_1(t)$$
$$dS_2(t) = \alpha_2 S_2(t)dt + S_2(t)\sigma_2 dW_2(t)$$

with $W_1(t)$ and $W_2(t)$ independent, and let $r$ be the risk free rate. Then the price at $t$ of an exchange option of maturity $T$, of payoff : $(S_2 - S_1)^+$ is given by:

$$S_1 \Phi(d_1) - S_2 \Phi(d_2)$$

$$d_1 = \frac{\ln \frac{S_1}{S_2} + \frac{\sigma_1^2 + \sigma_2^2}{2} (T - t)}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T - t)}}$$
$$d_2 = d_1 - \sqrt{(\sigma_1^2 + \sigma_2^2)(T - t)}$$

One can generalize this formula to the case where $\text{corr}(W_1(t), W_2(t)) = \rho$. 
Another method consists to apply the technique of change of numeraire.

The departure point consists to see that the price of the exchange option is equal to:

\[ E_Q[e^{-r(T-t)} (S_1(T) - S_2(T))_+ | \mathcal{F}_t] = E_Q \left[ e^{-r(T-t)} S_2(T) \left( \frac{S_1(T)}{S_2(T)} - 1 \right)_+ | \mathcal{F}_t \right] \]

The idea is to change measure: we will pass to a new equivalent measure under which the prices expressed w.r.t. the second asset \( S_2 \) are martingales.
A **numeraire** is an asset exchanged on the market having a strictly positive value and not paying any dividend.

Example: \( \beta_t = e^{rt} \) the saving/bank account.

Once we have a numeraire, we can express the prices of any financial product w.r.t. that numeraire:
If \( N_t \) is (the price at \( t \) of) a numeraire, and \( P(t) \) is the price of a financial product, the its price expressed w.r.t. that numeraire is: \( P(t)/N(t) \).

Example with the bank account: discounted prices \( P(t)e^{-rt} = \frac{P(t)}{\beta_t} \)
correspond actually to prices expressed w.r.t. the particular choice of numeraire of the bank account: \( N(t) = \beta_t \).

We know that under the risk neutral measure \( Q \), the discounted prices of assets are martingales. Measure \( Q \) is actually the martingale measure associated to the numeraire choice \( \beta_t \).

We will show that one can associate a martingale measure to any numeraire choice.
Generally, when we pass to an equivalent measure (e.g. from $Q_1$ to $Q_2$), the measure change can be done with the Radon-Nikodym derivative:

$$ Z = \frac{dQ^2}{dQ^1} $$

such that $E_{Q^2}[Y] = E_{Q^1}[ZY]$ for any random variable $Y$.

If we work in a probability space with filtration $(\mathcal{F}_t)_{t\in[0,T]}$, we can define a stochastic process $(Z_t)$ from the R-N derivative:

$$ Z_t := E_{Q^1}[Z|\mathcal{F}_t] $$

By construction, $Z_t$ is a $Q_1$-martingale adapted to this filtration, and

$$ Z_0 = E_{Q_1}[Z|\mathcal{F}_0] = E_{Q_1}[Z] = 1, \quad Z_T = E_{Q_1}[Z|\mathcal{F}_T] = E_{Q_1}[Z|\mathcal{F}] = Z. $$
Change of numeraire: change of measure

We have the first following result:

**Lemma**

*If $Y$ is a random variable $\mathcal{F}_t$–measurable, then:*

$$E_{Q^2}[Y] = E_{Q^1}[Z_t Y]$$

Indeed:

$$E_{Q^2}[Y] = E_{Q^1}[Z Y] = E_{Q^1}[E_{Q^1}[Z Y | \mathcal{F}_t]] = E_{Q^1}[Y E_{Q^1}[Z | \mathcal{F}_t]] = E_{Q^1}[YZ_t]$$

This lemma means in other words that:

$$\frac{dQ^2}{dQ^1}\bigg|_{\mathcal{F}_t} = Z_t$$
Change of numeraire: change of measure

We will see that we can use the process $Z_t$ to change measure in the conditional expectations:

Lemma

If $Y$ is $F_T$ measurable, then:

$$E_{Q^2}[Y | F_t] = E_{Q^1}\left[ Y \frac{Z_T}{Z_t} | F_t \right] = \frac{1}{Z_t} E_{Q^1}[YZ_T | F_t] \quad (*)$$

Proof:

By definition of the conditional expectation, $E_{Q^2}[Y | F_t]$ is the random variable $U - F_t$ measurable such that for any $A \in F_t$, we have:

$$E_{Q^2}[U \mathbb{1}_A] = E_{Q^2}[Y \mathbb{1}_A] \quad (**)$$

Let us show that $U = E_{Q^1}[Y \frac{Z_T}{Z_t} | F_t]$ well satisfies this condition.
Let $A \in \mathcal{F}_t$. Then $E_{Q^2}[U\mathbb{I}_A]$ is equal to:

$$
E_{Q^2} \left[ \mathbb{I}_A E_{Q^1} \left[ \frac{Y Z_T}{Z_t} | \mathcal{F}_t \right] \right] = E_{Q^1} \left[ Z_t \mathbb{I}_A E_{Q^1} \left[ \frac{Y Z_T}{Z_t} | \mathcal{F}_t \right] \right]
$$

by the preceding result. Since $1/Z_t$ is $\mathcal{F}_t$ measurable, we can put it out of the conditional expectation:

$$
= E_{Q^1} \left[ Z_t \mathbb{I}_A \frac{1}{Z_t} E_{Q^1} [YZ_T | \mathcal{F}_t] \right] = E_{Q^1} [\mathbb{I}_A E_{Q^1} [YZ_T | \mathcal{F}_t]]
$$

By definition of the conditional expectation $E_{Q^1} [YZ_T | \mathcal{F}_t]$, this is equal to:

$$
E_{Q^1} [YZ_T \mathbb{I}_A] = E_{Q^2} [Y \mathbb{I}_A]
$$

once again by applying the first lemma with now $t = T$, which is well the right-and side of (**).
Change of numeraire and martingale measure

When we have a numeraire $N(t)$, it is possible to associate to it a measure $Q^N \sim Q$ (and hence $Q^N \sim P$) such that the prices of financial products expressed with respect to that new numeraire are martingales under $Q^N$:

$$\mathbb{E}_{Q^N} \left[ \frac{S_T}{N_T} | \mathcal{F}_t \right] = \frac{S_t}{N_t}$$

Indeed, it suffices to define $Q^N$ from $Q$ as follows:

$$\frac{dQ^N}{dQ} = Z = \frac{M(0)N(T)}{N(0)M(T)}$$

(where $M$ is the numeraire associated to $Q$, $M(t) = \beta_t$ to fix ideas...).

The new measure $Q^N$ has the property that prices of assets expressed w.r.t. to numeraire $N(t)$ are martingales under $Q^N$. 
To show that, it suffices to use the second lemma:

1) Let us first search what is the form of $Z_t$, the associated R-N process:

$$Z_t = \mathbb{E}_Q \left[ \frac{M(0)N(T)}{N(0)M(T)} | F_t \right] = \frac{M(0)N(t)}{N(0)M(t)}$$

since $N(t)/M(t)$ is a $\mathbb{Q}$-martingale.
2) Let us now use the second lemma (change of measure in conditional expectations):

\[
\mathbb{E}_Q [S(T) \frac{N(T)}{N(t)} | \mathcal{F}_t] = \mathbb{E}_Q \left[ \frac{S(T)}{N(T)} \frac{Z_T}{Z_t} | \mathcal{F}_t \right] = \mathbb{E}_Q \left[ \frac{S(T)}{N(T)} \frac{M(0)N(T)}{N(0)M(T)} | \mathcal{F}_t \right]
\]

\[
= \mathbb{E}_Q \left[ \frac{S(T)}{M(T)} \frac{M(t)}{N(t)} | \mathcal{F}_t \right] = \frac{M(t)}{N(t)} \mathbb{E}_Q \left[ \frac{S(T)}{M(T)} | \mathcal{F}_t \right]
\]

\[
= \frac{M(t)}{N(t)} \frac{S(t)}{M(t)} = \frac{S(t)}{N(t)}
\]

and hence \( \frac{S(t)}{N(t)} \) is a martingale under \( Q^N \). □
Applications of Black-Scholes model

Call on forward - the Black formula
Exchange option - Margrabe formula
Foreign exchange options – Garman-Kohlagen formula

Change of numeraire and martingale measure

What we have seen can be summarized in the following result:

**Theorem (Change of numeraire)**

Let $\mathbb{Q}$ be a martingale measure associated to numeraire $M(t)$, and let $N(t)$ be another numeraire. Then there exists a measure $\mathbb{Q}^N \sim \mathbb{Q}$ such that prices of assets expressed in that numeraire, $S(t)/N(t)$, are martingales under $\mathbb{Q}^N$. This measure can be obtained from $\mathbb{Q}$ as:

$$
\frac{d\mathbb{Q}^N}{d\mathbb{Q}} = \frac{M(0)N(T)}{N(0)M(T)}
$$

and for all random variable $Y - \mathcal{F}_T$ measurable,

$$
\mathbb{E}_{\mathbb{Q}^N}[Y|\mathcal{F}_t] = \frac{M(t)}{N(t)} \mathbb{E}_{\mathbb{Q}} \left[ Y \frac{N(T)}{M(T)} | \mathcal{F}_t \right]
$$

Think about the case $\mathbb{Q} = \text{sirk neutral measure}$, associated to numeraire $\beta_t = e^{rt}$ of the saving account.
Price of the exchange option by change of numeraire technique

1. \[ E_Q \left[ e^{-r(T-t)} S_2(T) \left( \frac{S_1(T)}{S_2(T)} - 1 \right)_+ |\mathcal{F}_t \right] \]

This can be re-written as:

\[ M(t) E_Q \left[ \left( \frac{S_1(T)}{S_2(T)} - 1 \right)_+ \frac{N(T)}{M(T)} |\mathcal{F}_t \right] \]

if we take \( M(t) = \beta_t \) the saving account, and \( N(t) = S_2(t) \).

We will switch to martingale measure \( \mathbb{Q}_2 \) associated to numeraire \( N(t) = S_2(t) \).

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1. cf. Girsanov bi-dimensional
Indeed, the price of the option can be re-written by using this new measure thanks to the thm seen previously:

\[ M(t) E_Q \left( \left( \frac{S_1(T)}{S_2(T)} - 1 \right) + \frac{N(T)}{M(T)} |F_t \right) = N(t) E_{Q_2} \left( \left( \frac{S_1(T)}{S_2(T)} - 1 \right) + |F_t \right) \]

To compute this last expectation, we still miss:

- the RN derivative \( \frac{d}{dQ_2} \), in order to identify \( Q_2 \)
- the dynamics of \( Y(t) = \frac{S_1(t)}{S_2(t)} \) under \( Q_2 \)
Applications of Black-Scholes model

Price of the exchange option by change of numeraire technique

The R-N derivative:

\[
\frac{dQ_2}{dQ} = Z = Z_T = \frac{S_2(T)\beta_0}{S_2(0)\beta_T} = \frac{S_2(T)e^{-rT}}{S_2(0)} = e^{-\frac{\sigma_2^2}{2}T + \sigma_2 W_2(T)}
\]

where \( W_2(t) \) is a standard B.M. under \( Q \).

If we define \( Y(t) = \frac{S_1(t)}{S_2(t)} \), then by applying the Itô lemma to \( Y(t) = f(t, S_1(t), S_2(t)) \), we get:

\[
dY(t) = \frac{dS_1}{S_2} - \frac{S_1}{S_2^2} dS_2 + \frac{1}{2} \frac{S_1}{S_2} \frac{2S_2^2}{S_3^2} \sigma_2^2 dt
\]

\[
= \ldots = Y(t)\left[\sigma_2^2 dt + \sigma_1 dW_1(t) - \sigma_2 dW_2(t)\right]
\]

\[
= Y(t)\left[\sigma_1 dW_1(t) - \sigma_2 d(W_2(t) - \sigma_2 t)\right]
\]
By Girsanov thm applied to $Z_T = e^{-\sigma_2^2 T/2 + \sigma_2 W_T}$, we know that

$$W_2^*(t) = W_2(t) - \sigma_2 t$$

is a standard B.M. under $\mathbb{Q}_2$. Hence under this new measure, the dynamics of $Y(t)$ can be written:

$$dY(t) = Y(t)[\sigma_1 dW_1(t) - \sigma_2 dW_2^*(t)]$$

Moreover, we can see that $W_1(t)$ is still a standard B.M. under $\mathbb{Q}_2$, independent from $W_2(t)^2$.

We recover in particular the fact that $Y$ is a martingale under $\mathbb{Q}_2$ (...).
Price of the exchange option by change of numeraire technique

On the other hand:

\[ d(\ln Y) = \frac{1}{Y} dY - \frac{1}{2} \frac{1}{Y^2} dY \cdot dY = \sigma_1 dW_1(t) - \sigma_2 dW_2^*(t) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)dt \]

and hence:

\[ \frac{Y(T)}{Y(t)} = e^{\sigma_1(W_1(T) - W_1(t)) - \sigma_2(W_2^*(T) - W_2^*(t)) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T - t)} \]

where \( W_1, W_2^* \) are two independent standard M.B. under \( \mathbb{Q}_2 \), which implies:

\[ Y(T)/Y(t) \sim LN(a, b) \]

with

\[ a = -\frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T - t), \quad b = \sqrt{(\sigma_1^2 + \sigma_2^2)(T - t)} \]
Price of the exchange option by change of numeraire technique

In consequence, the price of the exchange option appears as:

$$S_2(t)E_{Q_2}[(Y(T) - 1)^+ | \mathcal{F}_t]$$

with \( \frac{Y(T)}{Y(t)} \sim LN(a, b) \) for \( a \) and \( b \) like in the Black-Scholes model with \( \sigma \) replaced by \( \sigma = \sqrt{(\sigma_1^2 + \sigma_2^2)} \).

We are hence exactly in the case of the Black-Scholes formula with:

\( r = 0, K = 1, \sigma = \sqrt{(\sigma_1^2 + \sigma_2^2)} \)

and we get the Margrabe formula.
Change of numeraire: remark

We can re-use the general results on measure and numeraire change in the case of interest rates models. An equivalent martingale measure that plays an important role will be the forward measure:

In the case of options on zero-coupons (or on options on coupon bonds or on swaps), we will often make the change of numeraire associated to a particular zero-coupon. This is what we will call the forward measure.

*If we take as numeraire the fixed leg of a swap, we will then talk about forward-swap measure.*
Assumptions:

- We suppose that there are 2 currencies: the domestic currency and a foreign currency.
- In each currency, we have a risk free rate:
  - $r_d$: domestic risk free rate
  - $r_f$: foreign risk free rate
- We suppose that the market is perfect.
- The exchange rate between the foreign and the domestic currencies follows a geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where $S(t) = \text{value at instant } t \text{ of one foreign currency unit expressed in the domestic currency.}$
We are interested in the price of a European call option of strike $K$ on that foreign currency:

$K$ represents in practice the maximum exchange rate at which we would accept to buy foreign currencies at a fixed future date. If we want to hedge ourselves against situations where the FX rate becomes superior to $K$, then we buy such a call.

\[ \text{Payoff} = (S(T) - K)^+ \]
Let us consider the following strategy:

- At t, we borrow an amount $S(t)$ at the risk-free rate $r_d$ in the domestic market, and we buy one foreign currency unit.
- At t, we invest this currency unit at the risk-free rate $r_f$ of the foreign market.
- At T, the risk-free investment in the foreign market becomes equal to $e^{r_f(T-t)}$ foreign currency units.
- We then take that money from the foreign bank account and exchange it against domestic currencies, at the FX rate that prevails at $T: S(T)$. We hence receive an amount $S(T)e^{r_f(T-t)}$.
- At T, we also redeem the borrowing at the risk-free rate: we need to pay $S(t)e^{r_d(T-t)}$ domestic currency units.
This strategy does not require any money at the beginning, so it is a “zero-cost” strategy, and its payoff at $T$ is:

$$S(T)e^{r_f(T-t)} - S(t)e^{r_d(T-t)}$$

Hence the price of a financial product delivering the same final payoff must also be zero if we assume that the market is arbitrage free. So if $Q$ is a (the) risk-neutral measure of the domestic market, necessarily:

$$e^{-r_d(T-t)}E_Q[S(T)e^{r_f(T-t)} - S(t)e^{r_d(T-t)} | \mathcal{F}_t] = 0$$

$$\iff E_Q[S(T)e^{(r_f-r_d)(T-t)} | \mathcal{F}_t] = S_t$$

$$\iff E_Q[S(T)e^{-(r_d-r_f)T} | \mathcal{F}_t] = S_t e^{-(r_d-r_f)t}$$

i.e. $S(t)e^{-(r_d-r_f)t}$ is a martingale under $Q$. 
Since $Q$ is equivalent to $P$, we know by Girsanov thm that the dynamics of $S(t)$ under $Q$ is identical to that under $P$ except that the drift can be different. The martingale condition above implies necessarily under $Q$,

$$dS(t) = (r_d - r_f)S(t)dt + \sigma S(t)dW(t)$$

i.e. the same equation as in the Black-Scholes model with dividends, in which the dividend rate is in fact the risk free rate in the foreign currency $r_f$.

Once again, we can see that the martingale measure is unique since the market is complete (...).
If we consider a European call on the exchange rate whose payoff at $T$ is $(S(T) - K)^+$, then its price is given by

$$C = E_Q[e^{-r_d(T-t)}(S(T) - K)^+ | \mathcal{F}_t] = e^{-r_f(T-t)}E_Q[e^{-(r_d-r_f)(T-t)}(S(T) - K)^+ | \mathcal{F}_t]$$

By applying the Black-Scholes formula in which we replace $r$ by $r_d - r_f$, and by multiplying the whole by $e^{-r_f(T-t)}$, we get:

$$C = e^{-r_f(T-t)}S(t)\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2)$$

$$d_1 = \frac{ln(S_t/K) + (r_d - r_f + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t}$$

(Garman-Kohlagen formula, 1983)