Propriétés Qualitatives de Fronts de Réaction-Diffusion

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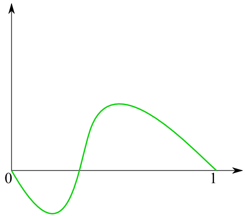
Reaction-diffusion equation

$$\left\{\begin{array}{rcll}\dfrac{\partial u}{\partial t} & = & \Delta u + f(u), & (t,x) \in \mathbb{R} \times \Omega, \\ \nabla u \cdot \nu & = & 0, & (t,x) \in \mathbb{R} \times \partial\Omega,\end{array}\right.$$

Bistable nonlinearity f: $f(0) = f(1) = 0$ and both 0 and 1 are stable

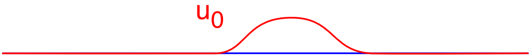


Typical example: cubic nonlinearity $f(s) = s(1-s)(s-\theta)$ with $0<\theta<1$.

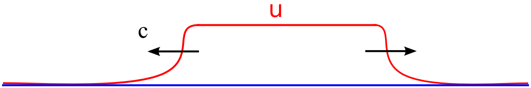
Classical solutions $0 < u(t,x) < 1$

One of the most important aspects: propagation phenomena

Formation of fronts

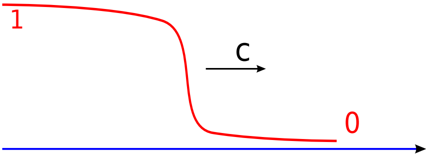


(a) $t = 0$

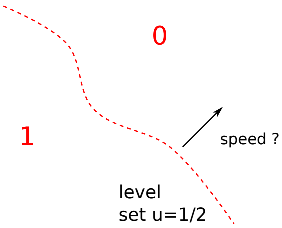


(b) $t >> 1$

A one-dimensional traveling front



More general propagating solutions



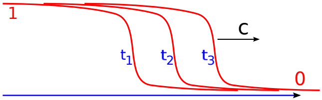
Limiting states 0 and 1

Questions

- Standard traveling fronts ?
- Notions of fronts and speeds ?
- Other fronts in $\mathbb{R}^N$?
- Characterization of fronts with planar level sets in $\mathbb{R}^N$?
- Characterization of the mean speed in $\mathbb{R}^N$?
- Other domains ?

One-dimensional case $\Omega = \mathbb{R}$

$$\boxed{u(t,x) = \phi(x-ct)}$$



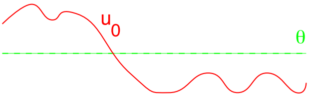
Existence and uniqueness of the speed $c = c_f$ and of the profile $\phi = \phi_f$

$$\phi_f'' + c_f\phi_f' + f(\phi_f) = 0 \text{ in } \mathbb{R}, \quad \phi_f(-\infty) = 1, \quad \phi_f(+\infty) = 0$$

[Aronson, Weinberger], [Fife, McLeod]

The speed c_f has the sign of $\int_0^1 f$.

Cauchy problem with initial condition u_0:



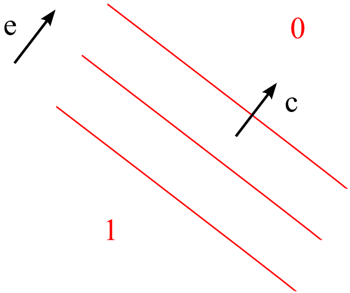
$$u(t,x) - \phi_f(x - c_f t + \xi) \to 0 \text{ unif. in } x \text{ as } t \to +\infty, \text{ with } \xi \in \mathbb{R}$$

Contrast with monostable $f > 0$ on $(0,1)$: continuum of speeds $[c_f^*, +\infty)$

Case of the whole space $\Omega = \mathbb{R}^N$: planar fronts

$$\boxed{u(t,x) = \phi_f(x \cdot e - c_f t)}$$

where e is any unit vector in $\mathbb{R}^N$



Uniqueness of the "planar" speed c_f, uniqueness of the profiles

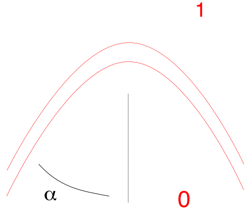
The level sets are parallel hyperplanes, moving with constant speed c_f

Stability: [Levermore,Xin], [Matano,Nara], [Matano,Nara,Taniguchi], [Xin]

Case of the whole space $\Omega = \mathbb{R}^N$: non-planar fronts with $c_f > 0$

- Axisymmetric conical-shaped level sets

$$\boxed{u(t,x) = \phi(|x'|, x_N + ct)}$$



where $x' = (x_1, \cdots, x_{N-1})$ and $0 < \alpha < \pi/2$

$$c = \frac{c_f}{\sin\alpha}$$

Constant speed, invariant profiles in the moving frame

[Fife], [Gui], [Hamel, Monneau, Roquejoffre], [Ninomiya, Taniguchi], [Roquejoffre, Roussier-Michon]

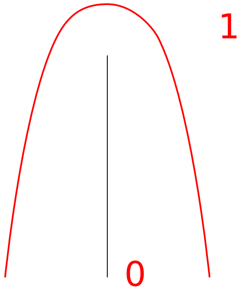
- Non-axisymmetric level sets: pyramidal fronts [Taniguchi]
- Systems with $N = 2$ and $\alpha \simeq \pi/2$: [Haragus, Scheel]

Case of the whole space $\Omega = \mathbb{R}^N$: non-planar fronts with $c_f = 0$

- Axisymmetric fronts

$$u(t,x) = \phi(|x'|, x_N + ct)$$

for all $c > 0$



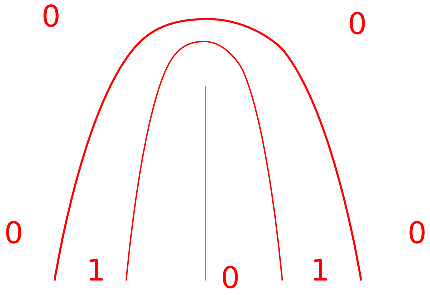
Dimension $N = 2$: the level sets have an exponential shape

Dimension $N \geq 3$: the level sets have a parabolic shape

[Chen, Guo, Hamel, Ninomiya, Roquejoffre]

Case of the whole space $\Omega = \mathbb{R}^N$: non-planar fronts with $c_f = 0$

- Other fronts $u(t,x) = \phi(|x'|, x_N + ct)$ with $c \simeq 0$ and $N \geq 3$



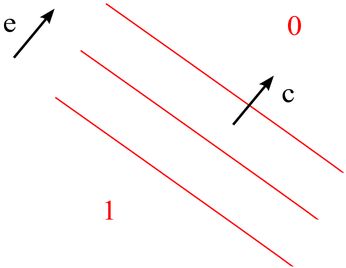
[Del Pino, Kowalczyk, Wei]

- Stationary solutions $u(t,x) = \phi(x)$ "connecting" 0 and 1:
  - x_N-monotone solutions: link with De Giorgi conjecture [Ambrosio, Cabré], [Berestycki, Caffarelli, Nirenberg], [Del Pino, Kowalczyk, Wei], [Ghoussoub, Gui], [Savin]
  - layered solutions with multiple ends [Del Pino, Kowalczyk, Pacard, Wei]
  - saddle-shaped solutions [Alessio, Calamai, Montecchiari], [Cabré], [Cabré, Terra], [Dang, Fife, Peletier]

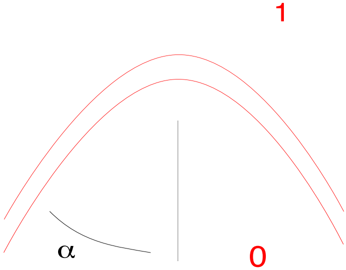
Notions of transition fronts and global mean speed

Many types of traveling fronts

(planar, conical, exponential, parabolic, saddle-shaped stationary, etc)



(c) Planar traveling front



(d) Curved front, $c = c_f \,/\, \sin\alpha$

Observations

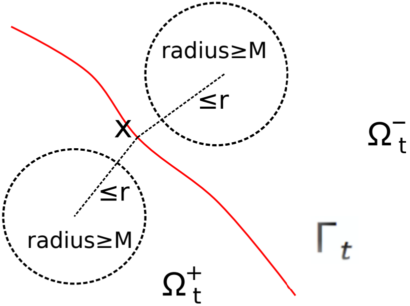
- All these fronts converge to 0 and 1 far away from their (moving or stationary) level sets, uniformly in time.
- The *level sets* of all these fronts move at the mean speed $|c_f|$

Families of open disjoint subsets $\Omega_t^\pm \subset \Omega$ and "interfaces" Γ_t

$$\left\{\begin{array}{l} \partial\Omega_t^- \cap \Omega = \partial\Omega_t^+ \cap \Omega =: \Gamma_t, \quad \Omega_t^- \cup \Gamma_t \cup \Omega_t^+ = \Omega, \\ \sup\left\{d_\Omega(x,\Gamma_t);\ x \in \Omega_t^+\right\} = \sup\left\{d_\Omega(x,\Gamma_t);\ x \in \Omega_t^-\right\} = +\infty \end{array}\right.$$

The interfaces between Ω_t^+ and Ω_t^- are "uniformly thin":

$$\left\{\begin{array}{ll} \sup\left\{d_\Omega(y,\Gamma_t);\ y \in \Omega_t^+,\ d_\Omega(y,x) \le r\right\} \to +\infty & \text{as } r \to +\infty \\ \sup\left\{d_\Omega(y,\Gamma_t);\ y \in \Omega_t^-,\ d_\Omega(y,x) \le r\right\} \to +\infty & \text{unif. in } t \in \mathbb{R} \text{ and } x \in \Gamma_t \end{array}\right.$$



The sets Γ_t are included in a bounded number of (moving) graphs

Dimension $N = 1$: $\Gamma_t = \{x_t^1, \cdots, x_t^p\}$

Notions of transition fronts and global mean speed

Definition [Berestycki, Hamel] (adapted to our equation)

A *transition front* connecting 0 and 1 is a solution u such that there are some sets $\Omega_t^\pm$ and Γ_t with

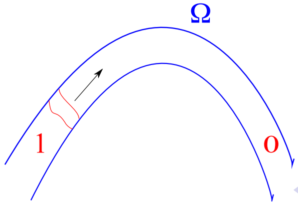
$$u(t,x) \to 0 \text{ (resp. } 1) \text{ as } d_\Omega(x,\Gamma_t) \to +\infty \text{ and } x \in \Omega_t^+ \text{ (resp. } \Omega_t^-)$$

(the transition between 0 and 1 has a uniformly bounded width)

The transition front has a *global mean speed* γ if

$$\frac{d_\Omega(\Gamma_t,\Gamma_s)}{|t-s|} \to \gamma \text{ as } |t-s| \to +\infty$$

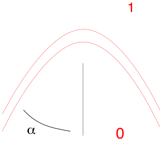
(mean normal speed of the interfaces Γ_t)



The standard traveling fronts in $\mathbb{R}^N$ are transition fronts

Example: conical-shaped front $u(t,x) = \phi(|x'|, x_N + ct)$

$$\Omega_t^+ = \{x_N > \psi(|x'|) - ct\} \text{ with } \phi(r, \psi(r)) = 1/2$$



First intrinsic property (satisfied by any given transition front):

The sets $\Omega_t^\pm$ and Γ_t are not uniquely determined, but the distance between Γ_t and any given level set of u is bounded in time

Second intrinsic property (satisfied by any given transition front):

The global mean speed γ, if any, is uniquely determined, that is it does not depend on the choice of the sets $\Omega_t^\pm$ and Γ_t.

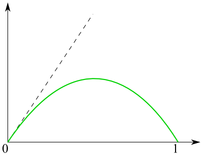
If $c_f > 0$, conical-shaped fronts with $c = c_f / \sin\alpha$: mean speed $\gamma = c_f$

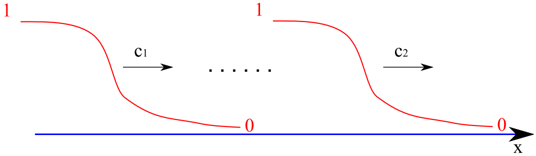
If $c_f = 0$, exponential or parabolic fronts: mean speed $\gamma = 0$

Further monotonicity and classification results in heterogeneous media [Berestycki, Hamel]

For general equations, there are transition fronts without speed !

$$u_t = u_{xx} + f(u) \quad \text{in } \mathbb{R} \quad \text{with KPP } f$$



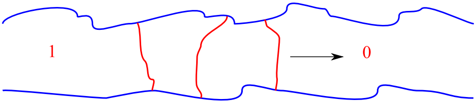


For $c_2 > c_1 \geq 2\sqrt{f'(0)}$, there are transition fronts with speed c_1 when $t \to -\infty$ and with speed c_2 when $t \to +\infty$

[Hamel, Nadirashvili]

These solutions are transition fronts connecting 0 and 1 without any global mean speed

Another definition by H. Matano



Another example : $u_t = u_{xx} + b(x)\,f(u)$

Define $\sigma_\xi b(\cdot) = b(\cdot{+}\xi)$ and assume that $\mathcal{H}{=}\overline{\{\sigma_\xi b\}}$ is compact in $L^\infty(\mathbb{R})$

A definition by H. Matano: u is a generalised front if there exists a continuous mapping $w : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ such that

$$\left\{\begin{array}{l} u(t, x+\xi(t)) = w(\sigma_{\xi(t)}b, x), \\ w(z,s) \to 1 \text{ when } s \to -\infty \text{ unif. w.r.t. } z \in \mathcal{H}, \\ w(z,s) \to 0 \text{ when } s \to +\infty \text{ unif. w.r.t. } z \in \mathcal{H}. \end{array}\right.$$

Then, u is a transition front connecting 0 and 1

Definition by W. Shen in random media

All aforementioned (bistable) traveling fronts $u(t,x) = \phi(x', x_N + ct)$ share a common property: they are invariant in a moving frame (in, say, the direction $-x_N$)

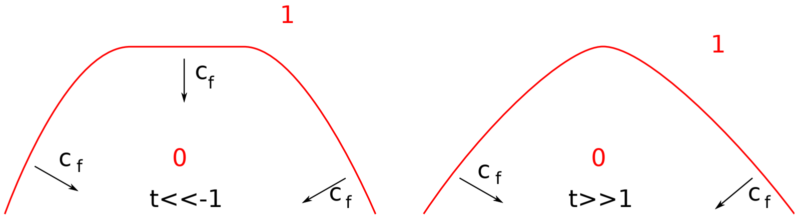
There are other transition fronts !

Theorem

There are transition fronts u connecting 0 and 1 for which

$$u(t,x) \neq \phi\big(R_t(x - x_t)\big)$$

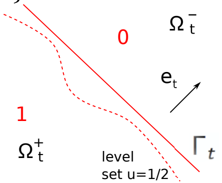
with $\phi : \mathbb{R}^N \to (0,1)$, $x_t \in \mathbb{R}^N$ and some rotations R_t.



Characterization of planar fronts in $\mathbb{R}^N$

Almost-planar transitions fronts: for every $t \in \mathbb{R}$, Γ_t can be chosen as

$$\Gamma_t = \{x \cdot e_t = \xi_t\} \quad \text{for some } e_t \in \mathbb{S}^{N-1} \text{ and } \xi_t \in \mathbb{R}$$



Proposition

In $\mathbb{R}^N$, almost-planar transition fronts u are planar fronts

$$u(t,x) = \phi_f(x \cdot e - c_f t + \xi) \;\; \textit{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N$$

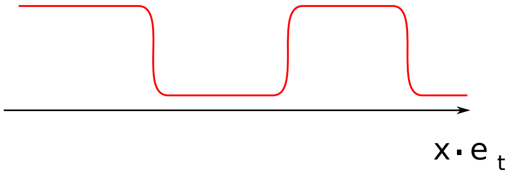
No abusive generalization: robustness of the definitions in classical cases

Almost-planar fronts have planar level sets moving in a constant direction with constant speed

In particular, in dimension $N = 1$, if all Γ_t are singletons or more generally if the diameters of Γ_t are bounded, then $u(t,x) = \phi_f(\pm x - c_f t + \xi)$

Case of a finite number of almost-planar interfaces

$$\Gamma_t = \bigcup_{1 \le i \le p} \{x \cdot e_t = \xi_t^i\}$$



Theorem

If $c_f \neq 0$, then u is a planar front $u(t,x) = \phi_f(x \cdot e - c_f t + \xi)$

Two or more non-trivial oscillations are not possible if $c_f \neq 0$

The condition $c_f \neq 0$ is necessary. If $c_f = 0$, more oscillations are possible: in dimension $N = 1$, there are transition fronts with

$$\Gamma_t = \{\xi_t^1, 0, \xi_t^3\} \text{ with } \xi_t^1 < 0 < \xi_t^3 \text{ for } t < 0, \quad \Gamma_t = \{0\} \text{ for } t \ge 0$$

and $\xi_t^3 = -\xi_t^1$ behave logarithmically as $t \to -\infty$ [Eckmann, Rougemont] [Ei]

Global mean speed of transition fronts in $\mathbb{R}^N$

Observation:

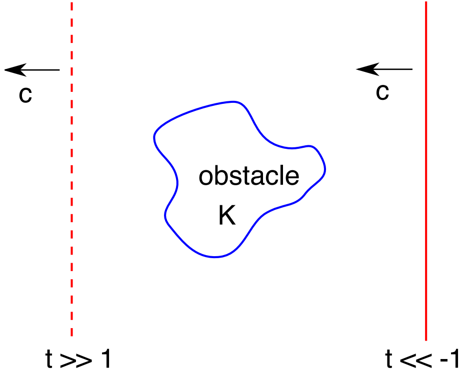
All aforementioned transition fronts (planar, conical-shaped, pyramidal, exponential, parabolic, stationary, not invariant in any moving frame...) share a common property: they have global mean speed equal to $|c_f|$

The existence and the uniqueness of the global mean speed hold whatever the shape of the level sets of the fronts may be and whatever the value of the planar speed c_f may be:

Theorem

In $\mathbb{R}^N$, any transition front connecting 0 and 1 has a global mean speed γ and this speed is equal to $|c_f|$

Example of a non-classical situation: front around an obstacle K



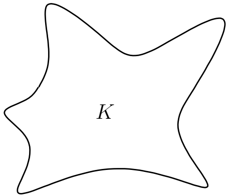
$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \Omega = \mathbb{R}^N \backslash K, \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega = \partial K \end{cases}$$

Simulations by Lionel Roques (INRA, Avignon)

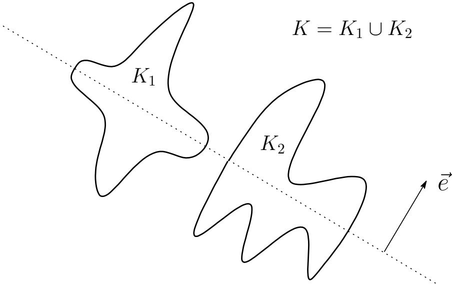
Theorem [Berestycki, Hamel, Matano]

Assume $c_f > 0$. If the obstacle K is star-shaped or strongly directionally convex, there exists an almost-planar transition front u connecting 0 and 1 such that

$$u(t,x) - \phi_f(x_1 + c_f t) \to 0 \text{ as } t \to \pm\infty \text{ unif. in } x, \text{ and as } |x| \to \infty \text{ unif. in } t$$



(g) Star-shaped obstacle



(h) Directionally convex obstacle

General case of a compact obstacle

Theorem [Berestycki, Hamel, Matano]

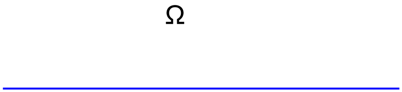
Assume $c_f > 0$. There exists an almost-planar transition front u connecting 0 and $p(x)$ where $0 < p(x) \leq 1$ is a stationary solution such that $p(x) \to 1$ as $|x| \to +\infty$, and

$$\Gamma_t = \{x_1 = -c_f t\}$$

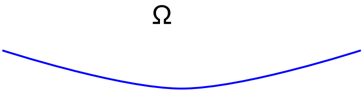
Theorem

For general obstacle K, any transition front connecting 0 and such a stationary solution p has a global mean speed γ and this speed is equal to $|c_f|$

Other domains



(i) Half-space

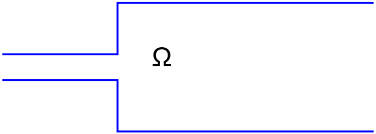


(j) Epigraph

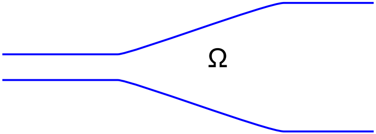
Theorem

Any transition front connecting 0 *and* 1 *has a global mean speed* γ *and this speed is equal to* $|c_f|$

Not true in general !



(k) Blocking



(l) Propagation

Blocking if too large variation [Chapuisat, Grenier]

Propagation if slow variation [Berestycki, Bouhours, Chapuisat]