Γ-convergence of gradient flows and applications

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Outline

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Extension of the scheme to the metric space setting

Illustrations

- Ginzburg-Landau vortices
- Allen-Cahn equation
- Cahn-Hilliard equation

The question

Given a family of energy functionals $(E_\varepsilon)_{\varepsilon>0}$ which Γ-converges to a functional F, when can we say that the solutions to the gradient flows

$$\begin{cases} \partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon) \\ u_\varepsilon(0) = u_\varepsilon^0 \end{cases}$$

converge as $\varepsilon \to 0$ to a solution to the limiting gradient flow

$$\begin{cases} \partial_t u = -\nabla F(u) \\ u(0) = u^0 \end{cases} \qquad ??$$

(Question raised by De Giorgi).

- it's not true in general, so additional conditions are needed
- in infinite dimensions it requires to specify in what sense the gradient is taken
- for an example where it is true think of Allen-Cahn equation (gradient flow of Modica-Mortola energy) converging to mean curvature flow (gradient flow of the perimeter functional).

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Abstract scheme in the Hilbert space setting (Sandier-S)

- Assume $E_\varepsilon \in C^1$ and $\nabla_{X_\varepsilon} E_\varepsilon$ denotes the gradient of E_ε with respect to a Hilbert space structure X_ε, defined by

$$dE_\varepsilon(u)\cdot\phi = \langle \nabla_{X_\varepsilon} E_\varepsilon(u), \phi\rangle_{X_\varepsilon}.$$

- Assume $F \in C^1$ is defined over a (finite-dimensional for simplicity) Hilbert space Y.
- Assume E_ε Γ-converges to F in the sense that if $u_\varepsilon \overset{S}{\rightharpoonup} u$ then

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq F(u).$$

We need two extra conditions:

(C1) (Lower bound on the velocity) If $\forall t \in (0,T)$, $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ then for every $s \in (0,T)$

$$\liminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon(t)\|_{X_\varepsilon}^2\, dt \geq \int_0^s \|\partial_t u(t)\|_Y^2\, dt.$$

(C2) (Lower bound for the slopes) If $u_\varepsilon \overset{S}{\rightharpoonup} u$ then

$$\liminf_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \geq \|\nabla_Y F(u)\|_Y^2.$$

Theorem (Sandier-S)

Let E_ε *and* F *be as above with (C1) and (C2) holding. Let then* $u_\varepsilon(t)$ *be a family of solutions to*

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \quad \text{on } [0,T)$$

with $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ *for all* $t \in [0,T)$*, such that*
$\forall t \in [0,T) \qquad E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2 \, ds$. *Assume also that*

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon(0)) = F(u(0)),$$

then u *is in* $H^1((0,T),Y)$ *(in particular continuous in time) and is a solution to*

$$\partial_t u = -\nabla_Y F(u) \quad \text{on } (0,T). \tag{1}$$

Moreover

$$\forall t \in (0,T) \quad E_\varepsilon(u_\varepsilon(t)) = F(u(t)) + o(1)$$
$$\|\partial_t u_\varepsilon\|_{X_\varepsilon} \to \|\partial_t u\|_Y \quad \text{in } L^2(0,T)$$
$$\|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \to \|\nabla_Y F\|_Y \quad \text{in } L^2(0,T).$$

The proof

$$
\begin{aligned}
E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) &= \int_0^t \|\partial_t u_\varepsilon(s)\|^2_{X_\varepsilon}\, ds \\
&= \frac{1}{2}\int_0^t \|\partial_t u_\varepsilon(s)\|^2_{X_\varepsilon} + \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(s))\|^2_{X_\varepsilon}\, ds \\
&\geq \frac{1}{2}\int_0^t \|\partial_t u(s)\|^2_Y + \|\nabla_Y F(u(s))\|^2_Y + o(1) \\
&\geq -\int_0^t \langle \partial_t u, \nabla_Y F(u)\rangle_Y\, ds + o(1) \qquad (2) \\
&= F(u(0)) - F(u(t)) + o(1).
\end{aligned}
$$

But since $u_\varepsilon(t)$ is a well-prepared solution, we have $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(0)) = F(u(0))$. Combining the above relations, we deduce

$$\liminf_{\varepsilon\to 0} \left(-E_\varepsilon(u_\varepsilon(t))\right) \geq -F(u(t))$$

But by Γ-convergence of E_ε to F the converse holds, so we must have equality everywhere, hence in (2) and for a.e. t

$$\partial_t u = -\nabla_Y F(u)$$

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&= \frac{1}{2}\int_0^t \|\partial_t u_\varepsilon(s)\|^2_{X_\varepsilon} + \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(s))\|^2_{X_\varepsilon}\, ds \\
&\geq \frac{1}{2}\int_0^t \|\partial_t u(s)\|^2_{Y} + \|\nabla_{Y} F(u(s))\|^2_{Y} + o(1) \\
&\geq -\int_0^t \langle \partial_t u, \nabla_Y F(u)\rangle_Y\, ds + o(1) \qquad (2) \\
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Comments

- The conditions (C1) and (C2) provide sufficient extra conditions for Γ-convergence of gradient flows. They correspond to a kind of C^1 notion of Γ-convergence: they allow to compare the C^1 structures of the energy landscapes of E_ε and F. Of course the spaces where these flows live being different, they cannot be compared, however the sizes of the slopes or derivatives can be compared, and this suffices.
- One does not really need to prove (C1)-(C2) for all u_ε but only for families of solutions to the gradient flow.
- The condition (C2) immediately implies that critical points of E_ε converge to critical points of F.
- A similar C^2 notion of Γ-convergence was introduced (S), providing sufficient conditions (based on the C^2 structure of the energy landscape) to ensure that stable critical points of E_ε converge to stable critical points of F
- The two conditions allow at the same time to "guess" for which structure Y the limiting equation is the gradient flow of F.

Extension of the scheme to the metric space setting

Notion of gradient flow on metric spaces (= curves of minimal slope, or minimizing movements of De Giorgi) is more general and can be better suited for applications. Framework by Ambrosio-Gigli-Savaré.
This definition of gradient flows is based on the remark that if u is a solution of the gradient flow

$$\partial_t u = -\nabla \phi(u)$$

u is characterized by the relation

$$\partial_t(\phi(u)) \leq -\frac{1}{2}\left(|\partial_t u|^2 + |\nabla \phi|^2\right) \tag{3}$$

Indeed the relation $\frac{1}{2}\left(|\partial_t u|^2 + |\nabla \phi|^2\right) \geq -\langle \partial_t u, \nabla \phi \rangle$ holds in all cases, and there is equality if and only if $\partial_t u = -\nabla \phi(u)$. Moreover (3) has a meaning even on metric spaces provided one gives a definition for $|\partial_t u|$ and for $|\nabla \phi|$.

Definition (Metric derivative)

Let v *be an absolutely continuous curve on* (a, b)*. Then the limit*

$$|v'|(t) := \lim_{s \to t} \frac{d(v(s), v(t))}{|s - t|}$$

exists for a.e. $t \in (a, b)$ *and is called the metric derivative of* v*.*

Definition (Strong upper gradient)

A function $g : \mathcal{S} \to [0, +\infty]$ *is a strong upper gradient for* ϕ *if for every absolutely continuous curve* v *on* (a, b)

$$|\phi(v(t)) - \phi(v(s))| \leq \int_s^t g(v(r))|v'|(r)\, dr \qquad a < s \leq t < b.$$

Definition (Curve of maximal slope)

v *absolutely continuous is a curve of maximal slope for the functional* ϕ *with respect to its strong upper gradient* g *if*

$$(\phi \circ u)'(t) \leq -\frac{1}{2}\left(|u'|^2(t) + g^2(u(t))\right) \quad a.e.\ t.$$

Theorem

Let Φ_ε *and* Φ *be functionals defined on metric spaces* $(\mathcal{S}_\varepsilon, d_\varepsilon)$ *and* $(\mathcal{S}, d)$ *respectively, and such that* $\Gamma - \liminf \Phi_\varepsilon \geq \Phi$. *Let* g_ε *and* g *be strong upper gradients of* Φ_ε *and* Φ *respectively. Assume in addition the relations*

1. (Lower bound on the metric derivatives) If $u_\varepsilon(t) \xrightharpoonup{S} u(t)$ *for* $s \in (0, T)$ *then*

$$\forall s \in [0, T) \quad \liminf_{\varepsilon \to 0} \int_0^s |u'_\varepsilon|^2_{d_\varepsilon}(t)\, dt \geq \int_0^s |u'|_d(t)\, dt. \tag{4}$$

2. (Lower bound on the upper gradients) If $u_\varepsilon \xrightharpoonup{S} u$ *then*

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon) \geq g(u). \tag{5}$$

Let then $u_\varepsilon(t)$ *be a curve of maximal slope on* $(0, T)$ *for* Φ_ε *with respect to* g_ε, *such that* $u_\varepsilon(t) \xrightharpoonup{S} u(t)$, *which is well-prepared in the sense that*

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u_\varepsilon(0)) = \Phi(u(0)).$$

Then u *is a curve of maximal slope with respect to* g.

The dynamics of Ginzburg-Landau vortices

Ginzburg-Landau energy functional without magnetic field

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{\varepsilon^2}, \tag{6}$$

Ω is a two-dimensional smooth bounded domain (simply connected), ε is a (small) material constant, and $u : \Omega \to \mathbb{C}$. Vortices = zeroes of u with winding number. If $F_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$ then configurations have bounded number of vortices and one may extract limiting vortices $a_i \in \Omega$ with degrees $d_i \in \mathbb{Z}$.
Γ-convergence result: there exists a limiting energy $F = W$: if

$$\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle \rightharpoonup 2\pi \sum_{i=1}^{n} d_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\Omega)$$

then

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - \pi \sum_{i=1}^{n} |d_i||\log\varepsilon| \geq W(\mathbf{a}, \mathbf{d}).$$

Bethuel-Brezis-Hélein

$$\frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2). \qquad (7)$$

is the gradient flow of F_ε for the structure $\|\cdot\|_{X_\varepsilon} = \frac{1}{\sqrt{|\log \varepsilon|}}\|\cdot\|_{L^2(\Omega)}$.

The limiting space of configurations $(\mathbf{a}, \mathbf{d})$ with d_i fixed to ± 1, can be identified to Ω^n, and we equip it with the rescaled Euclidean structure on $(\mathbb{R}^2)^n$ given by $\|\cdot\|_Y^2 = \frac{1}{\pi}|\cdot|^2_{\mathbb{R}^{2n}}$.

(C1)–(C2) are here:

1. if $u_\varepsilon(t) \overset{S}{\rightharpoonup} (\mathbf{a}(t), \mathbf{d})$ with $d_i = \pm 1$,

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_0^s \|\partial_t u_\varepsilon\|^2_{L^2(\Omega)}(t)\, dt \geq \frac{1}{\pi} \int_0^s |\partial_t a_i|^2\, dt$$

2. if $u_\varepsilon \overset{S}{\rightharpoonup} (\mathbf{a}, \mathbf{d})$ with $d_i = \pm 1$,

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_\Omega |\log \varepsilon|^2 \left| \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \right|^2 \geq \|\nabla_Y W(\mathbf{a}, \mathbf{d})\|_Y^2 = \pi |\nabla W(\mathbf{a}, \mathbf{d})|^2.$$

These two conditions were already proved to be true (Lin, Jerrard) along configurations such that $F_\varepsilon(u_\varepsilon) \leq \pi n |\log \varepsilon| + C$. We obtain a new variational proof of the known result (Lin, Jerrard-Soner)

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Theorem (Sandier-S)

Let u_ε *be a family of solutions to (7) with either Dirichlet or Neumann boundary condition, such that* $\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle(0) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i^0}$ *as* $\varepsilon \to 0$ *where* a_i^0 *are distinct points in* Ω *and* $d_i = \pm 1$*. Assume also* $u_\varepsilon(0)$ *is well-prepared in the sense*

$$F_\varepsilon(u_\varepsilon(0)) = \pi n|\log \varepsilon| + W(\mathbf{a}^0, \mathbf{d}) + o(1) \quad \text{as } \varepsilon \to 0.$$

Then there exists a time $T_* > 0$ *such that* $\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle(t) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i(t)}$ *for all* $t \in [0, T_*)$ *and*

$$\frac{da_i}{dt} = -\frac{1}{\pi}\partial_i W(\mathbf{a}(t), \mathbf{d}), \quad a_i(0) = a_i^0$$

with the d_i*'s remaining constant.* T_* *is the minimum of the collision time and the exit time (in the Neumann case) under this law.*

Application by Kurzke for dynamical law of boundary vortices in thin micromagnetic films.

Case of the Allen-Cahn equation

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) \tag{8}$$

where u is real valued and $f(u) = 2u(1-u^2)$. Gradient flow of

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}$$

for the structure $\|\cdot\|_{X_\varepsilon} = \sqrt{\varepsilon}\|\cdot\|_{L^2(\Omega)}$, with $W(u) = \frac{1}{2}(1-u^2)^2$.
E_ε Γ-converges to the perimeter functional

$$F(\Gamma) = 2\sigma\mathcal{H}^{N-1}(\Gamma)$$

$\sigma = \int_{-1}^{1}\sqrt{W(s)/2}\,ds = \frac{2}{3}$ Solutions to (8) converge to mean curvature flow

$$\partial_t \Gamma = H$$

(weak sense given by Brakke - varifold setting) where Γ is the limiting interface. Gradient flow of F for the (formal) structure $\|\cdot\|^2_{Y_\Gamma} = 2\sigma\|\cdot\|^2_{L^2_\Gamma}$.

(C1)–(C2) are in this setting:

1. if $u_\varepsilon(t) \xrightarrow{S} \Gamma(t)$,

$$\liminf_{\varepsilon\to 0} \int_0^s \varepsilon \|\partial_t u_\varepsilon\|^2_{L^2(\Omega)}(t)\, dt \geq 2\sigma \int_0^s \int_{\Gamma(t)} |\partial_t \Gamma|^2\, dt. \tag{9}$$

2. if $u_\varepsilon \xrightarrow{S} \Gamma$

$$\liminf_{\varepsilon\to 0} \int_\Omega \varepsilon \left| \Delta u_\varepsilon + \frac{1}{\varepsilon^2} f(u_\varepsilon) \right|^2 \geq 2\sigma \int_\Gamma |H|^2. \tag{10}$$

- First relation proved by Mugnai-Röger, in an appropriate weak sense (L^2 flows for rectifiable integer measures).
- Second relation proved by Röger-Schätzle in sense of varifolds. Corresponds to a De Giorgi conjecture (Γ-convergence of $\int_\Omega \varepsilon \left|\Delta u_\varepsilon + \frac{1}{\varepsilon^2} f(u_\varepsilon)\right|^2$ to the Wilmore energy $\int |H|^2$).
- With these two results at hand, another proof of convergence of AC to MC formally (and probably rigorously) follows.

Application to Cahn-Hilliard (by Nam Le)

Cahn-Hilliard equation

$$\begin{cases} \partial_t u_\varepsilon = -\Delta v_\varepsilon & \text{in } \Omega \\ v_\varepsilon = \varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} f(u_\varepsilon) & \text{in } \Omega \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) \end{cases} \tag{11}$$

Convergence to Mullins-Sekerka motion in the sense that v_ε converges to v solving the following free-boundary problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \backslash \Gamma(t) \\ v = \sigma H & \text{on } \Gamma(t) \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \partial_t \Gamma = \frac{1}{2} \left[\frac{\partial v}{\partial \nu}\right]_{\Gamma(t)} \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{12}$$

$\left[\frac{\partial v}{\partial \nu}\right]_{\Gamma(t)}$ denotes the jump of the normal derivative of v accross the hypersurface $\Gamma(t)$.

Cahn-Hilliard is the H^{-1} gradient flow of the Modica-Mortola energy E_ε so $X_\varepsilon = H^{-1}(\Omega)$. The limiting energy is $F(\Gamma) = 2\sigma\mathcal{H}^{N-1}(\Gamma)$.
For every $\tilde{f} \in H^1(\Omega)$ such that

$$\begin{cases} \Delta \tilde{f} = 0 & \text{in } \Omega\backslash\Gamma \\ \tilde{f} = f & \text{on } \Gamma \\ \frac{\partial \tilde{f}}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

we set

$$\|f\|_{H^{1/2}(\Gamma)} = \|\nabla \tilde{f}\|_{L^2(\Omega)}$$

By duality we define $H^{-1/2}(\Gamma)$ (assuming Γ is regular). The structure Y is then taken to be $\|\cdot\|_{Y_\Gamma} = 2\|\cdot\|_{H^{-1/2}(\Gamma)}$.

(C1)–(C2) are here :

1. if $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} \Gamma(t)$ on $[0, T)$ then for all $0 \leq s < T$

$$\int_0^s \|\partial_t u_\varepsilon\|^2_{H^{-1}(\Omega)}(t)\, dt \geq 4 \int_0^s \|\partial_t \Gamma(t)\|^2_{H^{-1/2}}\, ds \tag{13}$$

2. if $u_\varepsilon \stackrel{S}{\rightharpoonup} \Gamma$, then

$$\liminf_{\varepsilon \to 0} \int_\Omega \left| \nabla \left(\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) \right) \right|^2 \geq \sigma^2 \|H\|^2_{H^{1/2}(\Gamma)}. \tag{14}$$

(13) is easy to prove. Le proved that (14) holds if Γ is regular enough and $N \leq 3$, and that there is Γ-convergence of $\|\nabla \left(\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) \right)\|^2_{L^2(\Omega)}$ to $\sigma^2 \|\kappa\|^2_{H^{1/2}(\Gamma)}$. This is a higher derivative analogue to the De Giorgi conjecture. With this, Le obtains a theorem of convergence of well-prepared solutions to Cahn-Hilliard to classical solution of Mullins-Sekerka under regularity assumption on the limiting interface, until self-collision or exit time.

Other work in the same line by Bellettini-Bertini-Mariani-Novaga

Conclusions and perspectives

- This scheme of Γ-convergence of gradient flows is a simple tool to understand via a general principle why solutions to gradient flows converge to their limiting counterpart.
- It works formally or rigorously in some nontrivial examples. It would be interesting to find more.
- Proving whether the extra two conditions hold potentially leads to many open questions.
- The framework can be extended to metric spaces. It would be interesting to find examples where this setting is useful.
- It is also well adapted to study the Γ-convergence of action functionals

$$A_\varepsilon(u) = \int_0^T \|\partial_t u + \nabla_{X_\varepsilon} E_\varepsilon(u)\|_{X_\varepsilon}^2 \, dt$$

with $u(0) = u_\varepsilon^0$ and $u(T) = u_\varepsilon^T$. Cf. work of Kohn-Otto-Reznikoff-Vanden-Eijnden, Kohn-Reznikoff-Tonegawa.

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