Entire solutions for competition-diffusion systems and optimal partitions

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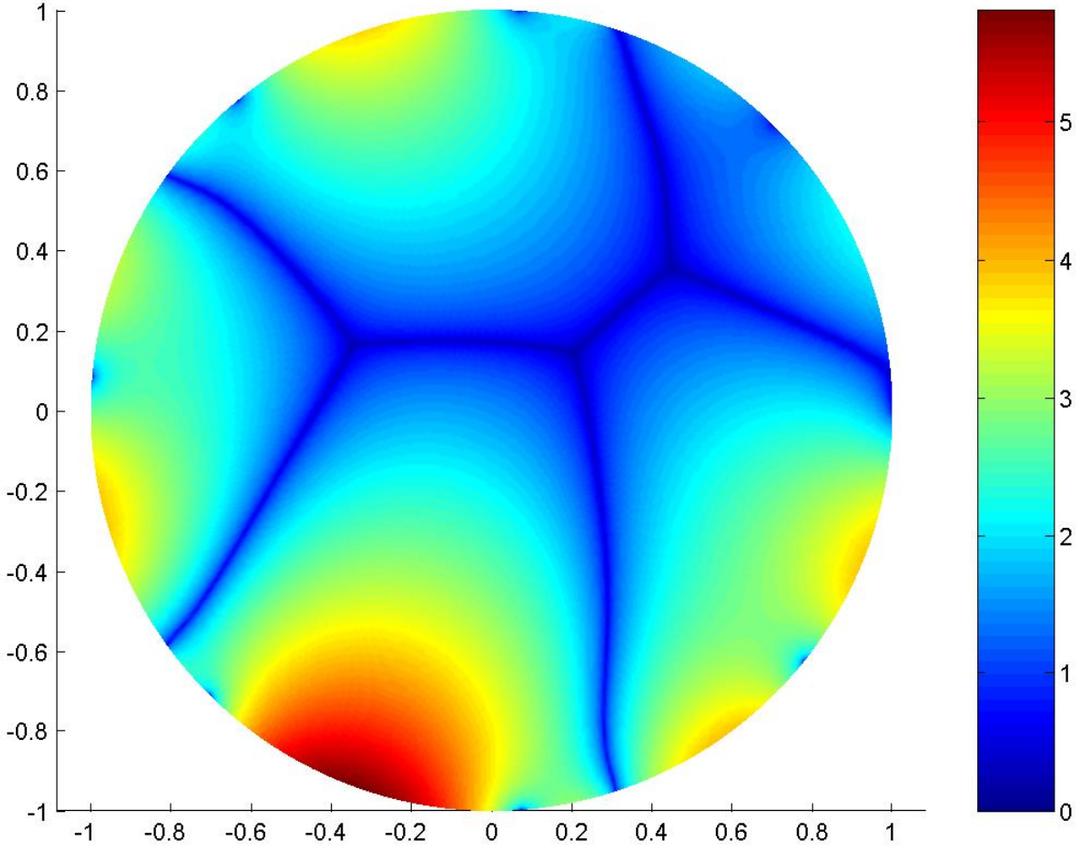
1 Competition diffusion systems with Lotka-Volterra interactions

➪ With symmetric interspecific competition rates $\beta_{i,j} = \beta_{j,i}$ large:

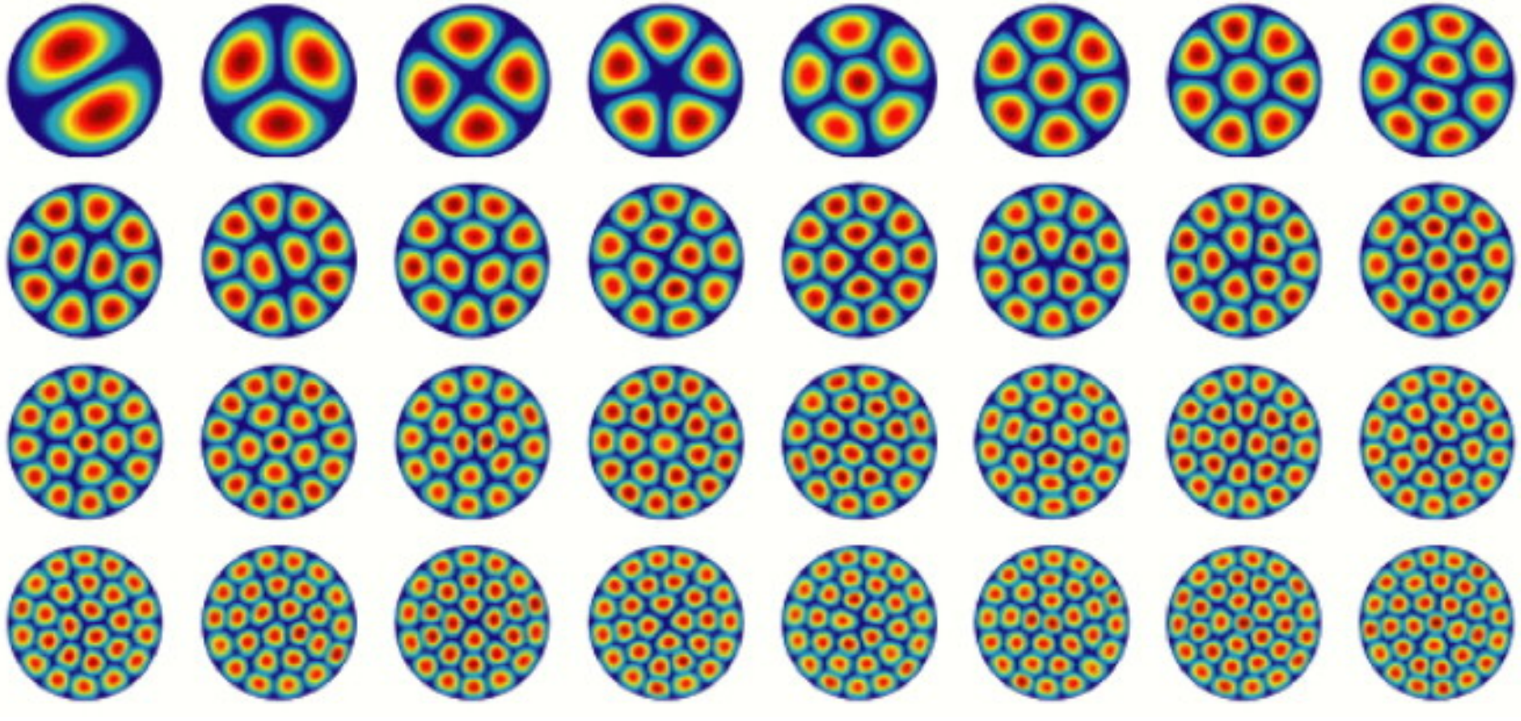
$$-\Delta u_i = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$



Energy minimizing configurations of Bose–Einstein condensates in multiple spin–states with repulsive interaction potentials

$$\mathcal{E}(\psi_1, \cdots, \psi_h) = \int \sum_i^h \tfrac{1}{2}|\nabla\psi_i|^2 + F_i(|\psi_i|^2) + \sum_{j\neq i}^h \beta_{i,j}|\psi_i|^2|\psi_j|^2 \text{ in } \Omega,$$

$$\int |\psi_i|^2 = m_i\ , \qquad i = 1, \ldots, h$$



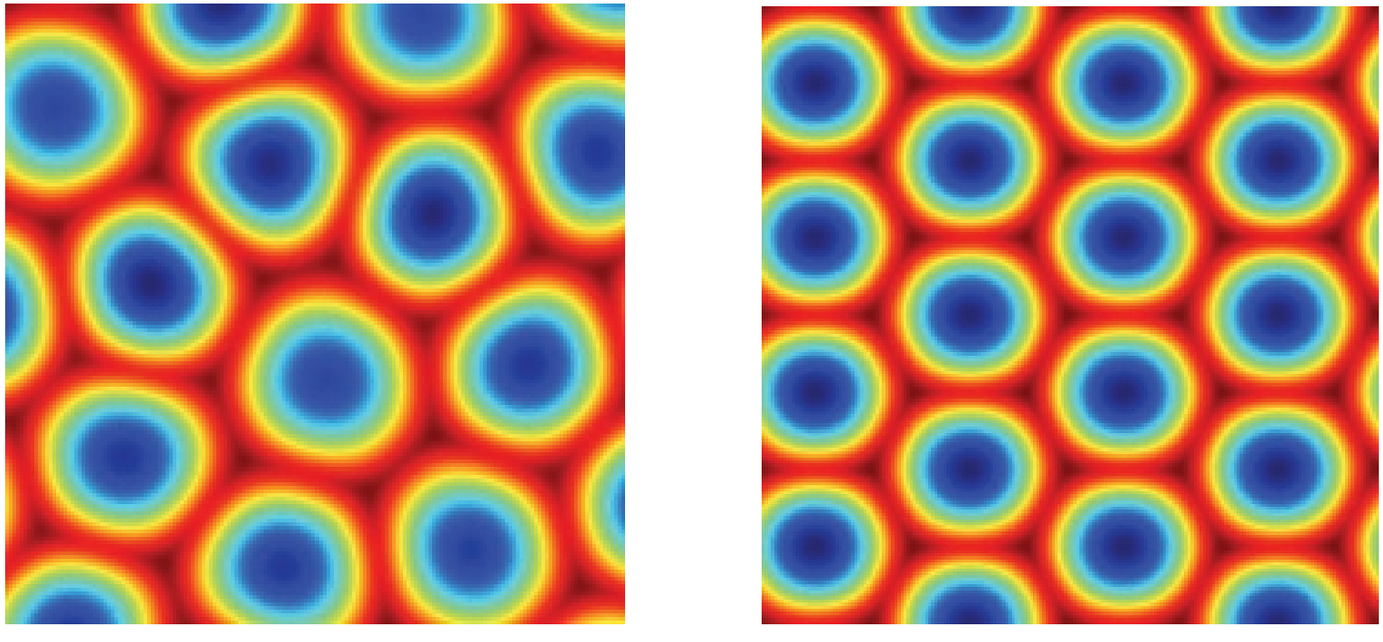
- Defocusing: S.M. Chang, C.S. Lin, T.C. Lin, and W.W. Lin, *Phys. D* **196**, 341–361 (2004)
- Focusing: Conti M., Terracini S., Verzini G., *J. Functional Analysis*, 198 (2003) 160-196

Optimal partition problems for Dirichlet eigenvalues

$$\min\left\{\sum_{i=1}^{h}\lambda_1^p(\omega_i)\ :\ (\omega_1,\cdots,\omega_k)\in\mathfrak{B}_h(\Omega)\right\}$$

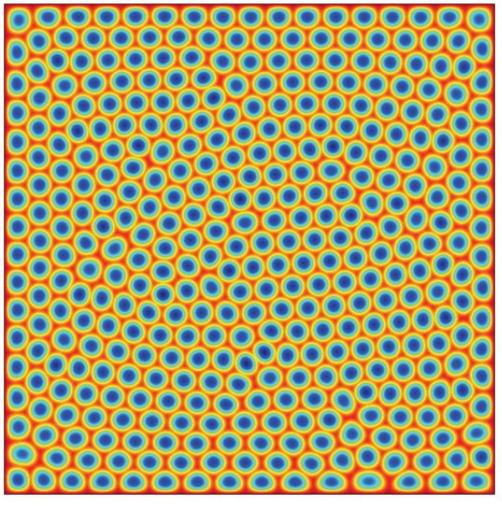
where

$$\mathfrak{B}_h=\{(\omega_1,\ldots,\omega_h):\ \omega_i \text{ open},\ |\omega_i\cap\omega_j|=0 \text{ for } i\neq j \text{ and } \cup_i\omega_i\subseteq\Omega\}\,.$$

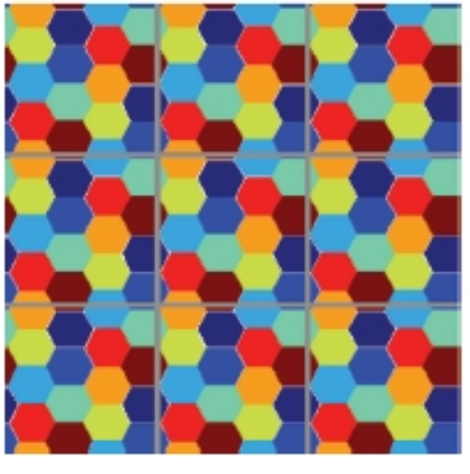


B. Bourdin, D. Bucur, and . Oudet, Optimal Partitions for Eigenvalues, *SIAM J. Sci. Comput.* 31, 2009/10 pp. 4100-4114

With more and more nodal components:



With higher eigenvalues:



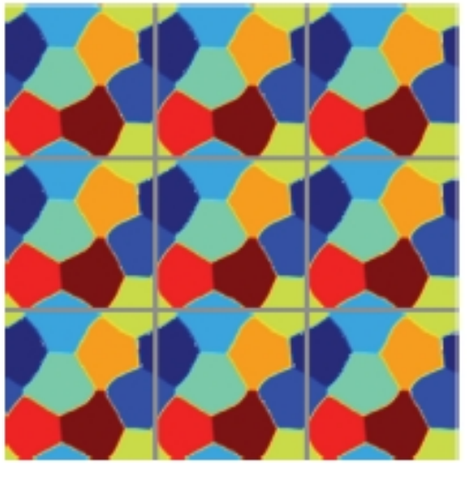


FIG. 3.8. *Optimal partitions of the sum of the second (left) and third (right) eigenvalues of the Dirichlet–Laplacian for $n = 8$ cells. The periodicity is highlighted by repeating the unit cell 9 times on a two-dimensional lattice.*

4 Uniform bounds in Hölder spaces

Consider strongly competing systems with either of the Lotka-Volterra or gradient type interactions

$$-\Delta u_i(x) = f(u_i) - \beta u_i(x) \sum_{j \neq i} u_j^2(x)$$

Theorem 1 (Conti-T-Verzini '06, Noris-Tavares-T-Verzini '10) *Let U_β be a family of H^1–bounded solutions. There exists $L_\alpha > 0$ such that*

$$\sup_{x,y \in \Omega} \frac{u_{i,\beta}(x) - u_{i,\beta}(y)}{|x-y|^\alpha} < L_\alpha$$

for all $i = 1, \dots, h$ and for all $\beta > 0$. Moreover, the limiting profiles are Lipschitz continuous.

Ingredients of the proof. Assuming the contrary, after scaling and performing the blow–up analysis, after a lot of work we would arrive to the existence of an entire solution of

$$-\Delta u_i(x) = -u_i(x) \sum_{j \neq i} u_j^2(x)$$

satisfying a global bound in Hölder norm, in contradiction with suitable perturbed monitonicity formulæ underlying the rules of the minimal spacial growth for competition systems.

5 A Liouville type theorem

The proof of Theorem 1 rests upon a blow–up argument and the following non existence theorem:

Theorem 2 (Conti-T-Verzini '06, Noris-Tavares-T-Verzini '10) *Let $k \geq 2$, $a_{ij} > 0$, and let $U = (u_1, \dots, u_k)$ be a solution of*

$$\begin{cases} -\Delta u_i(x) = -u_i(x) \displaystyle\sum_{j \neq i} a_{ij} u_j^2(x) & x \in \mathbb{R}^N \\ u_i(x) \geq 0 & x \in \mathbb{R}^N \end{cases}$$

for every i. Assume that, for some $\alpha \in (0,1)$, there holds

$$\max_{i=1,\dots,k} \sup_{x \in \mathbb{R}^N} \frac{|u_i(x)|}{1 + |x|^\alpha} < \infty.$$

Then all components (but possibly one) vanish.

6 Classification of entire solutions

We understand that a key step of the theory is the classification of the solutions of the system, with respect to their spatial growth. To this aim, we have at our disposal two major tools

- The perturbed Alt-Caffarelli-Friedman monotonicity formula (for Lotka-Volterra and all other types of interaction): let

$$\Phi(r) := \prod_i \frac{1}{r^{\alpha(k)-\varepsilon}} \int_{B_r(0)} K(x) \left(|\nabla u_i|^2 + \sum_{j\neq i} a_{ij} u_i^2 u_j^2 \right) ,$$

  where $K(x) = |x|^{2-N}$ for $|x| \geq 1$. The exponent $\alpha(k) \geq 2$ depends on a spectral optimal partition problem on the sphere. Then Φ is increasing.

- The Almgren monotonicity formula (only for gradient systems): let

$$N(r) := \frac{r \int_{B_r(0)} \sum_i |\nabla u_i|^2 + \sum_{j\neq i} u_i^2 u_j^2}{\int_{\partial B_r(0)} \sum_i u_i^2}$$

  Then N is increasing. Moreover,

$$\lim_{r\to+\infty} N(r) \geq 1$$

7 Two components in one space dimension

In order to understand the interplay of two neighboring components, we have to deal with the solutions to the system

$$\begin{cases} \Delta u = uv^2 \\ \Delta v = vu^2 \,, \\ u, v > 0 \quad \text{in} \quad \mathbb{R}^N \end{cases}$$

Of course, there are one variable solutions (depending on the energy $h > 0$):

$$\begin{cases} u'' = uv^2 \\ v'' = vu^2 \,, \\ |u'|^2 + |v'|^2 - u^2v^2 = h \\ u(x) = v(-x) \,, \qquad u, v > 0 \quad \text{in} \quad \mathbb{R}. \end{cases}$$

All these solution have the lowest possible growth

$$\lim_{r \to +\infty} N(r) = 1$$

Theorem 3 (Berestycki-Lin-Wei-Zhao '09, Berestycki-T-Wang-Wei '12)) *Up to translations and reflections, there is only one one-parameter family of solutions to*

$$\begin{cases} u'' = uv^2, \\ v'' = vu^2, \\ u, v > 0 \quad \text{in } \mathbb{R} \end{cases}$$

For this family we have $v(t^* - t) = u(t^* + t)$. *In addition they are all stable.*

Proof

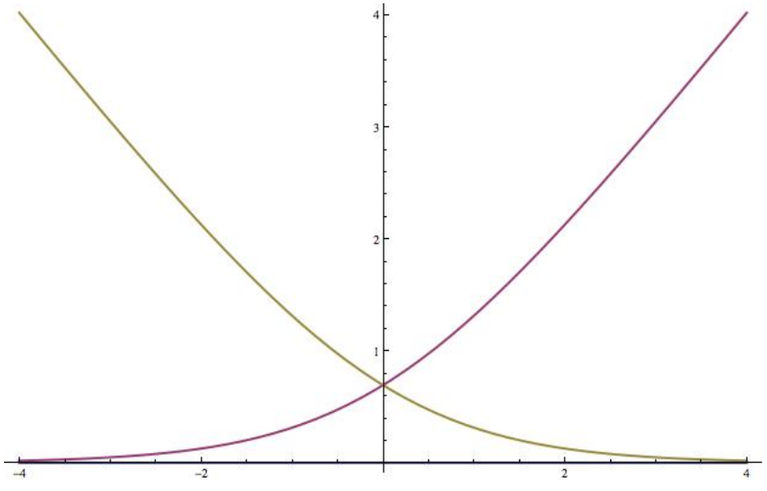
- Normalize the energy to one;
- observe that u and v are positive convex functions;
- assume, for instance, that u is increasing and v is decreasing. Moreover u decays superexponentially at $-\infty$, v at $+\infty$
- as $x \to +\infty$, $|u'(x) - 1|$ decays exponentially. This implies the existence of a positive constant A such that

$$|u(x) - x^+| + |v(x) - x^-| \leq A.$$

In addition, the limits

$$\lim_{x\to+\infty}(u(x)-x^+),\ \lim_{x\to-\infty}(v(x)-x^-)$$

exist. So there are asymptotes at infinity for both u and v



Now assume (u_1, v_1) and (u_2, v_2) are two entire solutions of the problem with the same unitary energy. For $t > 0$, denote

$$u_{1,t}(x) := u_1(x+t), v_{1,t}(x) := v_1(x+t).$$

Step 1. Thanks of the decay estimates, there exists $\bar{t}$ such that for all $t \geq \bar{t}$,

$$u_{1,t}(x) \geq u_2(x), v_{1,t}(x) \leq v_2(x) \ \text{ in } \ \mathbb{R}. \qquad \text{(Sl)}$$

Step 2. Decrease the t to an optimal value when (Sl) holds

$$\boxed{t_0 = \inf\{t' \,|\, \text{such that } (Sl) \text{ holds for all } t \geq t'\}.}$$

Note that there holds

$$\begin{cases} -(u_{1,t_0} - u_2)'' + v_{1,t_0}^2(u_{1,t_0} - u_2) \geq 0 \\ -(v_2 - v_{1,t_0})'' + u_{1,t_0}^2(v_2 - v_{1,t_0}) \leq 0 \end{cases}$$

Hence, by the strong maximum principle,

$$\boxed{\begin{array}{ll} \text{either} & u_{1,t_0}(x) \equiv u_2(x), v_{1,t_0}(x) \equiv v_2(x) \text{ in } \mathbb{R} \\ \text{or} & u_{1,t_0}(x) > u_2(x), v_{1,t_0}(x) < v_2(x) \text{ in } \mathbb{R} \end{array}}$$

Step 3. We show, using again some form of the maximum principle, that the second case can not occur. This is done in two steps:

- if the asymptotes do not coincide, then we work essentially on a bounded domain and we can use the usual continuation argument to obtain a contradiction;
- if some of the asymptotes coincides, the extremality of t_0 implies that, for $t_0 - \varepsilon$ there is a negative part of either $u_{1,t_0-\varepsilon} - u_2$ or $-v_{1,t_0-\varepsilon} + v_2$. This part must be placed close to infinity. Using as a weight function $g(x) = \log(x^2 + 1/2)$ we reach again a contradiction.

Theorem 4 (Berestycki-Lin-Wei-Zhao '09(+BTWW'12)) *Let (u, v) a solution to*

$$\Delta u = uv^2\,, \quad \Delta v = vu^2\,, \quad u, v > 0 \quad in \quad \mathbb{R}^2$$

such that

$$u(x) + v(x) \leq C(1 + |x|).$$

and which is monotone in one direction. Then (u, v) is one dimensional , (i.e., there exists $a \in \mathbb{R}^2, |a| = 1, b \in \mathbb{R}$ such that $(u, v) = (u_0(a \cdot x - b), v_0(a \cdot x - b))$ where (u_0, v_0) is the one-dimensional solution).

A quite standard argument shows that monotone $\Longrightarrow$ stable. Recall that a *stable* solution (u, v) is such that the linearization is weakly positive definite. That is, it satisfies

$$\int_{\mathbb{R}^n} [\nabla\varphi|^2 + |\nabla\psi|^2 + v^2\varphi^2 + u^2\psi^2 + 4uv\varphi\psi] \geq 0, \qquad \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^n).$$

11 Planar stable solutions and a theorem by Kelei Wang

Theorem 5 (Berestycki-T-Wang-Wei '12) *Let (u, v) be a stable solution to the system in $\mathbb{R}^2$ of*

$$\Delta u = uv^2\,, \quad \Delta v = vu^2\,, \quad u, v > 0 \quad in \quad \mathbb{R}^2$$

such that

$$u(x) + v(x) \leq C(1 + |x|).$$

Then (u, v) is one-dimensional, (i.e., there exists $a \in \mathbb{R}^2, |a| = 1, b \in \mathbb{R}$ such that $(u, v) = (u_0(a \cdot x - b), v_0(a \cdot x - b))$ where (u_0, v_0) is the one-dimensional solution)

Theorem 6 (Kelei Wang, preprint 2012) *In any space dimensions, let (u, v) a solution having at most linear growth and which is minimal (in the sense of Morse): the energy is minimized with respect to compact support variations. Then (u, v) is one dimensional .*

12 Solutions with polynomial growth

A natural question is whether or not all solutions to the system should satisfy the linear growth. For every integer d, there are solutions to the system with polynomial growth $|x|^d$. To describe the behavior at infinity, let us consider the harmonic polynomial Φ of degree d as

$$\Phi := \mathrm{Re}(z^d).$$

Note that Φ has some dihedral symmetry; indeed, let us take its d nodal lines $L_1, \cdots, L_d$ and denote the corresponding reflection with respect to these lines as $T_1, \cdots, T_d$: then there holds $\Phi(T_i z) = -\Phi(z)$.

Theorem 7 (B-T-W-W'12) *For each positive integer $d \geq 1$, there exists a solution (u, v) to the system, satisfying*

(1) $u - v > 0$ *in* $\{\Phi > 0\}$ *and* $u - v < 0$ *in* $\{\Phi < 0\}$;

(2) $u \geq \Phi^+$ *and* $v \geq \Phi^-$;

(3) $\forall i = 1, \cdots, d$, $u(T_i z) = v(z)$;

(4) $\forall r > 0$, *the Almgren frequency function satisfies*

$$N(r) := \frac{r \int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2} \leq d = \lim_{r \to +\infty} N(r);$$

13 Asymptotics at infinity

We consider the blow-down sequence

$$(u_R(x), v_R(x)) := (\frac{1}{L(R)} u(Rx) \frac{1}{L(R)} v(Rx)),$$

where $u(0) = v(0)$ and $L(R)$ is chosen so that

$$\int_{\partial B_1(0)} u_R^2 + v_R^2 = 1.$$

We have the following

Theorem 8 *Let (u, v) be a solution of the system such that $d := \lim_{r \to +\infty} N(r) < +\infty$. As $R \to \infty$, (u_R, v_R) defined above (up to a subsequence) converges to (Ψ^+, Ψ^-) uniformly on any compact set of $\mathbb{R}^N$. Here Ψ is a homogeneous harmonic polynomial of degree d. If $d = 1$ then (u, v) is asymptotically linear at infinity.*

In particular this applies to the solutions found by Theorem 6 to give

Corollary 1 *Let (u, v) be a solution of the system given by Theorem 6: then*

$$(u_R(x), v_R(x)) := (\frac{1}{R^d} u(Rx) \frac{1}{R^d} v(Rx))$$

converges uniformly on compact subsets of $\mathbb{R}^2$ to a multiple of (Φ^+, Φ^-), where $\Phi := Re(z^d)$.

Theorem 9 (Berestycki-T-Wang-Wei '12) *There exists a positive solution to the system*

$$\begin{cases} \Delta u_i = u_i \displaystyle\sum_{j\neq i, j=1}^{k} u_j^2, & in \ \ \mathbb{C}, i = 1, \dots, k \end{cases}$$

having the following symmetries (here $\overline{z}$ is the complex conjugate of z)

$$\begin{aligned} u_i(z) &= u_i(G^h z), && on\ \mathbb{C}\,, i = 1, \dots, k \\ u_i(z) &= u_{i+1}(Gz), && on\ \mathbb{C}\,, i = 1, \dots, k \\ u_{k+1}(z) &= u_1(z), && on\ \mathbb{C} \\ u_{k+2-i}(z) &= u_i(\overline{z}), && on\ \mathbb{C}\,, i = 1, \dots, k \end{aligned}$$

such that

$$\lim_{r\to\infty} \frac{1}{r^{1+2d}} \int_{\partial B_r(0)} \sum_1^k u_i^2 = b \in (0, +\infty)\ ;$$

and

$$\lim_{r\to\infty} \frac{r \int_{B_r(0)} \sum_1^k |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2}{\int_{\partial B_r(0)} \sum_1^k u_i^2} = d\ .$$

15 Limiting profiles

As the interspecific competition rate $\beta = \min_{ij} \beta_{ij}$ tends to infinity we find a vector $U = (u_1, \cdots, u_h)$ of functions

- having mutually disjoint supports: $u_i \cdot u_j \equiv 0$ in Ω for $i \neq j$,
- satisfying

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever } u_i \neq 0\ , \quad i = 1, \ldots, h,$$

Questions:

- Uniqueness vs multiplicity
- Extremality conditions at the interface of the nodal components
- Regularity

  of the minimizers
  of the interfaces

16 Segregated critical configuration

Let Ω be an open bounded subset of $\mathbb{R}^N$, with $N \geq 2$. Let $U = (u_1, \dots, u_h) \in (H^1(\Omega))^h$ be a vector of (real, complex, vector-valued)

- nontrivial Lipschitz functions in Ω,
- having mutually disjoint supports: $u_i \cdot u_j \equiv 0$ in Ω for $i \neq j$,
- satisfying

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever } u_i \neq 0\ , \quad i = 1, \dots, h,$$

where $f_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are C^1 functions such that $f_i(x, s) = O(s)$ when $s \to 0$, uniformly in x.

Our main interest is the study of the regularity of the nodal set of the segregated configurations $U = (u_1, \dots, u_h)$:

$$\Gamma_U = \{x \in \Omega : \ U(x) = 0\}$$

Obviously, without other conditions, there is no reason at all why the nodal set should be regular. We must add some information on the interaction between the components at the interface of their supports.

A weak reflection law

Theorem 10 (Tavares-T, 2010) *Let us define, for every $x_0 \in \Omega$ and $r \in (0, dist(x_0, \partial\Omega))$ the energy*

$$\tilde{E}(r) = \tilde{E}(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2 \, ,$$

then, $\tilde{E}(x_0, U, \cdot)$ is an absolutely continuous function of r, and we assume that it satisfies the following differential equation

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_i f_i(x, u_i) \langle \nabla u_i, x - x_0 \rangle .$$

Then, there exists a set $\Sigma_U \subseteq \Gamma_U$ the regular part, relatively open in Γ_U, such that

- $\mathcal{H}_{dim}(\Gamma_U \setminus \Sigma_U) \leq N - 2$, *and if $N = 2$ then actually $\Gamma_U \setminus \Sigma_U$ is a locally finite set;*
- *Σ_U is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$). Furthermore for every $x_0 \in \Sigma_U$*

$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface. Furthermore, if $N = 2$ then Σ_U consists in a locally finite collection of curves meeting with equal angles at singular points.

$$\frac{d}{dr}\,\frac{1}{r^{N-2}}\int_{B_r(x_0)}|\nabla U|^2=\frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}(\partial_\nu U)^2\,d\sigma+\frac{2}{r^{N-1}}\int_{B_r(x_0)}\sum_i f_i(x,u_i)\langle\nabla u_i,x-x_0\rangle. \quad \text{(WRL)}$$

- It is easy to check that equation (WRL) always holds for balls lying entirely inside one of the component supports, as a consequence of the elliptic equation. Hence, for our class systems, it represents the only interaction between the different components u_i through the common boundary of their supports;
- (WRL) is satisfied by the nodal components of solutions to a single semilinear elliptic equation of the form $-\Delta u = f(u)$.
- equation (WRL) can be seen as a weak form of a reflection property through the interfaces. Consider the following example: take two linear functions on complementary half-spaces:

$$u_1(x)=a_1x_1^+ \qquad u_2(x)=a_2x_1^- \ .$$

Then

$$\text{(WRL)} \qquad \Longleftrightarrow \qquad |a_1|=|a_2| \ .$$

More in general, when we have two components with a smooth interface between the supports, then

$$\text{(WRL)} \quad \Longleftrightarrow \quad \lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \ .$$

(WRL) as an extremality condition

Although this hypothesis may look weird and may seem hard to check in applications, it occurs naturally in many situations where the vector U appears as a limit configuration in problems of spatial segregation.

- It has to be noted indeed that a form of (WRL) always holds for solutions of systems of interacting semilinear equations and that it persists under strong H^1 limits.
- In addition, (WRL) holds for vector functions U minimizing Lagrangian functional associated with the system.
- It is fullfilled also for strong limits to competition–diffusion systems, both those possessing a variational structure and those with Lotka-Volterra type interaction.
- Our theorem extends also to sign changing, complex and vector valued functions u_i. Lipschitz continuity can be weakened into Hölder continuity for every $\alpha \in (0, 1)]$.

19 More remarks

This was brought to my attention by Kelei Wang. Assume U minimizes a Lagrangian energy with a pointwise constraint of the type $U(x) \in \Sigma$, for almost every $x \in \Omega$. Let $Y \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^N)$. Then, differentiation of the energy with respect to ε with $U(x) \mapsto U_\varepsilon(x) = U(x + \varepsilon Y(x))$ yields the well known identity

$$\int_\Omega \left\{ dY(x)\nabla U(x) \cdot \nabla U(x) - \mathrm{div} Y(x) \left[\frac{1}{2} |\nabla U(x)|^2 - F(U(x)) \right] \right\} dx = 0 \ , \forall Y \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^N) \ .$$

By localizing to a regular $\omega \subset \Omega$ this implies

$$\int_\omega \left\{ dY(x)\nabla U(x) \cdot \nabla U(x) - \mathrm{div} Y(x) \left[\frac{1}{2} |\nabla U(x)|^2 - F(U(x)) \right] \right\} dx$$
$$= \int_{\partial\omega} \left\{ Y(x) \cdot \nabla U(x) \nu(x) \cdot \nabla U(x) - \nu(x) \cdot Y(x) \left[\frac{1}{2} |\nabla U(x)|^2 - F(U(x)) \right] \right\} d\sigma \ ,$$
$$\forall \text{ smooth } \omega \text{ and } \forall Y \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^N) \ . \quad (*)$$

Next

$$(*) \quad + \quad \begin{pmatrix} Y(x) = x - x_0 \\ \omega = B_r(x_0) \end{pmatrix} \quad \Longrightarrow \quad (WRL)$$

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Asymptotic limits of a system of Gross-Pitaevskii equations

Consider the following system of nonlinear Schrödinger equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \omega_i u_i^3 - \beta u_i \sum_{j\neq i} \beta_{ij} u_j^2 \\ u_i \in H_0^1(\Omega),\ u_i > 0 \text{ in } \Omega. \end{cases} \qquad i = 1, \dots, h,$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. Such type of systems arises in the theory of Bose-Einstein condensation in multiple spin states. Here we consider $\beta_{ij} = \beta_{ji} \neq 0$ (which gives a variational structure to the problem) and take $\lambda_i, \omega_i \in \mathbb{R}$ and $\beta \in (0, +\infty)$ large. The existence of solutions for β large is still an open problem for some choices of λ_i, ω_i.

One of the many interesting questions about these systems is the asymptotic study of its solutions as $\beta \to +\infty$ (which represents an increasing of the interspecies scattering length) and study of the regularity of the limiting profiles. From the uniform Hölder bounds theorem we know that:

- $C^{0,\alpha}$– bounds (for all $0 < \alpha < 1$) for any given L^∞–bounded family of solutions $U_\beta = (u_{1,\beta}, \dots, u_{h,\beta})$ of the system;
- the possible limit configurations $U = \lim_{\beta\to+\infty} U_\beta$ are Lipschitz continuous.

As a byproduct, we have

Theorem 11 *Let U be a limit as $\beta \to +\infty$ of a family $\{U_\beta\}$ of L^∞–bounded solutions of the system. Then the conclusion of Theorem 7 holds.*

All the required assumptions are satisfied for such limiting profiles, with $f_i(x,s) = f_i(s) = \omega_i s^3 - \lambda_i s$, except for the weak reflection law. The procedure to verify it is the following: defining an approximated energy associated with system - which has a variational structure-,

$$E_\beta(r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U_\beta|^2 - \langle F(U_\beta), U_\beta \rangle \right) + \int_{B_r(x_0)} 2\beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2$$

by a direct calculation it holds

$$\begin{aligned} E'_\beta(r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U_\beta)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_i f_i(u_{i,\beta}) \langle \nabla u_{i,\beta}, x - x_0 \rangle + \\ + \frac{1}{r^{N-1}} \int_{B_r(x_0)} (N-2) \langle F(U_\beta), U_\beta \rangle - \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \langle F(U_\beta), U_\beta \rangle \, d\sigma + \\ + \frac{4-N}{r^{N-1}} \int_{B_r(x_0)} \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 + \int_{\partial B_r(x_0)} \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 \, d\sigma. \end{aligned}$$

We know the following facts:

- there holds strong convergence $U_\beta \to U$ in $H^1 \cap C^{0,\alpha}(\Omega)$ for every $0 < \alpha < 1$,
- and $\int_\Omega \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 \to 0$.

Hence, as $\beta \to +\infty$, we prove that U satisfies the weak reflection law.

23 Lotka-Volterra competitive interactions with symmetric competition rates

Consider the following Lotka-Volterra model for the competition between h different species.

$$\begin{cases} -\Delta u_i = f_i(u_i) - \beta u_i \sum_{j\neq i} a_{i,j} u_j \text{ in } \Omega, \\ u_i \geq 0 \text{ in } \Omega, \quad u_i = \varphi_i \text{ on } \partial\Omega. \end{cases}$$

with $\Omega \subset \mathbb{R}^N$ a smooth bounded domain and φ_i positive $W^{1,\infty}(\partial\Omega)$–functions with disjoint supports. We focus on the asymptotic study of solutions as $\beta \to +\infty$. It is not difficult to show that all the possible H^1–limits U of a given sequence of solutions $\{U_\beta\}_{\beta>0}$ (as $\beta \to +\infty$) belong to the class

$$\mathcal{S}(\Omega) = \Big\{ (u_1, \dots, u_h) \in \big(H^1(\Omega)\big)^h : \ u_i \geq 0 \text{ in } \Omega, \ u_i \cdot u_j = 0 \text{ if } i \neq j \text{ and } -\Delta u_i \leq f_i(u_i), \\ -\Delta(u_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} u_j) \geq f_i(x, u_i(x)) - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} f_j(x, u_j) \Big\}.$$

Theorem 12 *Let $U \in \mathcal{S}$, then if $a_{ij} = a_{ji}$, $\forall i, j$ the conclusion of Theorem 7 holds.*

Regularity of interfaces in optimal partition problems related to the first eigenvalue

Next we consider some optimal partition problems involving eigenvalues. For any integer $h \geq 0$, we define the set of h–partitions of Ω as

$$\mathfrak{B}_h = \{(\omega_1, \dots, \omega_h) : \ \omega_i \text{ measurable }, \ |\omega_i \cap \omega_j| = 0 \text{ for } i \neq j \text{ and } \cup_i \omega_i \subseteq \Omega\} .$$

Consider the following optimization problems: for any positive real number $p \geq 1$,

$$\mathfrak{L}_{h,p} := \inf_{\mathfrak{B}_h} \left(\frac{1}{h} \sum_{i=1}^{h} (\lambda_1(\omega_i))^p \right)^{1/p} ,$$

and, for $p = +\infty$ we find the limiting problem

$$\mathfrak{L}_h := \inf_{\mathfrak{B}_h} \max_{i=1,\dots,h} (\lambda_1(\omega_i)),$$

where $\lambda_1(\omega)$ denotes the first eigenvalue of $-\Delta$ in $H_0^1(\omega)$ in a generalized sense. We refer to the papers Conti, Verzini, T. and Helffer, Hoffmann-Ostenhof, T., for a more detailed description of these problems.

Our theorem applies to suitable multiples of the eigenfunctions associated with the optimal partition. More precisely, we proved that

- let $p \in [1, +\infty)$ and let $(\omega_1, \dots, \omega_h) \in \mathfrak{B}_h$ be any minimal partition associated with $\mathfrak{L}_{h,p}$ and let $(\phi_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\omega_i))_i$. Then there exist $a_i > 0$ such that the functions $u_i = a_i\phi_i$ verify in Ω, for every $i = 1, \dots, h$, the differential inequalities (in the distributional sense):

$$-\Delta u_i \le \lambda_1(\omega_i)u_i \quad \text{and} \quad -\Delta(u_i - \textstyle\sum_{j\neq i} u_j) \ge \lambda_1(\omega_i)u_i - \sum_{j\neq i} \lambda_1(\omega_i)u_j;$$

and:

- let $(\tilde\omega_1, \dots, \tilde\omega_h) \in \mathfrak{B}_h$ be any minimal partition associated with $\mathfrak{L}_h$ and let $(\tilde\phi_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\tilde\omega_i))_i$. Then there exist $a_i \ge 0$, not all vanishing, such that the functions $\tilde u_i = a_i\tilde\phi_i$ verify in Ω, for every $i = 1, \dots, h$, the differential inequalities (in the distributional sense):

$$-\Delta \tilde u_i \le \mathfrak{L}_h \tilde u_i \quad \text{and} \quad -\Delta(\tilde u_i - \textstyle\sum_{j\neq i} \tilde u_j) \ge \mathfrak{L}_h(\tilde u_i - \sum_{j\neq i} \tilde u_j).$$

In particular the functions $\tilde U = (\tilde u_1, \dots, \tilde u_h)$ and $U = (u_1, \dots, u_h)$ belong to $\mathcal{S}(\Omega)$. As consequence, we have the following result:

Theorem 13 *Let $(\omega_1, \dots, \omega_h) \in \mathfrak{B}_h$ be any minimal partition and let Γ be the union of the interfaces; then the conclusion of Theorem 7 holds.*

25 Extremality conditions for partitions involving higher eigenvalues

We would like to attack the optimal partition problem for higher eigenvalues ($k \geq 2$):

$$\mathfrak{L} = \min \frac{1}{h} \left(\sum_{i=1}^{h} \lambda_k(\omega_i) \right) .$$

➪ What are the extremality conditions? Hadamard domain variation requires simple eigenvalues. Introduce the penalized functional:

$$\mathcal{E}_\beta(u_1, \cdots, u_h) = \int_\Omega \sum_i |\nabla u_i|^2 + \beta \sum_{i \neq j} |u_i|^2 |u_j|^2$$

with constraints

$$\int_\Omega |u_i|^2 = 1 \quad \forall\, i = 1, \cdots, h \, .$$

As $\beta + \infty$, critical points of $\mathcal{E}_\beta$ converge to pairs of segregated eigenfunctions.

➪ Problem: How to define an appropriate critical level for the penalized functional?

Existence of the minimal partition has been proved by Bucur–Buttazzo.

Spectral partitions

➪ Consider an eigenfunction, and the associated nodal partition. how can we detect whether it possesses any minimization property?

We easily understand tha the most appropriate function of the eigenvalue to be considered is:

$$\mathfrak{L}_h := \inf_{\mathfrak{B}_h} \max_{i=1,\dots,h} (\lambda_1(\omega_i)),$$

A straightforward consequence of the Fisher–Courant theorem is that

$$\lambda_k \leq \mathfrak{L}_k \ .$$

For one dimensional problems we always have equality (Sturm oscillation principle) for every k. On the other hand, in more space dimensions, the k-th eigenfunction may possess less than k nodal domains.

Given k, we denote by L_k the smallest eigenvalue (if any) whose eigenspace contains an eigenfunction with k nodal domains ($L_k = +\infty$ if no such an eigenfunction exists).

In general, an easy consequence of the Courant nodal theorem for connected domains is that

$$\lambda_k \leq L_k \ .$$

So we have the inequalities

$$\lambda_k \leq \mathfrak{L}_k \leq L_k \ , \qquad \forall k.$$

➪ Can we characterize the equality cases?

Theorem 14 (Helffer, Hoffman-Ostenhof, T, 2009–2010) *Suppose $\Omega \subset \mathbb{R}^N$ regular. If either $\lambda_k = \mathfrak{L}_k$ or $\mathfrak{L}_k = L_k$ then*

$$\lambda_k = \mathfrak{L}_k = L_k \ .$$

In addition, one can find in the eigenspace associated to λ_k an eigenfunction u_k having extactly k nodal domains.

➪ The k–th eigenfunction has k nodal domains (i.e. is sharp with respect to the Courant nodal Theorem) if and only if the associated nodal k–partition is optimal.

As a consequence, every time we know (for instance for the symmetries of the problem) that the second eigenvalue is degenerate, then the minimal spectral 3-partition has necessarily a nontrivial clustering point.

27 The case of the sphere

We consider the Laplace-Beltrami operator on the two-sphere.

Conjecture 1 (Bishop 1992) *The minimal* 3*-partition for* $\frac{1}{3}(\sum_{i=1}^{3}\lambda_1(D_i))$ *corresponds to the* $\mathbf{Y}$*-partition, whose boundary is given by the intersection of* $\mathbb{S}^2$ *with the three half-planes defined respectively by* $\phi = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$

The conjecture can be restated as

$$\boxed{\mathfrak{L}_{3,1}(\mathbb{S}^2) = \frac{15}{4}}$$

and also

$$\boxed{\mathfrak{L}_{3,p}(\mathbb{S}^2) = \mathfrak{L}_{3}(\mathbb{S}^2) = \frac{15}{4}\ , \forall p}$$

Bishop's Conjecture was motivated by the analysis of the properties of harmonic functions in conic sets. A reference paper in this context is that by Friedland-Hayman. It is proved there that the optimal two-partition is achieved by the two half spheres.

Theorem 15 (Helffer, Hoffman-Ostenhof, T) *Any minimal spectral 3-partition of* $\mathbb{S}^2$ *is (up to a rotation) obtained by the* **Y***-partition. Hence*

$$\boxed{\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\,.}$$

Consider a homogeneous function in $\mathbb{R}^3$ of the form

$$\boxed{u(x) = r^\alpha g(\theta, \phi)}$$

which is harmonic outside its nodal set: $-\Delta u = 0$, $u > 0$ and such that the nodal set divides the sphere in three parts, then

$$\boxed{\alpha(\alpha+1) \geq \mathfrak{L}_3(\mathbb{S}^2)\,.}$$

Hence our theorem implies that $\alpha \geq 3/2$.

Ideas of the proof:

- first, minimal partitions on $\mathbb{S}^2$ in three parts exist and share the same properties as for planar domains: regularity and equal angle meeting property. Hence the nodal set is a finite union of arcs.
- because the second eigenvalue of the Laplace-Beltrami operator is singular (has nontrivial multiplicity), then the minimal 3-partition cannot be a nodal partition.
- use Euler's formula and deduce that the nodal line s of a minimal 3-partitions consists exactly two points x_1 and x_2 and three arcs joining these two points.
- use Borsuk (or Ljusternik-Schnirelman) theorem to prove that the nodal set contains a pair of antipodal points.
- the next point is that any minimal 3-partition which contains two antipodal points in its boundary can be lifted to a symmetric 6-partition on the double covering $\mathbb{S}^2_{\mathcal{C}}$.
- finally, the last point is to show that on the double covering a minimal symmetric 6-partition is necessarily the lifting on the double covering of the $\mathbf{Y}$-partition.
  - use the knowledge of the spectrum of the Laplace-Beltrami on the double covering and classify all odd and even spectrum.
  - use again the characterization of the eigenvalues whose nodal partition is minimal: this holds if and only the minimal eigenvalue whose nodal partition has k nodal domain is the minimal one,

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