Mathematical study of the stability of suspended bridges : focus on the fluid-structure interactions

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L'étude des ponts suspendus n'aurait pas été possible sans les progrès que l'analyse mathématique a faits dans ces derniers temps, et sans les institutions au moyen desquelles les personnes chargées de la direction des travaux publiques se trouvent initiées aux connaissances mathématiques les plus élevées.

Claude-Louis Navier (1785-1836)

1 Summary of the proposal

1.1 Scientific goals

Physical laws are used to model and understand the behaviour of nature. Usually, a mathematical model is based on two main ingredients: general laws and constitutive relations. The general laws mainly appear as conservation or balance laws whereas the constitutive relations are of an experimental nature and strongly depend on the features of the phenomena under examination. The outcome of the combination of the two ingredients is often a partial differential equation (PDE) or a system of them. The study of partial differential equations is a central topic in both pure and applied mathematics. This proposal is a research program of 3 years on the development of basic mathematics to understand the stability or instability of suspended bridges through the theoretical study of several models involving partial differential equations. **The innovative nature of this project is to study the interaction between Navier-Stokes equations and plate equations and its application to the stability of suspended bridges**. As an added-value, the study concerns important models from the real life and as a by product, we expect to shed some light and maybe propose some explanations on some observed phenomena in suspended bridges.

1.2 Members of the project

The research will be conducted at the Université libre de Bruxelles, in the Mathematics Department.

Principal investigator: Prof. Denis Bonheure, Director of the Research Unit *Analysis and Partial differential equations* at ULB.

Foreign members:
- Prof. Filippo Gazzola, Full professor at the Politecnico di Milano,
- Prof. Ederson Moreira dos Santos, Associate professor at the Universidade de São Paulo.

The present research project is a natural continuation of an ongoing collaboration on the torsional stability of a nonlocal plate equation (as a possible model for suspended bridges). We emphasize that Prof. Bonheure and Prof. dos Santos are co-authors of 6 publications and 1 preprint. Prof. Gazzola and Prof. dos Santos have a recent joint publication, namely [22].

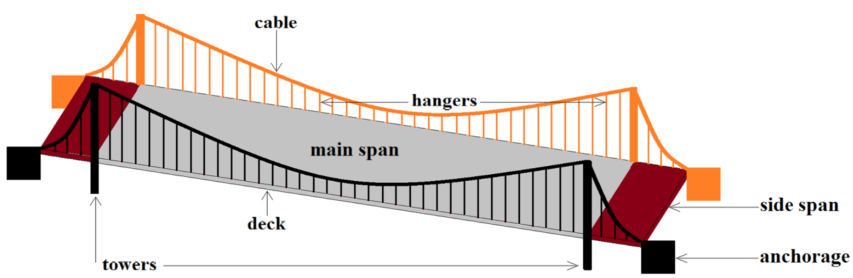
Other members could join the project during its course, as for instance phd students, post-doctoral researchers or some of our senior collaborators.

2 Description of the research program

2.1 Motivation and goals of the research

In order to explain the goals of the present project, it is mandatory to describe the main motivation which has its roots in an engineering problem: the dynamics and the stability of suspension bridges. The need of a purely theoretical approach is made clear in history by two celebrated mathematicians. Leonardo da Vinci (1452-1519) wrote that *Nessuna certezza delle scienze è dove non si pò applicare una delle scienze matematiche, ovver che non sono unite con esse matematiche* (no certitude exists in science if you cannot apply some mathematical science, that is if they are not connected to mathematics). Claude-Louis Navier (1785-1836) wrote that *L'étude des ponts suspendus n'aurait pas été possible sans les progrès que l'analyse mathématique a faits dans ces derniers temps, et sans les institutions au moyen desquelles les personnes chargées de la direction des travaux publiques se trouvent initiées aux connaissances mathématiques les plus élevées* (The study of suspension bridges would not have been possible without the progress made in mathematical analysis in recent times, and without the institutions by means of which those charged with the direction of public works are initiated into the most advanced ideas of mathematics). In this project we aim to follow these suggestions and to use purely theoretical mathematics to study a problem coming from applied sciences.

A suspension bridge is formed by several interacting elastic components. The deck (or roadway) is supported from below by a girder, which is often composed by stiffening trusses. Four high towers sustain two parallel cables, which, in turn, sustain the hangers. The hangers are hooked to the cables and the deck is hooked to the hangers; this deforms the cable and stretches the hangers, which start their restoring action on the deck (Section 15.23 in [47]). The most vulnerable part of the bridge is the main span, namely the part of the deck between the four towers. It has a rectangular shape with two long free edges (of the order of 1km) and two shorter edges (of the order of 20m) fixed and hinged between the towers. A sketch of a suspension bridge is given in the next picture: a challenging mathematical problem is to find simple and reliable models able to describe its static and dynamic behaviors.



The collapse of the Tacoma Narrows Bridge (TNB in the sequel), which occurred on November 7th, 1940, still stands nowadays as one of the most intriguing mysteries of structural engineering. We refer to [3] for the Federal Report and to [51] for a more recent description; see also [53] for a video of the collapse.

There have been many attempts, from Engineers, Physicists, and Mathematicians, to explain this amazing event. Some explanations claim that the collapse was due to a structural failure, some others to a resonance between the frequency of the wind and the oscillating modes of the bridge. Further explanations involve vortices, due both to the particular shape of the bridge and to the angle of attack of the wind. Finally, some explanations are based on flutter theory, linear parametric resonance, and self-excited oscillations, all due to the overcoming of the critical speed of the wind. These theories all agree that the main cause of the collapse was the instability that abruptly transformed longitudinal oscillations into torsional ones; they also agree that the extreme flexibility, slenderness, and lightness of the TNB allowed these oscillations to grow until they destroyed it. But all these theories leave without answer the main question on the origin of the instability, that is:

(Q) **why do longitudinal oscillations suddenly transform into torsional oscillations?**

Due to the vortex shedding, longitudinal oscillations are to be expected in suspension bridges, but the reason of the sudden transition from longitudinal to torsional oscillations is less clear. In fact, the appearance of torsional oscillations is not an isolated event occurred only at the TNB, see the survey in Chapter I in [26]. These accidents raised some fundamental questions of deep interest for several scientific communities. Early attempts [13, 48, 52] to give an answer to question (Q) were made soon after the TNB collapse. Von Karman was a member of the Board appointed for the Federal Report [3]: he was convinced that the torsional motion seen on the day of the collapse was due to the vortex shedding that amplified the already present torsional oscillations and caused the center span to violent twist until the collapse [19, p.31]. But the civil and aeronautical engineer Scanlan [50, p.841] proved that the frequency of the torsional mode had nothing to do with the natural frequency of the vortex shedding following the von Karman vortex pattern. Moreover, the conclusion on [11, p.122] is that the vortex trail is a consequence and not a primary cause of the torsional oscillation. Also the physicists Green-Unruh [28, Section III] believe that vortices form independently of the motion and are not responsible for the catastrophic oscillations of the TNB. The vortex theory was later revisited by Larsen [33, p.247] who stated that "vortices may only cause limited torsional oscillations, but cannot be held responsible for divergent large-amplitude torsional oscillations". McKenna [40] noticed that the behavior described by Larsen was never observed at the TNB while Green-Unruh [28] claim that "the Larsen model does not adequately explain data or simulations at around 23m/s". In 1948, Bleich [12] suggested a possible connection between instability in suspension bridges and flutter of aircraft wings; on the contrary, Billah-Scanlan [11, p.122] believe that it is a great mistake to relate these two phenomena. Billah-Scanlan also claim that their own work proves that the failure of the TNB was in fact related to an aerodynamically induced condition of self-excitation in a torsional degree of freedom; but Larsen [33, p.244] believes that their work fails to connect the vortex pattern to the switch of damping from positive to negative. Moreover, McKenna [40] states that [11] "is a perfectly good explanation of something that was never observed, namely small torsional oscillations, and no explanation of what really occurred, namely large vertical oscillations followed by torsional oscillations".

A linear parametric resonance method was adapted to a TNB model by Pittel-Yakubovich [45, 46], see also Chapter VI in [58] for the English translation and a more general setting. The conclusion on [58, p.457] claims that "the most dangerous phenomenon for the stability of suspension bridges is a combination of parametric resonance". But Scanlan [50, p.841] comments these attempts by writing that "Others have added to the confusion. A recent mathematics text [58], for example, seeking an application for a developed theory of parametric resonance, attempts to explain the Tacoma Narrows failure through this phenomenon". To conclude this quick survey of attempts for aeroelastic explanations, let us mention that Scanlan [49, p.209] writes that "the original TNB withstood random buffeting for some hours with relatively little harm until some fortuitous condition broke the bridge action over into its low antisymmetrical torsion flutter mode"; the words *fortuitous condition* tell us that he also had no answer to question (Q).

Due to all these controversial discussions, McKenna [39, Section 2.3] writes that "there is no consensus on what caused the sudden change to torsional motion", whereas Scott [51] writes that "opinion on the exact cause of the TNB collapse is even today not unanimously shared". Summarizing, all the attempts to find a purely aeroelastic answer to (Q) fail either because the quantitative parameters do not fit the theoretical explanations or because the experiments in wind tunnels do not confirm the underlying theory.

This confusion motivates the present project. Following the suggestions by Leonardo da Vinci and Navier, we wish to use pure mathematics to try to give an answer to question (Q). We will need to create a reliable model for suspension bridges and for the surrounding air. After showing that the related problems are well-posed we will seek qualitative properties of the solutions of the related equations. This project requires the use of a wide spectrum of skills and techniques: classical methods from theoretical and applied analysis in ODE's and PDE's, qualitative and quantitative behavior of solutions of ODE's and PDE's, Galerkin-type approximations for PDE's, stability techniques from the Floquet theory such as the use of Hill equations and Poincaré maps, variational methods and shape optimization, mathematical modeling in elasticity and fluid mechanics, tools from nonlinear structural mechanics. By properly combining all these techniques, we expect to upgrade and use the existing models for the study of the fluid-structure interaction in suspension bridges. The models will be attacked with strategies going from the field of Hamiltonian systems in finite dimension to PDE's and systems in infinite dimension. We are confident that the quantitative responses obtained from this project will help the scientific community to gain more insight into the instability of bridges and how to lower the impact of the air on the bridges.

2.2 State of the art

Prior to the TNB collapse, the only mathematical treatises of suspension bridges were the celebrated report by Navier [43] and the monograph by Melan [41]. The collapse highlighted the lack of rigorous mathematical and physical theories of bridges and suggested to the scientific community to seek models able to describe the behavior of bridges under the action of the wind. The engineering community of the 20th century [13] derived some rough formulas for computing the natural frequencies and modal shapes of the components of a bridge: all the models considered were linear. Only much later the attention of Engineers has turned to the nonlinear behavior of structures, starting from [1, 2] and ending up to the most updated point of view [31]. This delay in introducing nonlinear models has an evident justification: the theoretical tools necessary for a nonlinear analysis are by far more complicated than the linear ones and the numerical methods need a reliable and solid theoretical framework. A fully reliable structural model for suspension bridges is useless if not accompanied by a likewise reliable model for the behavior of the wind and of its interaction with the bridge. The mathematical modeling of the fluid-structure interaction is a fairly recent research topic. According to the MathSciNet, the first paper was published in 1971 but it is only in the 21st century that the Mathematical community has obtained important breakthroughs. This modeling has to face several extremely difficult problems, the most relevant being the correct functional setting in order to obtain well-posedness and the subsequent regularity theory. In the case of bridges, very violent wind-forces attack the structure and this yields further difficulties. Moreover, the formation of vortices creates many complicated phenomena. In a model of fluid-structure interaction, an elastic solid is immersed in a fluid occupying a larger domain and the interaction takes place on the boundary of the solid. Simplest models are given by a linear elastic hyperbolic-type equation describing the dynamics of the solid, by the incompressible Navier-Stokes equations [7, 8] or the linearized Stokes equations [34] for the dynamic of the fluid, and by suitable Neumann-type transmission boundary conditions. A challenging problem is to deal with the mismatch between parabolic and hyperbolic regularity. However, satisfactory regularity results may be achieved [5, 8, 34], which proves that the setting is correct. Also the Euler equations are used to model the fluid [15]. Nonlinear plates interacting with fluids have also been studied [16, 17].

Our intention is to use nonlinear equations for the structural behavior of the bridge, combined with either the Navier-Stokes, the Stokes, or the Euler equations. For several fairly different models, in literature one can find some attempts [35], including with mechanical damping [36] and stochastic forcing [18]: these effects will be weighted in our models. The equations of motion for suspension bridges were employed for aeroelastic investigations in [4], where the analysis is focused on experimentally determined flutter derivatives and a three-dimensional modal analysis of the structure. But the present state-of-the-art is insufficient to guarantee fully reliable responses: a precise model would allow to replace part of the expensive wind tunnel experiments with numerical analysis. Moreover, wind tunnel experiments sometimes predict incorrect responses up to 20% of the real behavior of bridges [20, p.32]. In a complete aeroelastic model, the external forces represent any unsteady aerodynamic action such as gusts, buffeting or vortex shedding. The latter is perhaps one of the most studied phenomena of fluid mechanics, especially in fluid interactions with circular cylinders. When a vortex is formed on one side of the immersed body, it immediately increases the flow velocity on the opposite side, which results, according to the Bernoulli theory, in a pressure reduction. The process of vortex shedding can be emphasized only if the effect of viscosity is considered, since only a viscous fluid satisfies the non-slip condition of its particles on the surface of the body immersed in the flow. Even if the viscosity is very small, this condition holds, but its influence on the flow regime will be confined to a small region, the so-called boundary layer along the body. Within this layer the velocity of the fluid rapidly changes from zero on the surface of the structure to the free-stream velocity of the flow. While the free stream is pulling the boundary layer forward, the skin friction at the solid wall is retarding it. At surfaces with high curvature, there can also be an adverse pressure gradient adding contributions to the retarding action, which may cause the flow to be interrupted entirely and the boundary layer may detach from the wall: this phenomenon is called separation. Also streamlined bodies can experience separation if the angle of attack between the free stream and the surface is large enough. It is clear from the physical understanding of the separation process, that viscosity and free stream velocity have an important influence and can be collected in the Reynolds number (Re in the sequel) which expresses the ratio between inertial forces and viscous forces of the flow. Considering the flow past a circular cylinder, a great variety of changes in the nature of the flow occur with increasing Re. At very small Re, the inertial effects are negligible and the flow pattern is very similar to that of a laminar flow, the pressure recovery being nearly complete: the pressure drag is also negligible and the effective drag on the structure is entirely due to skin friction. At increased Re, a separation of the boundary layer occurs at two points at the back of the cylinder where symmetrical eddies are formed and rotate in opposite directions. They basically remain steady and the flow closes behind them. A further increase of Re elongates the fixed vortices, which then start to oscillate until they break away for larger Re. The breaking away occurs alternatively from one and the other side, and then the eddies travel downstream. This process is intensified with a further increase of Re while the shedding of vortices from alternate sides of the cylinder is regular. This leads to formation of the characteristic wake, which is known as the von Karman vortex street. The eddying motion is periodic, both in space and time. The pressure drag at this stage is already larger than the profile drag. Having passed a transition range where the regularity of shedding decreases, for very large Re the vortex shedding becomes irregular: there still is a predominant frequency but the amplitude appears to be random. The critical threshold for Re decreases as the roughness of the body surface increases. For extremely large Re, the separation point moves rearward on the cylinder, consequently the drag coefficient decreases appreciably. The flow in the wake becomes so turbulent that the vortex street pattern is no longer recognizable. These facts show that the process of vortex shedding around a cylinder and its dependence on the Reynolds number is highly complex, which makes the analytical as well as the numerical treatments very challenging. Due to its irregular and nonsmooth shape, even more challenging appears the modeling of vortex shedding around the deck of a bridge.

2.3 Our new model

In the space $\mathbb{R}^3$ we consider the deck of the bridge to be a thin plate defined by

$$D = (0, \pi) \times (-\ell, \ell) \times (-d, d) \tag{1}$$

where $d \ll \ell \ll \pi$. To have an idea, one could take $\ell = \pi/150$ and $d = \pi/1000$. Then we consider the region where the air surrounds the deck

$$\Omega = (0, \pi) \times (-L, L)^2 \setminus D \tag{2}$$

where $L \gg \pi$, for instance $L = 100\pi$. The domain Ω and its intersection with the plane $x = \frac{\pi}{2}$ are represented in Figure 1 (not in scale!).

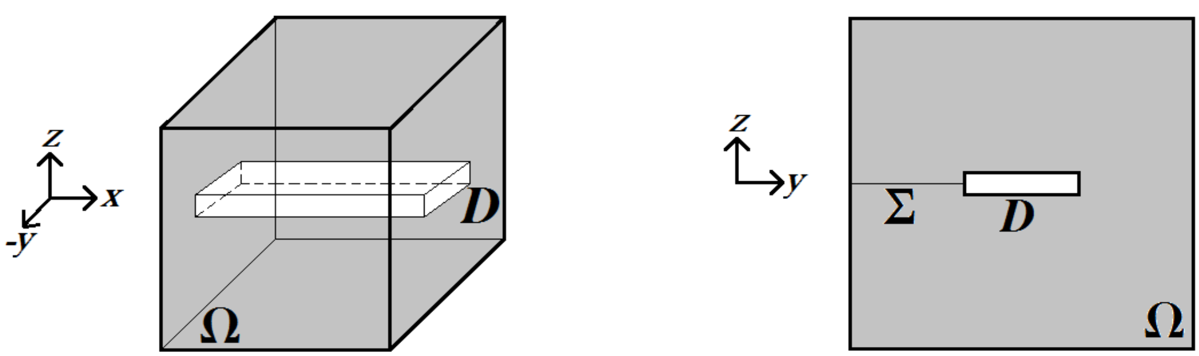


Figure 1: The domain Ω and its intersection with the plane $x = \frac{\pi}{2}$.

We couple the Navier-Stokes equations with a nonlinear nonlocal plate equation. This study is motivated by a phenomenon which is visible in actual bridges and we mention that, according to the Federal Report [3] (see also [51]), the main reason for the Tacoma Narrows Bridge collapse was the sudden transition from longitudinal to torsional oscillations. Since the origin of this instability is of structural nature (see [26] for a survey of modeling and results), we analyze in detail how a solution of the plate equation initially oscillating in an almost purely longitudinal fashion can suddenly start oscillating in a torsional fashion. Before doing this we study the mechanism of energy input from the air into the bridge. Overall, our results should reproduce, at least qualitatively, the oscillations and the instability appeared during the Tacoma collapse [3, 51].

2.3.1 Some features of Helmholtz-Weyl decomposition

In the sequel, bold capital letters ($\mathbf{L}^2$, $\mathbf{H}^1$,...) represent functional spaces of vector functions and usual capital letters (L^2, H^1,...) represent spaces of scalar functions: we set $L^2 := L^2(\Omega)$,.... and we specify the set only when it is not Ω.

We recall here a decomposition of $\mathbf{L}^2$ introduced in 1870 by Helmholtz [29] and complemented in 1940 by Weyl [56]. In 1978, Foias-Temam [23] refined this decomposition in the case of domains which are not simply connected, as the domain Ω in (2). We consider the spaces

$$\begin{array}{lll} \mathbf{G}_1 := \{f \in \mathbf{L}^2;\ \nabla \cdot f = 0,\ \gamma_n f = 0\}, & \mathbf{E} := \{f \in \mathbf{L}^2;\ \nabla \cdot f \in L^2\}, & \mathbf{M} := \{f \in \mathbf{H}^1;\ \Delta f \in \mathbf{L}^2\}, \\ \mathbf{G}_2 := \{f \in \mathbf{L}^2;\ \nabla \cdot f = 0,\ \exists g \in H^1,\ f = \nabla g\}, & \mathbf{E}_0 := \{f \in \mathbf{E};\ \gamma_n f = 0\}, & \\ \mathbf{G}_3 := \{f \in \mathbf{L}^2;\ \exists g \in H^1_0,\ f = \nabla g\}, & \mathbf{V} := \{f \in \mathbf{H}^1_0;\ \nabla \cdot f = 0\}, & \end{array}$$

where γ_n denotes the normal trace operator; γ_n is linear continuous and surjective from $\mathbf{E}$ onto $H^{-1/2}(\partial\Omega)$ and its kernel is $\mathbf{E}_0$. Obviously $\mathbf{V} \subset \mathbf{G}_1 \subset \mathbf{E}_0$ and $\mathbf{G}_1 \oplus \mathbf{G}_2 \subset \mathbf{E}$; the spaces $\mathbf{V}$ and $\mathbf{E}$ are Hilbert spaces when endowed with the scalar products $(u, v)_{\mathbf{V}} := (\nabla u, \nabla v)_{\mathbf{L}^2}$ and $(u, v)_{\mathbf{E}} = (u, v)_{\mathbf{L}^2} + (\nabla \cdot u, \nabla \cdot v)_{L^2}$. It is well-known (see [29, 54, 56]) that

$$\mathbf{L}^2 = \mathbf{G}_1 \oplus \mathbf{G}_2 \oplus \mathbf{G}_3 \tag{3}$$

and that the spaces $\mathbf{G}_i$ $(i = 1, 2, 3)$ are mutually orthogonal: we denote by P_i $(i = 1, 2, 3)$ the orthogonal projectors of $\mathbf{L}^2$ onto $\mathbf{G}_i$. The decomposition of a function $f \in \mathbf{L}^2$ following $\mathbf{G}_1 \oplus \mathbf{G}_2 \oplus \mathbf{G}_3$ is determined by solving the homogeneous Dirichlet problem for a Poisson equation and a Neumann problem for Laplace equation: let $f \in \mathbf{L}^2$ and put $\varphi = \nabla \cdot f$ $(\varphi \in H^{-1})$, let ψ be the unique solution of

$$\left\{ \begin{array}{ll} \Delta\psi = \varphi & \text{in} \quad \Omega \\ \psi \in H^1_0\ , & \end{array} \right. \tag{4}$$

then $P_3 f = \nabla\psi$. Let θ be the unique solution (up to the addition of constants) of the problem

$$\left\{ \begin{array}{ll} \Delta\theta = 0 & \text{in} \quad \Omega \\ \frac{\partial\theta}{\partial n} = \gamma_n(f - P_3 f) & \text{on} \quad \partial\Omega\ , \end{array} \right. \tag{5}$$

then $P_2 f = \nabla\theta$. Since $\nabla \cdot (f - P_3 f) = 0$, then $f - P_3 f \in \mathbf{E}$ and $\gamma_n(f - P_3 f)$ makes sense; using the generalized Stokes formula

$$\forall f \in \mathbf{E} \quad \forall g \in H^1 \qquad (f, \nabla g)_{\mathbf{L}^2} + (\nabla \cdot f, g)_{L^2} = \langle \gamma_n f, \gamma g \rangle\ , \tag{6}$$

where γ is the usual trace operator and $\langle \cdot, \cdot \rangle$ is the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$: one can easily verify that the compatibility condition for (5) is fulfilled. Finally, we have $P_1 f = f - P_2 f - P_3 f$.

Since Ω is connected but not simply connected the space $\mathbf{G}_1$ can be decomposed as direct sum of $\mathbf{G}_0 = \ker(curl) \cap \mathbf{G}_1$ and its orthogonal complement $\mathbf{G}_0^{\perp}$ in $\mathbf{G}_1$, see [23]. Since this is a crucial point, let us explain how to characterize the space $\mathbf{G}_0$ in our case. The domain Ω can be made simply connected by subtracting a portion of a plane. More precisely, let

$$\Sigma := (0, \pi) \times (-L, -\ell) \times \{0\} \tag{7}$$

then $\Omega \setminus \Sigma$ is simply connected, which means that the order of connection of the domain Ω is 1. Then it can be proved (see [23, Lemme 1.1]) that $\dim(\mathbf{G}_0) = 1$ and that

$$\begin{array}{c} \mathbf{G}_0 \text{ contains the vectors proportional to the gradient of the (unique) function } q \text{ such that} \\ \Delta q = 0 \text{ in } \Omega \setminus \Sigma, \quad \frac{\partial q}{\partial n} = 0 \text{ on } \partial\Omega, \quad q = \pm 1 \text{ on } \Sigma_{\pm}, \quad \frac{\partial q}{\partial n} = 0 \text{ on } \Sigma\ , \end{array}$$

where Σ_- (resp. Σ_+) represents the side of Σ towards the half plane $z > 0$ (resp. $z < 0$). It is also known [23, Lemme 1.4] that

$$\mathbf{G}_0^\perp = \left\{ w \in \mathbf{G}_1;\ \int_\Sigma \gamma_n v \, d\Sigma = 0 \right\}$$

and that Σ may be chosen differently; here, we chose Σ as in (7) both for symmetry reasons and because the flow will somehow be parallel to such Σ. Therefore, (3) can be refined with

$$\mathbf{L}^2 = \mathbf{G}_0 \oplus \mathbf{G}_0^\perp \oplus \mathbf{G}_2 \oplus \mathbf{G}_3\,. \tag{8}$$

From (8), we immediately infer that all the spaces of this decomposition are closed subspaces of $\mathbf{L}^2$. As direct consequences of the definitions of $\mathbf{G}_2$ and $\mathbf{G}_3$ we have the following facts, see [25]:

- If $f \in \mathbf{G}_2$ then f is harmonic in Ω; if $f \in \mathbf{G}_3$ and $\nabla \cdot f = 0$ then $f \equiv 0$.
- $\mathbf{G}_2 \cap \mathbf{E}_0 = \{0\}$.
- $\mathbf{G}_3 \cap \mathbf{E} = \{f \in \mathbf{L}^2;\ \exists g \in H_0^1 \cap H^2,\ f = \nabla g\}$.
- $\mathbf{G}_3 \cap \mathbf{E}_0 = \{f \in \mathbf{L}^2;\ \exists g \in H_0^2,\ f = \nabla g\}$.
- γ_n is an isomorphism from $\mathbf{G}_2$ onto $H_0^{-1/2}(\partial\Omega) := \{\Phi \in H^{-1/2}(\partial\Omega);\ \langle \Phi, 1\rangle = 0\}$.
- Let $i \in \{1,3\}$ and let $f \in \mathbf{M} \cap (\mathbf{G}_i \oplus \mathbf{G}_2)$; then $\Delta f \in \mathbf{G}_i \oplus \mathbf{G}_2$.
- Let $f \in \mathbf{M}$ and $\varphi = \nabla \cdot f$; then $\Delta f \in \mathbf{G}_1 \oplus \mathbf{G}_2$ if and only if $\Delta\varphi = 0$.
- Let $f \in \mathbf{M} \cap \mathbf{G}_3$; then $\Delta f = 0$ if and only if $\nabla \cdot f = k$ ($k \in \mathbb{R}$).

2.4 Step-by-step work plan

2.4.1 Step 1: the Navier-Stokes equations

We model the case where the wind is blowing in the y-direction, so that the forcing term f and its potential F have the form

$$f = f(y) = (0, w, 0)\,, \qquad F(x,y,z) = wy\,, \tag{9}$$

where $w > 0$ is the wind velocity. Clearly, f is conservative and $\nabla F = f$ so that $f \in \mathbf{G}_2 \oplus \mathbf{G}_3$. We then consider the Navier-Stokes equations

$$\begin{cases} u_t - \eta\Delta u + (u\cdot\nabla)u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases} \qquad (x,y,z) \in \Omega\,, \quad t > 0\,. \tag{10}$$

The *vorticity vector* ω is the curl of the velocity u, namely

$$\omega = \nabla \wedge u\,.$$

To (10) we associate a number of boundary conditions on different parts of the boundary $\partial\Omega$. A possible choice is to take some of the boundary conditions introduced in [9, 27] that are related to the Helmholtz-Weyl decomposition and hence have their roots in Functional Analysis. One can take

$$\begin{array}{ll} u = 0 & \text{for } x \in (0,\pi)\,,\ y = -\ell\,,\ z \in (-d,d) \\ \gamma_n(u) = \gamma_n(\omega) = \gamma_n(\nabla \wedge \omega) = 0 & \text{for } x \in (0,\pi)\,,\ y = \ell\,,\ z \in (-d,d) \\ \gamma_n(u) = 0\,, \quad \omega \wedge n = 0 & \text{for } x \in (0,\pi)\,,\ y \in (-\ell,\ell)\,,\ z \in \{-d,d\} \\ \gamma_n(u) = 0\,, \quad \omega \wedge n = 0 & \text{for } x \in (0,\pi)\,,\ y \in (-L,L)\,,\ z \in \{-L,L\} \\ \gamma_n(u) = 0\,, \quad \omega \wedge n = 0 & \text{for } x \in \{0,\pi\}\,,\ y \in (-L,L)\,,\ z \in (-L,L) \\ u \wedge n = 0\,, \quad \gamma p = \pm\lambda\ (\lambda < 0) & \text{for } x \in (0,\pi)\,,\ y \in \{-L,L\}\,,\ z \in (-L,L)\,. \end{array} \tag{11}$$

Let us interpret these conditions. The first line states that on the side of the deck hit by the wind, the air is still (non-slip boundary condition). The second line states that on the opposite side the velocity u is tangential as well as the vorticity ω and its curl: this means that u and ω are "very tangential", namely tangential of higher order (indeed, since $\nabla \cdot u = 0$, one has $\nabla \wedge \omega = \Delta u$). The third line in (11) tells us that the flow is tangential on the upper and lower faces of the deck D. Similarly, the fourth and fifth lines tell us that the flow is tangential on the four faces of Ω where the wind has a laminar motion. On the contrary, the sixth line models the fact that the flow is normal on these faces of $\partial\Omega$: these are the faces where the flow is entering and exiting and therefore the pressure is constant with opposite signs according to whether we are in the inflow or outflow face.

With f as in (9), we prove existence and uniqueness of (10)-(11) under the initial condition

$$u(x,y,z,0) = 0 \quad \text{in } \Omega\,. \tag{12}$$

To this end, we need to find the correct variational formulation following [9] and a careful use of the spaces defined in the previous section. We emphasize that there are at least two challenging difficulties. First, due to these "strange" boundary conditions, the variational characterization is not straightforward. Secondly, there is a lack of regularity of the solution due to the concave right corner in the domain.

2.4.2 Step 2: Navier-Stokes equations in a variable domain

When a fluid hits a bluff body its flow is modified and goes around the body. Behind the body, or a "hidden part" of the body, the flow creates vortices, see the sketch in the left picture of Figure 2. In general, asymmetric vortices appear and this asymmetry generates a forcing lift which starts the vertical oscillations of the body, see the right picture in Figure 2.

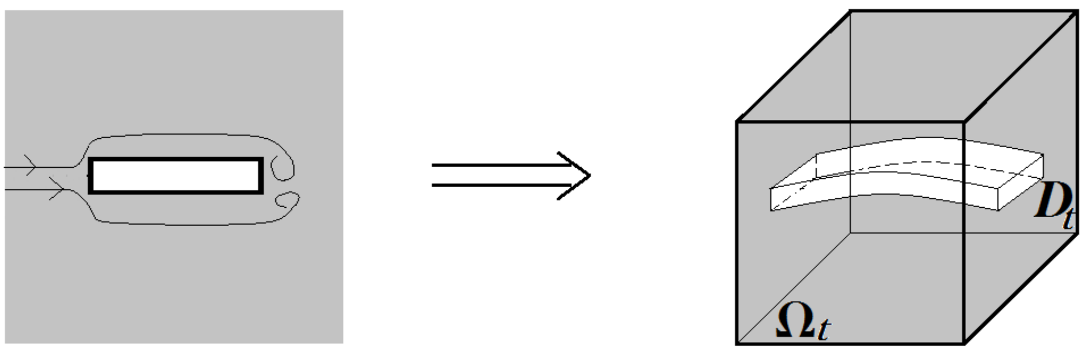


Figure 2: Vortex shedding and its effect on the movement of the deck D.

This explanation is shared by the whole scientific community and, for bridges, it has been studied with great precision in wind tunnel tests, see e.g. [33, 51]. This means that it is not correct to assume that the deck D is still, instead it evolves in time. We are therefore led to consider domains such as

$$D_t = \Big\{(x,y,z) \in \mathbb{R}^3;\ (x,y) \in (0,\pi) \times (-\ell,\ell),\ v(x,y,t) - d < z < v(x,y,t) + d\Big\}\ , \tag{13}$$

where $v = v(x,y,t)$ is a given continuous function satisfying $v(0,y,t) = v(\pi,y,t) = 0$.

Then one should prove existence and uniqueness for the Navier-Stokes equations in

$$\Omega_t = (0,\pi) \times (-L,L)^2 \setminus D_t\,.$$

Equations (10) then become

$$\left\{\begin{array}{l} u_t - \eta\Delta u + (u\cdot\nabla)u + \nabla p = f \\ \nabla\cdot u = 0 \end{array}\right. \qquad (x,y,z) \in \Omega_t\,, \quad t > 0\,. \tag{14}$$

We will also have to add the (suitably modified) boundary conditions (11) and the initial condition (12).

2.4.3 Step 3: estimates of the vorticity and its impact on the deck

We will do this first in Ω and then in Ω_t. In particular, we must find the *force* created by ω on D and D_t. We will call this force $g(\omega)$ in the next steps.

2.4.4 Step 4: the equation of the deck

We now view the deck of the bridge D in (1) as a thin narrow rectangular plate where the two short edges are hinged whereas the two long edges are free. This plate is subject to the vertical force $g(\omega)$ generated by the vortices of the solution u of (10) so that it moves in the z-direction. Since the thickness $2d$ remains constant, we focus our attention on the behavior of the middle cross section. The deflections of this plate are fully described by a function $v = v(x, y, t)$ with $(x, y) \in D_0$ where

$$D_0 := (0, \pi) \times (-\ell, \ell)\,. \tag{15}$$

In absence of forces, the plate is in the horizontal position $D_0 \times \{0\}$ in the 3D space. The plate is also subject to edge loads, the so-called buckling loads, that are compressive forces along the edges: this means that the plate is subject to prestressing [42]. We will then follow the plate model suggested by Berger [10]; see also the previous beam model suggested by Woinowsky-Krieger [57] and, independently, by Burgreen [14]. The nonlocal evolution equation modeling the deformation of the plate reads

$$\begin{cases} v_{tt} + \delta v_t + \Delta^2 v - \varphi(v) v_{xx} = g(\omega) & \text{in } D_0 \times (0, T) \\ v = v_{xx} = 0 & \text{on } \{0, \pi\} \times (-\ell, \ell) \\ v_{yy} + \sigma v_{xx} = v_{yyy} + (2 - \sigma) v_{xxy} = 0 & \text{on } (0, \pi) \times \{-\ell, \ell\} \\ v(x, y, 0) = v_0(x, y), \qquad v_t(x, y, 0) = v_1(x, y) & \text{in } D_0\,. \end{cases} \tag{16}$$

The term $g(\omega)$ represents the vertical load over the deck generated by the vorticity of the solution of (10): clearly, it depends on time. The parameter $\delta > 0$ measures the damping due to internal friction in the structure. The functions v_0 and v_1 are, respectively, the initial position and velocity of the deck. The boundary conditions on the short edges are named after Navier [44] and model the fact that the plate is hinged in connection with the ground. The boundary conditions on the long edges model the fact that the plate is free, see [38, 55]. Finally, φ is defined by

$$\varphi(v) = -P + S \int_{D_0} v_x^2\,, \tag{17}$$

and carries a nonlinear nonlocal effect into the model. Here $S > 0$ depends on the elasticity of the material composing the deck, $S \int_{D_0} v_x^2$ measures the geometric nonlinearity of the plate due to its stretching, and $P > 0$ is the prestressing constant: one has $P > 0$ if the plate is compressed and $P < 0$ if the plate is stretched. For a partially hinged plate such as D_0, the buckling load only acts in the x-direction and therefore one obtains the term $\int_{D_0} v_x^2$; see [30]. The constant σ is the Poisson ratio: for metals its value lies around 0.3 while for concrete it is between 0.1 and 0.2. Since the deck of a bridge is a mixture of concrete and metal, we take $\sigma = 0.2$.

We refer to [21] for the derivation of (16) and to the monograph [26] for the complete updated story. Existence, uniqueness and asymptotic behavior for the solutions of (16) were proved in [22]. We will mainly deal with weak solutions, although with little effort one could extend the results to more regular solutions (including classical solutions) by arguing as in the seminal paper by Ball [6] for the beam equation.

By separating variables, it is known [22] that (16) admits solutions with a finite number of nontrivial Fourier components. This will happen if the vorticity ω has certain frequencies or if the initial data are concentrated only on a few components. This step will finally require to write and prove all the related results about existence, uniqueness and modes involved.

2.4.5 Step 5: an evolution fixed point?

Unfortunately, neither (14) nor (16) describe the fluid-structure interaction between the air and the deck. The former merely describes the flow around a moving deck whose movement D_t is known whereas the latter merely describes the oscillations of the deck when the forcing term $g(\omega)$ is known. Overall, assuming we have prove all the above results, we have obtained so far that

(i) for a *given* function v, we know the movement of D_t and we can solve (10);

(ii) for a *given* vorticity ω, we know the force $g(\omega)$ and we can solve (16).

But nothing between v and ω is known a priori. Therefore, the natural strategy will be to use an iterative method, hoping that it will converge to some fixed point. More precisely, fixing some $v^1 = v^1(t)$ and the related movement D_t: through (10) we will find a velocity u^1 and its vorticity ω_1. By taking the force $g(\omega_1)$, we can solve (16) and find a new function $v^2(t)$, possibly different from v^1. We iterate this procedure hoping that the sequence of solutions $\{v^n, u^n\}$ converges to some fixed point.

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