Optimal Procedures Based on Interdirections and pseudo-Mahalanobis Ranks for Testing multivariate elliptic White Noise against ARMA Dependence

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Abstract

We propose a multivariate generalization of signed-rank tests for testing elliptically symmetric white noise against ARMA serial dependence. These tests are based on Randles (1989)'s concept of interdirections and the ranks of pseudo-Mahalanobis distances. They are affine-invariant, and asymptotically equivalent to strictly distribution-free statistics. Depending on the score function considered (van der Waerden, Laplace,...), they allow for locally asymptotically maximin tests at selected densities (multivariate normal, multivariate double-exponential,...). Local powers and asymptotic relative efficiencies with respect to the Gaussian procedure are derived. We extend to the multivariate serial context the Chernoff-Savage result, showing that classical correlogram-based procedures are uniformly dominated by the van der Waerden version of our tests, so that correlogram methods are not admissible in the Pitman sense. We also prove an extension of the celebrated Hodges-Lehmann ".864 result", providing, for any fixed space dimension, the lower bound for the asymptotic relative efficiency of the proposed multivariate Spearman type tests with respect to the Gaussian tests. These asymptotic results are confirmed by a Monte-Carlo investigation.

1 Introduction.

Much attention has been devoted in the past ten years to multivariate extensions of the classical, univariate theory of rank and signed-rank tests; see Oja (1999) for an insightful review of the abundant literature on this subject. Emphasis in this literature however has been put, essentially if not exclusively, on invariance and robustness rather than on asymptotic optimality issues. Hallin and Paindaveine (2002) recently showed that invariance and asymptotic efficiency, in this context, are not irreconciliable objectives, and that locally asymptotically optimal procedures (in the Le Cam sense), in the context of multivariate one-sample location models, can be based on the robust tools that have been developed in the area, such as Randles' interdirections, and (pseudo-)Mahalanobis ranks.

Although the need for non-Gaussian and robust procedures in multivariate time series problems is certainly as strong as in the context of independent observations, little has been done to extend this strand of statistical investigation beyond the traditional case of linear models with

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independent errors. A similar phenomenon has been observed in the development of univariate rank-based methods which, for quite a long period, were restricted to nonserial models (involving independent observations)—despite the serial nature of some of the earliest nonparametric and rank-based procedures, such as the tests based on runs or turning points.

A rank-based test for randomness against multivariate serial dependence of the ARMA type was proposed by Hallin, Ingenbleek, and Puri (1989), but suffers the same lack of invariance with respect to affine transformations as all methods based on componentwise rankings (the reader is referred to the monograph by Puri and Sen (1971) for an extensive description, in the nonserial case, of this approach). The purpose of this paper is to attack the same problem from an entirely different perspective, in the light of the nonserial contributions of Möttönen et al. (1995, 1997, and 1998), Hettmansperger et al. (1994, 1997), Randles (1989), Peters and Randles (1990), or Jan and Randles (1994), to quote only a few. Basically, we show that tests of randomness that are locally and asymptotically optimal, in the Le Cam sense, against ARMA dependence can be based on the tools developed in some of these papers, namely, interdirections and pseudo-Mahalanobis ranks, which jointly provide a multivariate extension of (univariate) signed ranks.

Testing for randomness or white noise of course is the simplest of all serial problems one can think of. In view of the crucial role of white noise in most time series models, it is also an essential one, as most hypothesis testing problems, in time-series analysis, more or less reduce to testing that some transformation (possibly involving nuisances) of the observed process is white noise.

The paper is organized as follows. Section 2 introduces the main technical assumptions, states the LAN result to be used throughout, and provides the locally and asymptotically maximin parametric procedures for the problem under study. Section 3 presents the class of multivariate serial rank statistics to be used in the sequel. Section 4 provides the asymptotic relative efficiencies of the proposed procedures with respect to their Gaussian counterparts, and multivariate serial extensions of two classical results by Chernoff-Savage (1958) and by Hodges-Lehmann (1956), respectively. In Section 5, we investigate finite-sample performance via a Monte-Carlo study. Proofs are concentrated in the appendix.

2 Local asymptotic normality, and parametric optimality.

2.1 Main assumptions.

The testing procedures we are proposing here constitute a multivariate generalization of the classical univariate signed-rank methods. So, they require some symmetry condition : throughout, we will restrict our attention to *elliptically symmetric white noise*. Let $\mathbf{X}^{(n)} := \left(\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)}\right)$ be an observed k-dimensional series of length n. Denote by $\mathbf{\Sigma}$ a symmetric positive definite $k \times k$ matrix, and by $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ a nonnegative function. We say that $\mathbf{X}^{(n)}$ is a finite realization of an elliptic white noise process with scatter matrix $\mathbf{\Sigma}$ and radial density f, if and only if its probability density at $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{nk}$ is of the form $\prod_{t=1}^n \underline{f}(\mathbf{x}_t; \mathbf{\Sigma}, f)$, where

$$\underline{f}(\mathbf{x}_1; \mathbf{\Sigma}, f) := c_{k,f} \frac{1}{(\det \mathbf{\Sigma})^{1/2}} f(\|\mathbf{x}_1\|_{\mathbf{\Sigma}}), \quad \mathbf{x}_1 \in \mathbb{R}^k. \tag{1}$$

Here, $\|\mathbf{x}\|_{\mathbf{\Sigma}} := (\mathbf{x}' \mathbf{\Sigma}^{-1} \mathbf{x})^{1/2}$ denotes the norm of $\mathbf{x}$ in the metric associated with $\mathbf{\Sigma}$. The constant $c_{k,f}$ is the normalization factor $(\omega_k \, \mu_{k-1;f})^{-1}$, where ω_k stands for the $(k-1)$-dimensional

Lebesgue measure of the unit sphere $\mathcal{S}^{k-1} \subset \mathbb{R}^k$, and $\mu_{l;f} := \int_0^\infty r^l f(r)\, dr$. The following assumption is thus required on f :

Assumption (A1). The radial density f satisfies $f > 0$ a.e. and $\mu_{k-1;f} < \infty$.

Note that the scatter matrix $\mathbf{\Sigma}$ in (1) needs not be (a multiple of) the covariance matrix of the observations, which may not exist, and that f is not, strictly speaking, a probability density; see Hallin and Paindaveine (2002) for a discussion. Moreover, $\mathbf{\Sigma}$ and f are identified up to an arbitrary scale transformation only. More precisely, for any $a > 0$, letting $\mathbf{\Sigma}_a := a^2\mathbf{\Sigma}$ and $f_a(r) := f(ar)$, we have $\underline{f}(\mathbf{x}; \mathbf{\Sigma}, f) = \underline{f}(\mathbf{x}; \mathbf{\Sigma}_a, f_a)$. This will not be a problem in the sequel, where estimated scatter matrices are always defined up to a positive factor a (see Assumption (A4) below).

Under (A1), $\tilde{f}_k(r) := (\mu_{k-1;f})^{-1}\, r^{k-1} f(r)$ is a probability density over $\mathbb{R}_0^+$. More precisely, $\tilde{f}_k$ is the density of $\|\mathbf{X}\|$, where $\mathbf{X}$ is a random k-vector with density $\underline{f}(\,.\,; \mathbf{I}_k, f)$ ($\mathbf{I}_k$ denotes the k-dimensional identity matrix). In the sequel, we denote by $\tilde{F}_k$ the distribution function associated with $\tilde{f}_k$.

Whenever local asymptotic normality is needed, or when Gaussian procedures are to be used, or, more generally, whenever finite second-order moments are required, Assumption (A1) has to be strengthened into

Assumption (A1′). Same as Assumption (A1), but we further assume that $\mu_{k+1;f} < \infty$.

Null hypotheses of elliptical white noise will be tested against alternatives of multivariate ARMA(p, q) dependence. As usual, denoting by L the lag operator, consider the multivariate ARMA(p, q) model

$$\mathbf{A}(L)\, \mathbf{X}_t = \mathbf{B}(L)\, \boldsymbol{\varepsilon}_t, \tag{2}$$

where $\mathbf{A}(L) := \mathbf{I}_k - \sum_{i=1}^p \mathbf{A}_i L^i$ and $\mathbf{B}(L) := \mathbf{I}_k + \sum_{i=1}^q \mathbf{B}_i L^i$ for some $k \times k$ real matrices $\mathbf{A}_1, \ldots, \mathbf{A}_p, \mathbf{B}_1, \ldots, \mathbf{B}_q$ such that $|\mathbf{A}_p| \neq 0 \neq |\mathbf{B}_q|$. Writing

$$\boldsymbol{\theta} := ((\mathrm{vec}\, \mathbf{A}_1)', \ldots, (\mathrm{vec}\, \mathbf{A}_p)', (\mathrm{vec}\, \mathbf{B}_1)', \ldots, (\mathrm{vec}\, \mathbf{B}_q)')' \in \mathbb{R}^{k^2(p+q)},$$

we denote by $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$ the hypothesis under which the observed n-tuple $\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)}$ is a finite realization of some solution of (2), where $\{\boldsymbol{\varepsilon}_t, t \in \mathbb{Z}\}$ is elliptic white noise with scatter parameter $\mathbf{\Sigma}$ and radial density f. Writing $\mathcal{H}^{(n)}(\boldsymbol{\theta}, ., .)$ for the hypothesis $\bigcup_{\mathbf{\Sigma}} \bigcup_f \mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$ (where unions are taken over the largest sets that are compatible with the assumptions), our objective is to test $\mathcal{H}^{(n)}(\mathbf{0}, ., .)$ against $\cup_{\boldsymbol{\theta} \neq \mathbf{0}} \mathcal{H}^{(n)}(\boldsymbol{\theta}, ., .)$.

Under some further regularity assumptions on the radial density f, the local asymptotic normality (for fixed $\mathbf{\Sigma}$ and f) of ARMA models follows from a more general result in Garel and Hallin (1995). The elliptic version of these assumptions, which essentially guarantee the quadratic mean differentiability of $\underline{f}^{1/2}$, takes the following form (see Hallin and Paindaveine (2002) for a discussion).

Considering the space $L^2(\mathbb{R}_0^+, \mu_l)$ of all functions that are square-integrable w.r.t. the Lebesgue measure with weight r^l on $\mathbb{R}_0^+$ (i.e. the space of measurable functions $h : \mathbb{R}_0^+ \to \mathbb{R}$ satisfying $\int_0^\infty [h(r)]^2 r^l\, dr < \infty$), denote by $\mathcal{W}^{1,2}(\mathbb{R}_0^+, \mu_l)$ the subspace containing all functions of $L^2(\mathbb{R}_0^+, \mu_l)$ admitting a weak derivative that also belongs to $L^2(\mathbb{R}_0^+, \mu_l)$. The following assumption is strictly equivalent to the quadratic mean differentiability of $\underline{f}^{1/2}$ (see Hallin and Paindaveine (2002), Proposition 1), but has the important advantage of involving univariate quadratic mean differentiability only.

Assumption (A2). The square root $f^{1/2}$ of the radial density f is in $\mathcal{W}^{1,2}(\mathbb{R}_0^+, \mu_{k-1})$; denote by $(f^{1/2})'$ its weak derivative in $L^2(\mathbb{R}_0^+, \mu_{k-1})$, and let $\varphi_f := -2\frac{(f^{1/2})'}{f^{1/2}}$.

Assumption (A2) guarantees the finiteness of the *radial Fisher information* $\mathcal{I}_{k,f} := (\mu_{k-1;f})^{-1} \int_0^\infty [\varphi_f(r)]^2 r^{k-1} f(r)\, dr$.

Whenever ranks and rank-based statistics come into the picture, they will be defined from score functions K_1 and K_2; suitable score functions will be required to satisfy the following conditions :

Assumption (A3). The score functions $K_\ell :]0,1[\to \mathbb{R}$, $\ell = 1,2$, are continuous, satisfy $\int_0^1 |K_\ell(u)|^{2+\delta}\, du < \infty$ for some $\delta > 0$, and can be expressed as differences of two monotone increasing functions.

The score functions yielding locally and asymptotically optimal procedures, as we shall see, are of the form $K_1 := \varphi_{f_\star} \circ \tilde{F}_{\star k}^{-1}$ and $K_2 := \tilde{F}_{\star k}^{-1}$, for some radial density $f_\star$. Assumption (A3) then takes the form of an assumption on $f_\star$:

Assumption (A3′). The radial density $f_\star$ satisfies Assumption (A2), and $\mu_{k+1+\delta;f_\star} < \infty$ for some $\delta > 0$. The associated function $\varphi_{f_\star}$ is continuous, satisfies $\int_0^\infty |\varphi_{f_\star}(r)|^{2+\delta}\, r^{k-1} f_\star(r)\, dr < \infty$ for some $\delta > 0$, and can be expressed as the difference of two monotone increasing functions.

The assumption of being the difference of two monotone functions, which characterizes the functions with bounded variation, is extremely mild. In most cases ($f_\star$ normal, double exponential, ...), $\varphi_{f_\star}$ itself is monotone increasing, and, without loss of generality, this will be assumed to hold for the proofs. The multivariate t-distributions considered in the sequel however are an example of non-monotone score functions $\varphi_{f_\star}$ satisfying Assumption (A3).

Finally, the matrix $\mathbf{\Sigma}$ in practice is never known, and has to be estimated from the observations. We assume that a sequence of statistics $\widehat{\mathbf{\Sigma}}^{(n)}$ is available, with the following properties.

Assumption (A4). The sequence $\widehat{\mathbf{\Sigma}}^{(n)}$ is invariant under permutations and reflections with respect to the origin in $\mathbb{R}^k$ of the observations; $\sqrt{n}(\widehat{\mathbf{\Sigma}}^{(n)} - a\,\mathbf{\Sigma})$ is $O_{\rm P}(1)$ as $n \to \infty$ under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{\Sigma}, f)$ for some $a > 0$. The corresponding *pseudo-Mahalanobis distances* $(\mathbf{X}_i^{(n)\prime}(\widehat{\mathbf{\Sigma}}^{(n)})^{-1} \mathbf{X}_i^{(n)})^{1/2}$ are *quasi-affine-invariant* in the sense that, if $\mathbf{Y}_i^{(n)} = \mathbf{M}\mathbf{X}_i^{(n)}$ for all i, where $\mathbf{M}$ is an arbitrary non-singular $k \times k$ matrix, denoting by $\widehat{\mathbf{\Sigma}}_{\mathbf{X}}^{(n)}$ and $\widehat{\mathbf{\Sigma}}_{\mathbf{Y}}^{(n)}$ the estimators computed from $(\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)})$ and $(\mathbf{Y}_1^{(n)}, \ldots, \mathbf{Y}_n^{(n)})$, respectively, we have

$$(\mathbf{Y}_i^{(n)\prime}(\widehat{\mathbf{\Sigma}}_{\mathbf{Y}}^{(n)})^{-1}\mathbf{Y}_i^{(n)})^{1/2} = d \times (\mathbf{X}_i^{(n)\prime}(\widehat{\mathbf{\Sigma}}_{\mathbf{X}}^{(n)})^{-1}\mathbf{X}_i^{(n)})^{1/2},$$

for some positive scalar d that may depend on $\mathbf{M}$ and the sample $(\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)})$.

In the sequel, we write $\widehat{\mathbf{\Sigma}}$ and $\mathbf{X}_i$ instead of $\widehat{\mathbf{\Sigma}}^{(n)}$ and $\mathbf{X}_i^{(n)}$. Note that quasi-affine-invariance of pseudo-Mahalanobis distances implies strict affine-invariance of their ranks.

Under Assumption (A1′), $\underline{f}$ has finite second moments, and the empirical covariance matrix $n^{-1}\sum_{i=1}^n \mathbf{X}_i\mathbf{X}_i'$ satisfies Assumption (A4). The resulting pseudo-Mahalanobis distances are then equivalent to the classical ones, and are of course strictly affine-invariant. However, if (as in Assumption (A1)) no assumption is made about the moments of the radial density, the empirical covariance matrix may not be root-n consistent. Other affine-equivariant estimators of scatter then have to be considered, such as the one proposed by Tyler (1987). For the k-dimensional sample $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$, this estimator is defined as $\widehat{\mathbf{\Sigma}}^{(n)}_{\rm Tyl} := ((\mathbf{C}^{(n)}_{\rm Tyl})'\mathbf{C}^{(n)}_{\rm Tyl})^{-1}$, where $\mathbf{C}^{(n)}_{\rm Tyl}$ is the (unique for $n > k(k-1)$) upper triangular $k \times k$ matrix with positive diagonal elements and a "1" in the upper left corner that satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathbf{C}^{(n)}_{\rm Tyl}\mathbf{X}_i}{\|\mathbf{C}^{(n)}_{\rm Tyl}\mathbf{X}_i\|}\right)\left(\frac{\mathbf{C}^{(n)}_{\rm Tyl}\mathbf{X}_i}{\|\mathbf{C}^{(n)}_{\rm Tyl}\mathbf{X}_i\|}\right)' = \frac{1}{k}\,\mathbf{I}_k. \tag{3}$$

See Tyler (1987) and Randles (2000) for the invariance and consistency properties of this empirical measure of scatter. Unless otherwise stated, we always use this estimator in the sequel (under the notation $\widehat{\mathbf{\Sigma}}^{(n)}$ or $\widehat{\mathbf{\Sigma}}$) to compute pseudo-Mahalanobis distances.

2.2 Local asymptotic normality (LAN).

Writing $\mathbf{A}^{(n)}(L) := \mathbf{I}_k - \sum_{i=1}^{p} n^{-1/2}\mathbf{A}^{(n)}_i L^i$ and $\mathbf{B}^{(n)}(L) := \mathbf{I}_k + \sum_{i=1}^{q} n^{-1/2}\mathbf{B}^{(n)}_i L^i$, consider the sequence of experiments associated with the (sequence of) stochastic difference equations

$$\mathbf{A}^{(n)}(L)\,\mathbf{X}_t = \mathbf{B}^{(n)}(L)\,\boldsymbol{\varepsilon}_t,$$

where the parameter vector $\boldsymbol{\tau}^{(n)} := ((\mathrm{vec}\,\mathbf{A}^{(n)}_1)', \ldots, (\mathrm{vec}\,\mathbf{A}^{(n)}_p)', (\mathrm{vec}\,\mathbf{B}^{(n)}_1)', \ldots, (\mathrm{vec}\,\mathbf{B}^{(n)}_q)')' \in \mathbb{R}^{k^2(p+q)}$ is such that $\sup_n (\boldsymbol{\tau}^{(n)})'\boldsymbol{\tau}^{(n)} < \infty$. With the notation above, this sequence of parameters characterizes a sequence of local alternatives $\mathcal{H}^{(n)}(n^{-\frac{1}{2}}\boldsymbol{\tau}^{(n)}, \mathbf{\Sigma}, f)$.

Let $d_t(\mathbf{\Sigma}) = d^{(n)}_t(\mathbf{\Sigma}) := \|\mathbf{X}_t\|_{\mathbf{\Sigma}}$ and $\mathbf{U}_t(\mathbf{\Sigma}) = \mathbf{U}^{(n)}_t(\mathbf{\Sigma}) := \mathbf{\Sigma}^{-1/2}\mathbf{X}_t/d_t(\mathbf{\Sigma})$, where $\mathbf{\Sigma}^{-1/2}$ denotes an arbitrary symmetric square root of $\mathbf{\Sigma}^{-1}$. Writing $\varphi_{\underline{f}}$ for $-2\,(\mathbf{D}\underline{f}^{1/2})/\underline{f}^{1/2}$, where $\mathbf{D}\underline{f}^{1/2}$ denotes the quadratic mean gradient of $\underline{f}^{1/2}$, define, as in Garel and Hallin (1995), the *$\underline{f}$-cross covariance matrix* of lag i as

$$\begin{aligned}
\mathbf{\Gamma}^{(n)}_{i;\mathbf{\Sigma},f} &:= (n-i)^{-1}\sum_{t=i+1}^{n}\varphi_{\underline{f}}(\mathbf{X}_t)\,\mathbf{X}'_{t-i}\\
&= (n-i)^{-1}\,\mathbf{\Sigma}^{-1/2}\left(\sum_{t=i+1}^{n}\varphi_f(d_t(\mathbf{\Sigma}))\,d_{t-i}(\mathbf{\Sigma})\;\mathbf{U}_t(\mathbf{\Sigma})\mathbf{U}'_{t-i}(\mathbf{\Sigma})\right)\mathbf{\Sigma}^{1/2}.
\end{aligned}$$

The maximum lag we will need is $\pi := \max(p,q)$. Considering the $k^2\pi \times k^2(p+q)$ matrix

$$\mathbf{M} := \begin{pmatrix} \mathbf{I}_{k^2p} & \mathbf{I}_{k^2q} \\ \mathbf{0}_{k^2(\pi-p)\times(k^2p)} & \mathbf{0}_{k^2(\pi-q)\times(k^2q)} \end{pmatrix},$$

let $\mathbf{d}^{(n)} := \mathbf{M}\boldsymbol{\tau}^{(n)}$. Note that $\mathbf{d}^{(n)} = ((\mathrm{vec}\,\mathbf{D}^{(n)}_1)', \ldots, (\mathrm{vec}\,\mathbf{D}^{(n)}_\pi)')' \in \mathbb{R}^{k^2\pi}$, where

$$\mathbf{D}^{(n)}_i := \begin{cases} \mathbf{A}^{(n)}_i + \mathbf{B}^{(n)}_i & \text{if} \quad i = 1, \ldots, \min(p,q) \\ \mathbf{A}^{(n)}_i & \text{if} \quad i = q+1, \ldots, \pi \\ \mathbf{B}^{(n)}_i & \text{if} \quad i = p+1, \ldots, \pi. \end{cases}$$

When $\mathbf{A}^{(n)} = \mathbf{A}$, $\mathbf{B}^{(n)} = \mathbf{B}$, hence $\boldsymbol{\tau}^{(n)} = \boldsymbol{\tau}$, are constant sequences, we also write $\mathbf{D}_i$ instead of $\mathbf{D}^{(n)}_i$.

Local asymptotic normality, for given $\mathbf{\Sigma}$ and f, then takes the following form.

Proposition 1 *Assume that Assumptions (A1′) and (A2) hold. Then, the logarithm* $L^{(n)}_{n^{-\frac{1}{2}}\boldsymbol{\tau}^{(n)}/\mathbf{0};\boldsymbol{\Sigma},f}$ *of the likelihood ratio associated with the sequence of local alternatives* $\mathcal{H}^{(n)}(n^{-\frac{1}{2}}\boldsymbol{\tau}^{(n)},\boldsymbol{\Sigma},f)$ *with respect to* $\mathcal{H}^{(n)}(\mathbf{0},\boldsymbol{\Sigma},f)$ *is such that*

$$L^{(n)}_{n^{-\frac{1}{2}}\boldsymbol{\tau}^{(n)}/\mathbf{0};\boldsymbol{\Sigma},f}(\mathbf{X}) = (\mathbf{d}^{(n)})'\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f} - \frac{1}{2}(\mathbf{d}^{(n)})'\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f}\,\mathbf{d}^{(n)} + o_{\mathrm{P}}(1),$$

as $n\to\infty$*, under* $\mathcal{H}^{(n)}(\mathbf{0},\boldsymbol{\Sigma},f)$*, with the* central sequence

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f} := ((\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f})'_1,\ldots,(\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f})'_\pi)',$$

where

$$(\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f})_i := (n-i)^{1/2}\,\mathrm{vec}\,\boldsymbol{\Gamma}^{(n)}_{i;\boldsymbol{\Sigma},f},$$

and

$$\boldsymbol{\Gamma}^{(n)}_{\boldsymbol{\Sigma},f} := \frac{\mu_{k+1;f}\,\mathcal{I}_{k,f}}{k^2\,\mu_{k-1;f}}\;\mathbf{I}_\pi\otimes(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}^{-1})$$

Moreover, $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f}$*, still under* $\mathcal{H}^{(n)}(\mathbf{0},\boldsymbol{\Sigma},f)$*, is asymptotically* $\mathcal{N}_{k^2\pi}(\mathbf{0},\boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f})$*.*

Proof. The result is a very particular case of the LAN result in Garel and Hallin (1995). The required quadratic mean differentiability of $\mathbf{x}\mapsto f^{1/2}(\|\mathbf{x}\|)$ follows from Assumption (A2) (see Hallin and Paindaveine 2002). Note that the causality and invertibility conditions are trivially satisfied. One can also verify that LAN in this purely serial model only requires the finiteness of radial Fisher information, rather than the stronger condition $\int_0^1[\varphi_f(\tilde{F}_k^{-1}(u))]^4\,du<\infty$ as for the more general model considered in Garel and Hallin (1995), which includes a linear trend. □

2.3 Parametric optimality.

The above LAN property straightforwardly allows for building locally and asymptotically optimal testing procedures, under fixed $\boldsymbol{\Sigma}$ and f, for the problem under study. More precisely, the test that rejects the null hypothesis $\mathcal{H}^{(n)}(\mathbf{0},\boldsymbol{\Sigma},f_\star)$ whenever

$$\begin{aligned} Q^{\mathrm{par}}_{\boldsymbol{\Sigma},f_\star} &:= \boldsymbol{\Delta}^{(n)\prime}_{\boldsymbol{\Sigma},f_\star}\boldsymbol{\Gamma}^{-1}_{\boldsymbol{\Sigma},f_\star}\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f_\star} \\ &= \frac{k^2\,\mu_{k-1;f_\star}}{\mu_{k+1;f_\star}\,\mathcal{I}_{k,f_\star}}\sum_{i=1}^{\pi}(n-i)^{-1}\sum_{t,\tilde{t}=i+1}^{n}\varphi_{f_\star}(d_t(\boldsymbol{\Sigma}))\,\varphi_{f_\star}(d_{\tilde{t}}(\boldsymbol{\Sigma})) \\ &\qquad d_{t-i}(\boldsymbol{\Sigma})\,d_{\tilde{t}-i}(\boldsymbol{\Sigma})\,\mathbf{U}'_t(\boldsymbol{\Sigma})\mathbf{U}_{\tilde{t}}(\boldsymbol{\Sigma})\,\mathbf{U}'_{t-i}(\boldsymbol{\Sigma})\mathbf{U}_{\tilde{t}-i}(\boldsymbol{\Sigma}) \end{aligned} \tag{4}$$

exceeds the α-upper quantile $\chi^2_{k^2\pi,1-\alpha}$ of a chi-square distribution with $k^2\pi$ degrees of freedom, is locally asymptotically maximin, at asymptotic level α, for $\mathcal{H}^{(n)}(\mathbf{0},\boldsymbol{\Sigma},f_\star)$ against alternatives of the form $\bigcup_{\boldsymbol{\theta}\neq\mathbf{0}}\mathcal{H}^{(n)}(\boldsymbol{\theta},\boldsymbol{\Sigma},f_\star)$: see Le Cam (1986), Section 11.9.

This procedure is of course highly parametric; in particular, it is only valid if the underlying radial density is $f_\star$. This can be improved by replacing the exact asymptotic variance of $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f_\star}$ with an estimation. Consider, for instance, the Gaussian case $f_\star(z)=\phi(z):=\exp(-z^2/2)$, and the test statistic

$$Q^{\mathrm{par}}_{\mathcal{N}} := \boldsymbol{\Delta}^{(n)\prime}_{\mathbf{I}_k,\phi}\hat{\boldsymbol{\Gamma}}^{-1}_{\mathbf{I}_k,\phi}\boldsymbol{\Delta}^{(n)}_{\mathbf{I}_k,\phi}, \tag{5}$$

where

$$\hat{\boldsymbol{\Gamma}}_{\mathbf{I}_k,\phi} := \mathbf{I}_\pi\otimes\hat{\boldsymbol{\Gamma}}^{(1)}_{\mathbf{I}_k,\phi},$$

with

$$\hat{\mathbf{\Gamma}}^{(1)}_{\mathbf{I}_k,\phi} := (n-1)^{-1}\sum_{t=2}^{n} \text{vec}\,\left(\mathbf{X}_t\mathbf{X}'_{t-1}\right)\left(\text{vec}\,\left(\mathbf{X}_t\mathbf{X}'_{t-1}\right)\right)'.$$

Note that $Q^{\text{par}}_{\mathcal{N}}$ is affine-invariant. The ergodic theorem (see e.g., Hannan (1970), Theorem 2, p. 203) yields $\hat{\mathbf{\Gamma}}_{\mathbf{I}_k,\phi} = \mathbf{\Gamma}_{\mathbf{I}_k,\phi;f} + o_P(1)$ under $\mathcal{H}^{(n)}(\mathbf{0},\mathbf{I}_k,f)$, where, denoting by U a random variable uniform over $]0,1[$,

$$\mathbf{\Gamma}_{\mathbf{I}_k,\phi;f} := \frac{1}{k^2}\,\mathrm{E}^2[(\tilde{F}_k^{-1}(U))^2]\;\mathbf{I}_{k^2\pi}$$

is the asymptotic variance of $\mathbf{\Delta}^{(n)}_{\mathbf{I}_k,\phi}$ under $\mathcal{H}^{(n)}(\mathbf{0},\mathbf{I}_k,f)$, so that $Q^{\text{par}}_{\mathcal{N}}$ is asymptotically equivalent, under $\mathcal{H}^{(n)}(\mathbf{0},.,\phi)$ and under contiguous alternatives, to the Gaussian version of the parametric statistic (4).

The following result then is easy to derive.

Proposition 2 *Assume that Assumptions (A1′) and (A2) hold. Consider the test $\phi^{(n)}_{\mathcal{N}}$ that rejects the null hypothesis $\mathcal{H}^{(n)}(\mathbf{0},.,.)$ whenever $Q^{\text{par}}_{\mathcal{N}}$ exceeds the α-upper quantile $\chi^2_{k^2\pi,1-\alpha}$ of a chi-square distribution with $k^2\pi$ degrees of freedom. Then,*

(i) $Q^{\text{par}}_{\mathcal{N}}$ is asymptotically chi-square with $k^2\pi$ degrees of freedom under $\mathcal{H}^{(n)}(\mathbf{0},.,.)$, and asymptotically noncentral chi-square, still with $k^2\pi$ degrees of freedom but with noncentrality parameter

$$\frac{1}{k^2}\,\mathrm{E}^2[\tilde{F}_k^{-1}(U)\,\varphi_f(\tilde{F}_k^{-1}(U))]\sum_{i=1}^{\pi}\text{tr}\left(\mathbf{\Sigma}^{1/2}\,\mathbf{D}_i'\mathbf{\Sigma}^{-1}\mathbf{D}_i\mathbf{\Sigma}^{1/2}\right),$$

(here and in the sequel, U stands for a random variable uniform over $]0,1[$) under $\mathcal{H}^{(n)}(n^{-1/2}\boldsymbol{\tau},\mathbf{\Sigma},f)$.

(ii) The sequence of tests $\phi^{(n)}_{\mathcal{N}}$ has asymptotic level α.

(iii) The sequence of tests $\phi^{(n)}_{\mathcal{N}}$ is locally asymptotically maximin, at asymptotic level α, for $\mathcal{H}^{(n)}(\mathbf{0},.,.)$ against alternatives of the form $\bigcup_{\boldsymbol{\theta}\neq\mathbf{0}}\mathcal{H}^{(n)}(\boldsymbol{\theta},.,\phi)$.

3 Test statistics and their asymptotic distributions.

3.1 Group invariance, interdirections, and pseudo-Mahalanobis ranks.

In this short section, we briefly review the invariance features of the problem under study justifying the (invariant) statistics to be used in the sequel : interdirections, and pseudo-Mahalanobis ranks. Denote by $\mathbf{Z}_t(\mathbf{\Sigma}) = \mathbf{Z}^{(n)}_t(\mathbf{\Sigma}) := \mathbf{\Sigma}^{-1/2}\mathbf{X}_t$, $t=1,\ldots,n$ the standardized residuals associated with the null hypothesis of randomness. Under $\mathcal{H}^{(n)}(\mathbf{0},\mathbf{\Sigma},.)$, the vectors $\mathbf{U}_t(\mathbf{\Sigma}) = \mathbf{U}^{(n)}_t(\mathbf{\Sigma}) = \mathbf{Z}^{(n)}_t(\mathbf{\Sigma})/\|\mathbf{Z}^{(n)}_t(\mathbf{\Sigma})\|$ are independent and uniformly distributed over the unit sphere $\mathcal{S}^{k-1}$. The notation $\hat{\mathbf{Z}}_t$ will be used for the residuals $\mathbf{Z}^{(n)}_t(\widehat{\mathbf{\Sigma}}^{(n)})$ associated with the estimator $\widehat{\mathbf{\Sigma}}^{(n)}$ considered in (A4).

The interdirection $c^{(n)}_{t,\tilde{t}}$ associated with the pair $(\hat{\mathbf{Z}}_t,\hat{\mathbf{Z}}_{\tilde{t}})$ in the n-tuple of residuals $\hat{\mathbf{Z}}_1,\ldots,\hat{\mathbf{Z}}_n$ is defined (Randles 1989) as the number of hyperplanes in $\mathbb{R}^k$ passing through the origin and

$(k-1)$ out of the $(n-2)$ points $\hat{\mathbf{Z}}_1, \ldots, \hat{\mathbf{Z}}_{t-1}, \hat{\mathbf{Z}}_{t+1}, \ldots, \hat{\mathbf{Z}}_{\tilde{t}-1}, \hat{\mathbf{Z}}_{\tilde{t}+1}, \ldots, \hat{\mathbf{Z}}_n$, that are *separating* $\hat{\mathbf{Z}}_t$ and $\hat{\mathbf{Z}}_{\tilde{t}}$: obviously, $0 \leq c^{(n)}_{t,\tilde{t}} \leq \binom{n-2}{k-1}$. Interdirections are invariant under linear transformations, so that they indifferently can be computed from the residuals $\hat{\mathbf{Z}}_t$ as from the residuals $\mathbf{Z}_t(\mathbf{\Sigma})$, or from the observations $\mathbf{X}_t$ themselves. Finally, let $p_{t,\tilde{t}} = p^{(n)}_{t,\tilde{t}} := c^{(n)}_{t,\tilde{t}} / \binom{n-2}{k-1}$ for $t \neq \tilde{t}$, and $p_{t,t} := 0$.

Interdirections provide affine-invariant estimations of the Euclidean angles between the unobserved residuals $\mathbf{Z}_t(\mathbf{\Sigma})$, that is, they estimate the quantities $\pi^{-1} \arccos\left(\mathbf{U}'_t(\mathbf{\Sigma})\mathbf{U}_{\tilde{t}}(\mathbf{\Sigma})\right)$. The following consistency result is proved in Hallin and Paindaveine (2002), by using a U-statistic representation.

Lemma 1 *Let* $(\mathbf{X}_1, \mathbf{X}_2, \ldots)$ *be an i.i.d. process of* k*-variate random vectors with spherically symmetric density. For any fixed* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^k$*, denote by* $\alpha(\mathbf{v}, \mathbf{w}) := \arccos(\mathbf{v}'\mathbf{w}/(\|\mathbf{v}\|\|\mathbf{w}\|))$ *the angle between* $\mathbf{v}$ *and* $\mathbf{w}$*, and by* $c^{(n)}(\mathbf{v}, \mathbf{w})$ *the interdirection associated with* $\mathbf{v}$ *and* $\mathbf{w}$ *in the sample* $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$*. Then,* $c^{(n)}(\mathbf{v}, \mathbf{w})/\binom{n-2}{k-1}$ *converges in quadratic mean to* $\pi^{-1}\alpha(\mathbf{v}, \mathbf{w})$ *as* $n \to \infty$*.*

The ranks of pseudo-Mahalanobis distances between the observations and the origin in $\mathbb{R}^k$ are the other main tool used in this paper. Let $R_t(\mathbf{\Sigma}) = R^{(n)}_t(\mathbf{\Sigma})$ denote the rank of $d_t(\mathbf{\Sigma})$ among the distances $d_1(\mathbf{\Sigma}), \ldots, d_n(\mathbf{\Sigma})$; write $\hat{R}_t$ and $\hat{d}_t$ for $R_t(\widehat{\mathbf{\Sigma}})$ and $d_t(\widehat{\mathbf{\Sigma}})$, respectively, where $\widehat{\mathbf{\Sigma}}$ is the estimator considered in (A4). It will be convenient to refer to $\hat{R}_t$ as the *pseudo-Mahalanobis rank* of $\mathbf{X}_t$. The following result is proved in Peters and Randles (1990).

Lemma 2 (Peters and Randles 1990) *For all* $t \in \mathbb{N}$, $\left(\hat{R}_t - R_t(\mathbf{\Sigma})\right)/(n+1)$ *is* $o_{\mathrm{P}}(1)$ *as* $n \to \infty$*, under* $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{\Sigma}, .)$*.*

For each $\mathbf{\Sigma}$ and n, consider the group of transformations $\mathcal{G}^{(n)}_{\mathbf{\Sigma}} = \{\mathcal{g}^{(n)}_g\}$, acting on $(\mathbb{R}^k)^n$, such that $\mathcal{g}(\mathbf{X}_1, \ldots, \mathbf{X}_n) := (g(d_1(\mathbf{\Sigma}))\, \mathbf{\Sigma}^{1/2}\mathbf{U}_1(\mathbf{\Sigma}), \ldots, g(d_n(\mathbf{\Sigma}))\, \mathbf{\Sigma}^{1/2}\mathbf{U}_n(\mathbf{\Sigma}))$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing, and such that $g(0) = 0$ and $\lim_{r\to\infty} g(r) = \infty$. The group $\mathcal{G}^{(n)}_{\mathbf{\Sigma}}$ is a generating group for the submodel $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{\Sigma}, .)$. Interdirections clearly are invariant under the action of $\mathcal{G}^{(n)}_{\mathbf{\Sigma}}$, and so are the ranks $R_t(\mathbf{\Sigma})$. Lemma 2 thus entails the asymptotic invariance of the pseudo-Mahalanobis ranks $\hat{R}_t/(n+1)$.

Another group of interest is the group of affine transformations acting on $(\mathbb{R}^k)^n$—more precisely, the group $\mathcal{G}^{(n)} = \{\mathcal{g}^{(n)}_{\mathbf{M}}\}$, where $|\mathbf{M}| > 0$, and $\mathcal{g}_{\mathbf{M}}(\mathbf{X}_1, \ldots, \mathbf{X}_n) := (\mathbf{M}\mathbf{X}_1, \ldots, \mathbf{M}\mathbf{X}_n)$. This group of affine transformations is a generating group for the submodel $\mathcal{H}^{(n)}(\mathbf{0}, ., f)$; unlike the componentwise ranks considered in Hallin, Ingenbleek, and Puri (1989), interdirections and pseudo-Mahalanobis ranks clearly are invariant under this group (see Assumption (A4)).

For $k = 1$, interdirections (or more precisely, the cosines $\cos(\pi p_{t,\tilde{t}})$) reduce to signs, and pseudo-Mahalanobis ranks to the ranks of absolute values. The statistics we are considering in the next section thus are a multivariate generalization of the signed rank statistics of the serial type considered in Hallin and Puri (1991).

3.2 A class of statistics based on interdirections and pseudo-Mahalanobis ranks.

Denoting by K_1 and $K_2 :]0,1[\to \mathbb{R}$ two *score functions*, consider quadratic test statistics of the form

$$Q_K^{(n)} := \frac{k^2}{\mathrm{E}[K_1^2(U)]\mathrm{E}[K_2^2(U)]} \sum_{i=1}^{\pi} (n-i)^{-1} \sum_{t,\tilde{t}=i+1}^{n} K_1\Big(\frac{\hat{R}_t}{n+1}\Big) K_1\Big(\frac{\hat{R}_{\tilde{t}}}{n+1}\Big) \tag{6}$$

$$K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big) K_2\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big) \cos(\pi p_{t,\tilde{t}}) \cos(\pi p_{t-i,\tilde{t}-i}),$$

where U is uniform over $]0,1[$. The form of $Q_K^{(n)}$ of course is closely related to that of (4). Of course, exact variances $(\sigma_K^{(n)})^2$ can be substituted, in (6), for the asymptotic ones; see Hallin and Puri (1991, page 12) for an explicit form of $(\sigma_K^{(n)})^2$.

Specific choices of the scores K_1 and K_2 yield a variety of statistics generalizing some well-known univariate ones. If we let $K_1(u) = K_2(u) = 1$ for all u, (6) reduces to the quadratic "multivariate sign" test statistic

$$Q_S^{(n)} := k^2 \sum_{i=1}^{\pi} (n-i)^{-1} \sum_{t,\tilde{t}=i+1}^{n} \cos(\pi p_{t,\tilde{t}}) \cos(\pi p_{t-i,\tilde{t}-i}), \tag{7}$$

a serial version of Randles' multivariate sign test statistic (Randles 1989) but also a multivariate extension of the quadratic generalized runs statistics proposed in Dufour, Hallin, and Mizera (1998). Linear scores K_1, K_2 yield

$$Q_{SP}^{(n)} := \frac{9k^2}{(n+1)^4} \sum_{i=1}^{\pi} (n-i)^{-1} \sum_{t,\tilde{t}=i+1}^{n} \hat{R}_t \, \hat{R}_{\tilde{t}} \, \hat{R}_{t-i} \, \hat{R}_{\tilde{t}-i} \cos(\pi p_{t,\tilde{t}}) \cos(\pi p_{t-i,\tilde{t}-i}), \tag{8}$$

a multivariate version of the Spearman-autocorrelation type test statistics. This is the serial version of Peters and Randles' Wilcoxon-type test statistic (see Peters and Randles 1990).

When local asymptotic optimality is required under some specified radial density $f_\star$, the adequate score functions K_1 and K_2 are $K_1 = J_{k,f_\star} := \varphi_{f_\star} \circ \tilde{F}_{\star k}^{-1}$ and $K_2 = \tilde{F}_{\star k}^{-1}$, yielding

$$Q_{f_\star}^{(n)} := \frac{k^2 \, \mu_{k-1;f_\star}}{\mu_{k+1;f_\star} \mathcal{I}_{k,f_\star}} \sum_{i=1}^{\pi} (n-i)^{-1} \sum_{t,\tilde{t}=i+1}^{n} J_{k,f_\star}\Big(\frac{\hat{R}_t}{n+1}\Big) J_{k,f_\star}\Big(\frac{\hat{R}_{\tilde{t}}}{n+1}\Big) \tag{9}$$

$$\tilde{F}_{\star k}^{-1}\Big(\frac{\hat{R}_{t-i}}{n+1}\Big) \tilde{F}_{\star k}^{-1}\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big) \cos(\pi p_{t,\tilde{t}}) \cos(\pi p_{t-i,\tilde{t}-i}).$$

Particular cases are

- the van der Waerden test statistic, associated with Gaussian densities ($f_\star(r) := \phi(r)$),

$$Q_{vdW}^{(n)} := \sum_{i=1}^{\pi} (n-i)^{-1} \sum_{t,\tilde{t}=i+1}^{n} \sqrt{\Psi_k^{-1}\Big(\frac{\hat{R}_t}{n+1}\Big)} \sqrt{\Psi_k^{-1}\Big(\frac{\hat{R}_{\tilde{t}}}{n+1}\Big)} \tag{10}$$

$$\sqrt{\Psi_k^{-1}\Big(\frac{\hat{R}_{t-i}}{n+1}\Big)} \sqrt{\Psi_k^{-1}\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big)} \cos(\pi p_{t,\tilde{t}}) \cos(\pi p_{t-i,\tilde{t}-i}),$$

where Ψ_k stands for the chi-square distribution function with k degrees of freedom, or

– the Laplace statistic,

$$Q^{(n)}_{\mathcal{L}} := \frac{k}{k+1}\sum_{i=1}^{\pi}(n-i)^{-1}\sum_{t,\tilde{t}=i+1}^{n}\tilde{F}^{-1}_{\star k}\Big(\frac{\hat{R}_{t-i}}{n+1}\Big)\tilde{F}^{-1}_{\star k}\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big)\cos(\pi p_{t,\tilde{t}})\cos(\pi p_{t-i,\tilde{t}-i}), \quad (11)$$

associated with double-exponential radial densities ($f_\star(r) := \exp(-r)$; $\tilde{F}_{\star k}(r) = \frac{\Gamma(k,r)}{\Gamma(k)}$, where Γ stands for the incomplete gamma function, defined by $\Gamma(k,r) := \int_0^r s^{k-1}\exp(-s)\,ds$). Note that, as usual in time series problems, the Laplace statistic (11) does not coincide with the sign test statistic (7).

Before investigating the asymptotic behavior of (6) and describing the associated asymptotic tests, let us point out that, for small samples, $Q^{(n)}_K$ allows for particularly pleasant permutational procedures. Exact permutational critical values indeed can be obtained from enumerating the 2^n possible values $s_1\mathbf{X}_1,\ldots,s_n\mathbf{X}_n$ ($\mathbf{s} := (s_1,\ldots,s_n)\in\{-1,1\}^n$), which are equally probable under the null. What makes these exact procedures so pleasant is that the (at most) 2^n corresponding possible values of the test statistics can be based on a unique evaluation of the interdirections and the (pseudo-)Mahalanobis ranks. Indeed, denoting by $p_{t,\tilde{t}}(\mathbf{s})$ the interdirection associated with the pair $(s_t\mathbf{X}_t, s_{\tilde{t}}\mathbf{X}_{\tilde{t}})$ in the n-tuple $s_1\mathbf{X}_1,\ldots,s_n\mathbf{X}_n$, it can be easily checked that

$$\cos(\pi\, p_{t,\tilde{t}}(\mathbf{s})) = s_t\, s_{\tilde{t}}\, \cos(\pi\, p_{t,\tilde{t}}). \quad (12)$$

It follows that the value of the test statistic $Q^{(n)}_K(\mathbf{s})$ computed at $s_1\mathbf{X}_1,\ldots,s_n\mathbf{X}_n$ is simply

$$\frac{k^2}{\mathrm{E}[K_1^2(U)]\mathrm{E}[K_2^2(U)]}\sum_{i=1}^{\pi}(n-i)^{-1}\sum_{t,\tilde{t}=i+1}^{n} s_t\, s_{\tilde{t}}\, s_{t-i}\, s_{\tilde{t}-i}\; K_1\Big(\frac{\hat{R}_t}{n+1}\Big)K_1\Big(\frac{\hat{R}_{\tilde{t}}}{n+1}\Big)$$
$$K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big)K_2\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big)\cos(\pi p_{t,\tilde{t}})\cos(\pi p_{t-i,\tilde{t}-i}).$$

3.3 Asymptotic behavior of statistics based on interdirections and pseudo-Mahalanobis ranks.

We now turn to the asymptotic behavior of $Q^{(n)}_K$ as $n\to\infty$, both under the null hypothesis of randomness as under local alternatives of ARMA dependence. Proofs are given in the Appendix.

The following lemma provides an asymptotic representation result for $Q^{(n)}_K$.

Lemma 3 *Assume that (A1) through (A4) hold. Then, under* $\mathcal{H}^{(n)}(\mathbf{0},\mathbf{\Sigma},f)$,

$$Q^{(n)}_K = \tilde{Q}^{(n)}_{K;\mathbf{\Sigma},f} + o_{\mathrm{P}}(1)$$

as $n\to\infty$, *where*

$$\tilde{Q}^{(n)}_{K;\mathbf{\Sigma},f} := \frac{k^2}{\mathrm{E}[K_1^2(U)]\mathrm{E}[K_2^2(U)]}\sum_{i=1}^{\pi}(n-i)^{-1}\sum_{t,\tilde{t}=i+1}^{n} K_1(\tilde{F}_k(d_t(\mathbf{\Sigma})))\,K_1(\tilde{F}_k(d_{\tilde{t}}(\mathbf{\Sigma})))$$
$$\times K_2(\tilde{F}_k(d_{t-i}(\mathbf{\Sigma})))\,K_2(\tilde{F}_k(d_{\tilde{t}-i}(\mathbf{\Sigma})))\;\mathbf{U}'_t(\mathbf{\Sigma})\mathbf{U}_{\tilde{t}}(\mathbf{\Sigma})\,\mathbf{U}'_{t-i}(\mathbf{\Sigma})\mathbf{U}_{\tilde{t}-i}(\mathbf{\Sigma}).$$

Let $D_k(K;f) := \int_0^1 K(u)\, \tilde{F}_k^{-1}(u)\, du$ and $C_k(K;f) := \int_0^1 K(u)\, J_{k,f}(u)\, du$, where K denotes some score function defined over $]0,1[$. When K is a density over $\mathbb{R}_0^+$ rather than a score function, we write $D_k(f_1, f_2)$ and $C_k(f_1, f_2)$ for $D_k(\tilde{F}_{1k}^{-1}; f_2)$ and $C_k(J_{k,f_1}; f_2)$ respectively; for simplicity, we also write $C_k(f)$ and $D_k(f)$ instead of $C_k(f,f)$ and $D_k(f,f)$. We then have the following results.

Proposition 3 *Assume that (A1) through (A4) hold. Then, under $\mathcal{H}^{(n)}(\mathbf{0}, ., .)$, $Q_K^{(n)}$ is asymptotically chi-square with $k^2\pi$ degrees of freedom, as $n \to \infty$. Under $\mathcal{H}^{(n)}(n^{-1/2}\boldsymbol{\tau}, \boldsymbol{\Sigma}, f)$, and provided that (A1) is reinforced into (A1′), $Q_K^{(n)}$ is asymptotically noncentral chi-square, still with $k^2\pi$ degrees of freedom but with noncentrality parameter*

$$\frac{1}{k^2}\, \frac{D_k^2(K_2; f)}{\mathrm{E}[K_2^2(U)]}\, \frac{C_k^2(K_1; f)}{\mathrm{E}[K_1^2(U)]} \sum_{i=1}^{\pi} \mathrm{tr}\left(\boldsymbol{\Sigma}^{1/2}\mathbf{D}_i'\boldsymbol{\Sigma}^{-1}\mathbf{D}_i\boldsymbol{\Sigma}^{1/2}\right).$$

Proposition 4 *Assume that (A1) through (A4) hold. Consider the test $\phi_K^{(n)}$ (resp. $\phi_{f_\star}^{(n)}$) that rejects the null hypothesis $\mathcal{H}^{(n)}(\mathbf{0}, ., .)$ whenever $Q_K^{(n)}$ (resp. $Q_{f_\star}^{(n)}$) exceeds the $(1-\alpha)-$quantile $\chi^2_{k^2\pi, 1-\alpha}$ of a chi-square distribution with $k^2\pi$ degrees of freedom. Then,*

(i) the sequences of tests $\phi_K^{(n)}$ and $\phi_{f_\star}^{(n)}$ have asymptotic level α;

(ii) provided that (A1) is reinforced into (A1′), the sequence of tests $\phi_{f_\star}^{(n)}$ is locally asymptotically maximin, at asymptotic level α, for $\mathcal{H}^{(n)}(\mathbf{0}, ., .)$ against alternatives of the form $\bigcup_{\boldsymbol{\theta} \neq \mathbf{0}} \mathcal{H}^{(n)}(\boldsymbol{\theta}, ., f_\star)$.

4 Asymptotic performances.

4.1 Asymptotic relative efficiencies.

We now turn to asymptotic relative efficiencies of the tests $\phi_K^{(n)}$ with respect to the Gaussian test $\phi_{\mathcal{N}}^{(n)}$. For the sake of simplicity, we drop useless superscripts, writing ϕ_K, $\phi_{\mathcal{N}}$, etc. for $\phi_K^{(n)}$, $\phi_{\mathcal{N}}^{(n)}$, etc.

Proposition 5 *Assume that (A1′) through (A4) hold. Then, the asymptotic relative efficiency of ϕ_K with respect to the Gaussian test $\phi_{\mathcal{N}}$, under radial density f, is*

$$\mathrm{ARE}_{k,f}^{(\mathrm{ser})}(\phi_K/\phi_{\mathcal{N}}) = \frac{1}{k^2}\frac{D_k^2(K_2; f)}{\mathrm{E}[K_2^2(U)]}\,\frac{C_k^2(K_1; f)}{\mathrm{E}[K_1^2(U)]}.$$

For the $f_\star$-scores procedures, this yields

$$\mathrm{ARE}_{k,f}^{(\mathrm{ser})}(\phi_{f_\star}/\phi_{\mathcal{N}}) = \frac{1}{k^2}\frac{D_k^2(f_\star, f)}{D_k(f_\star)}\,\frac{C_k^2(f_\star, f)}{C_k(f_\star)}.$$

These ARE values directly follow as the ratios of the corresponding noncentrality parameters in the asymptotic distributions of ϕ_K ($\phi_{f_\star}$) and $\phi_{\mathcal{N}}$ under local alternatives (see Propositions 2 and 3).

4.2 A generalized Chernoff-Savage result.

Denote by $\mathrm{ARE}_{k,f}^{(\mathrm{loc})}(\phi_{f_\star}/\phi_{\mathcal{N}})$ the asymptotic relative efficiency, under radial density f, for the multivariate one-sample location problem, of the generalized signed-rank test associated with radial density $f_\star$, with respect to the corresponding Gaussian procedure $\phi_{\mathcal{N}}$, namely the Hotelling test T^2. Then Hallin and Paindaveine (2002) show that

$$\mathrm{ARE}_{k,f}^{(\mathrm{loc})}(\phi_{f_\star}/\phi_{\mathcal{N}}) = \frac{D_k(f)}{k^2}\frac{C_k^2(f_\star, f)}{C_k(f_\star)}.$$

It directly follows from the Cauchy-Schwarz inequality that

$$\mathrm{ARE}_{k,f}^{(\mathrm{ser})}(\phi_{f_\star}/\phi_{\mathcal{N}}) \leq \mathrm{ARE}_{k,f}^{(\mathrm{loc})}(\phi_{f_\star}/\phi_{\mathcal{N}}), \tag{13}$$

with equality iff the radial densities f and $f_\star$ are of the same density type, that is, iff $f(r) = \lambda f_\star(ar)$ for some $\lambda, a > 0$.

Like in the univariate case (see Hallin 1994), the van der Waerden procedure is uniformly more efficient than the Gaussian procedure. More precisely, we show the following generalization of the serial Chernoff-Savage result of Hallin (1994).

Proposition 6 *Denote by ϕ_{vdW} and $\phi_{\mathcal{N}}$ the van der Waerden test, based on the test statistic (10), and the Gaussian test based on (5), respectively. For any f satisfying Assumptions (A1′) and (A2),*

$$\mathrm{ARE}_{k,f}^{(\mathrm{ser})}(\phi_{vdW}/\phi_{\mathcal{N}}) \geq 1,$$

where equality holds if and only if f is normal.

Some numerical values of $\mathrm{ARE}_{k,f}^{(\mathrm{ser})}(\phi_{vdW}/\phi_{\mathcal{N}})$ are provided in Table 2, where it appears that the advantage of the van der Waerden procedure over the Gaussian parametric grows with the dimension k of the observation, and with the importance of the tails of underlying densities (an ARE value of 1.535 is reached for a 10-variate Student density with 3 degrees of freedom). Note that the multivariate extension of the Chernoff-Savage (1958) theorem presented in Hallin and Paindaveine (2002) for the location problem appears, via inequality (13), as a corollary of Proposition 6. Interestingly enough, although Proposition 6 is stronger than its location model counterpart, its proof is simpler than the direct proof of its (weaker) location model counterpart (see Appendix B for the proof).

4.3 A multivariate serial version of Hodges and Lehmann's ".864 result".

Although it is never optimal (there is no density $f_\star$ such that $Q_{f_\star}$ coincides with Q_{SP}), the Spearman-type procedure ϕ_{SP}, based on (8), exhibits excellent asymptotic efficiency properties, especially for relatively small dimensions k. To show this, we extend the ".856 result" of Hallin and Tribel (2000) (the serial analog of Hodges and Lehmann (1956)'s celebrated ".864 result") by computing, for any dimension k, the lower bound for the asymptotic relative efficiency of ϕ_{SP} with respect to the Gaussian procedure $\phi_{\mathcal{N}}$. More precisely, we prove the following result (see Appendix B for the proof).

Proposition 7 *Denote by* ϕ_{SP} *the Spearman procedure based on the test statistic (8). Then, denoting by* J_r *the first-kind Bessel function of order* r,

$$c(r) := \min\left\{x > 0 \;\middle|\; (\sqrt{x}\, J_r(x))' = 0\right\} = \min\left\{x > 0 \;\middle|\; x\,\frac{J_{r+1}(x)}{J_r(x)} = r + \frac{1}{2}\right\},$$

the lower bound for the asymptotic relative efficiency of ϕ_{SP} *with respect to* $\phi_{\mathcal{N}}$ *is*

$$\inf_f \mathrm{ARE}^{(\mathrm{ser})}_{k,f}(\phi_{SP}/\phi_{\mathcal{N}}) = \frac{9\left(2\,c^2(\sqrt{2k-1}/2) + k - 1\right)^4}{2^{10}\,k^2\,c^4(\sqrt{2k-1}/2)}, \tag{14}$$

where the infimum is taken over all radial densities f *satisfying Assumptions (A1′) and (A2).*

Some numerical values are presented in Table 1 and Figure 1, along with the corresponding bounds for the location model. Note that the sequence of lower bounds (14) is monotonically decreasing for $k \geq 2$; as the dimension k tends to infinity, it tends to $9/16 = 0.5625$.

The reader is referred to the proof of Proposition 7 in Appendix B for an explicit form, and a graph, of the densities achieving the infimum (cf. Figure 2).

4.4 Asymptotic performance under heavy-tailed densities.

The tests we are proposing can be expected to exhibit better performances than the traditional Gaussian ones under heavy-tailed densities. In order to evaluate the impact of heavy tails on asymptotic performances, we consider the particular case of a multivariate Student density with $\nu_\star$ degrees of freedom. Recall that a k-dimensional random vector $\mathbf{X}$ is multivariate Student with ν degrees of freedom if and only if there exist a vector $\boldsymbol{\mu} \in \mathbb{R}^k$ and a symmetric $k \times k$ positive definite matrix $\boldsymbol{\Sigma}$ such that the density of $\mathbf{X}$ can be written as

$$\frac{\Gamma((k+\nu)/2)}{(\pi\nu)^{k/2}\,\Gamma(\nu/2)}\,(\det\boldsymbol{\Sigma})^{-1/2} f_\nu(\|\mathbf{x} - \boldsymbol{\mu}\|_{\boldsymbol{\Sigma}}),$$

with $f_\nu(r) := (1 + r^2/\nu)^{-(k+\nu)/2}$. Fixing $\nu_\star > 2$, consider the test $\phi_{f_{\nu_\star}}$ associated with the radial density $f_{\nu_\star}$. Since $\varphi_{f_{\nu_\star}}(r) = (k+\nu_\star)r/(\nu_\star + r^2)$, and since the distribution of $\|\mathbf{X}\|^2/k$ under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{I}_k, f_{\nu_\star})$ is Fisher-Snedecor with k and $\nu_\star$ degrees of freedom, the test statistic $Q_{f_{\nu_\star}}$ is

$$\frac{(k+\nu_\star)(k+\nu_\star+2)(\nu_\star-2)}{\nu_\star}\sum_{i=1}^{\pi}(n-i)^{-1}\sum_{t,\tilde{t}=i+1}^{n}\frac{T_t}{\nu_\star + T_t^2}\,\frac{T_{\tilde{t}}}{\nu_\star + T_{\tilde{t}}^2}\,T_{t-i}\,T_{\tilde{t}-i}\cos(\pi p_{t,\tilde{t}})\cos(\pi p_{t-i,\tilde{t}-i})$$

where, denoting by $G_{k,\nu}$ the Fisher-Snedecor distribution function (k and ν degrees of freedom),

$$T_t := \sqrt{k\,G^{-1}_{k,\nu_\star}\Big(\frac{\hat{R}_t}{n+1}\Big)}.$$

Table 2 reports the AREs of the tests ϕ_{f_5}, ϕ_{f_8}, and $\phi_{f_{15}}$, as well as those of the van der Waerden tests ϕ_{vdW}, with respect to the Gaussian test $\phi_{\mathcal{N}}$, under k-variate Student densities with various degrees of freedom ν, including the Gaussian density obtained for $\nu = \infty$. Inspection of Table 2 reveals that $\phi_{f_{\nu_\star}}$ (respectively, ϕ_{vdW}), as expected, performs best when the underlying density itself is Student with $\nu_\star$ degrees of freedom (respectively, normal). In that case, the AREs for the serial and nonserial cases coincide. All tests however exhibit rather good performance, particularly under heavy tailed densities. Note that the van der Waerden test performs uniformly better than the Gaussian test, which provides an empirical confirmation of Proposition 6.

Since $D_k(f_{\nu_\star}) = k\nu_\star/(\nu_\star - 2)$ and $C_k(f_{\nu_\star}) = k(k+\nu_\star)/(k+\nu_\star+2)$, we obtain that

$$\mathrm{ARE}^{(\mathrm{ser})}_{k,f_{\nu_\star}}[\phi_{f_{\nu_\star}}/\phi_{\mathcal{N}}] = \frac{(k+\nu_\star)\,\nu_\star}{(k+\nu_\star+2)(\nu_\star-2)}, \tag{15}$$

a quantity that increases with k, and tends to $\nu_\star/(\nu_\star - 2)$ as $k \to \infty$. The advantage of $\phi_{f_{\nu_\star}}$ over the Gaussian test thus increases with the dimension k of the observations. Table 3 presents some of these limiting ARE values.

4.5 The multivariate sign and Spearman tests.

In view of their simplicity, the multivariate sign test against randomness (S) and the multivariate Spearman (SP) test, which are the serial counterparts of Randles' multivariate sign test and Peters and Randles' Wilcoxon-type multivariate signed-rank test, respectively, deserve special interest.

Table 4 provides the asymptotic relative efficiencies, still with respect to the Gaussian tests, and under the same densities as in Table 2, of these tests, based on the statistics (7), (8) and (11), respectively. Because of its relation to the sign test (S), the multivariate Laplace test (L) also has been included in this study.

For the multivariate sign test (S), the following closed-form expressions are obtained :

$$\mathrm{ARE}^{(\mathrm{ser})}_{k,f_\nu}[S/\phi_{\mathcal{N}}] = \frac{16}{k^2\,(\nu-1)^2}\left[\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right]^4, \tag{16}$$

$$\mathrm{ARE}^{(\mathrm{ser})}_{k,\phi}[S/\phi_{\mathcal{N}}] = \frac{4}{k^2}\left[\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right]^4. \tag{17}$$

The asymptotic relative efficiencies of the corresponding one-sample location tests with respect to Hotelling's test, namely Randles' multivariate sign test ($\mathrm{S}^{(\mathrm{loc})}$) and Peters and Randles' Wilcoxon-type multivariate signed-rank test (W), are also provided in Table 4, thus allowing for a comparison between the serial and the nonserial cases.

5 Simulations.

The following Monte-Carlo experiment was conducted in order to investigate the finite-sample behavior of the tests proposed in Section 3 for $k = 2$: $N = 2,500$ independent samples $(\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_{400})$ of size $n = 400$ have been generated from bivariate standard Student densities with 3, 8, and 15 degrees of freedom, and from the bivariate standard normal distribution. The simulation of bivariate Student variables $\boldsymbol{\varepsilon}_i$ was based on the fact that (for ν degrees of freedom; $=_{\mathrm{d}}$ stands for equality in distribution) $\boldsymbol{\varepsilon}_i =_{\mathrm{d}} \mathbf{Z}_i/\sqrt{Y_i/\nu}$, where $\mathbf{Z}_i \sim \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ and $Y_i \sim \chi^2_\nu$ are independent. Autoregressive alternatives of the form

$$\mathbf{X}_t - (m\mathbf{A})\,\mathbf{X}_{t-1} = \boldsymbol{\varepsilon}_t, \quad \mathbf{X}_0 = \mathbf{0}, \tag{18}$$

were considered, with $\mathbf{A} = \begin{pmatrix} 0.05 & 0.02 \\ -0.01 & 0.04 \end{pmatrix}$, and $m = 0, 1, 2, 3$. For each replication, the following seven tests were performed at nominal asymptotic probability level $\alpha = 5\%$: the Gaussian test $\phi_{\mathcal{N}}$, ϕ_{f_5}, ϕ_{f_8}, $\phi_{f_{15}}$, ϕ_{vdW}, the sign test for randomness (S), and the Spearman type test (SP). Tyler's estimator of scatter was used whenever pseudo-Mahalanobis ranks had to be computed. The estimator was obtained via the iterative scheme described in Randles (2000). Iterations were stopped as soon as the Frobenius distance between the left and right hand sides of (3) fell below 10^{-6}.

Rejection frequencies are reported in Table 5. Note that the corresponding standard errors are (for $N = 2,500$ replications) .0044, .0080, and .0100 for frequencies (size or power) of the order of $p = .05$ ($p = .95$), $p = .20$ ($p = .80$), and $p = .50$, respectively.

All tests apparently satisfy the 5% probability level constraint (a 95%-confidence interval has approximate half-length .01). Power rankings are essentially consistent with the ARE values given in Tables 2 and 4. For instance, under Gaussian densities, the powers of the $\phi_{f_{\nu_\star}}$ tests are increasing with $\nu_\star$, as expected, whereas the asymptotic optimality of $\phi_{f_{\nu_\star}}$ under the Student distribution with $\nu_\star$ degrees of freedom is confirmed.

6 Appendix : proofs.

6.1 Appendix A : Proofs of Section 3.

The main task here consists in proving Lemma 3. Let us first establish the following "serial" result about interdirections.

Lemma 4 *Let $i \in \{1, \ldots, \pi\}$ and $s, \tilde{s}, t, \tilde{t} \in \{i+1, \ldots, n\}$ be such that at least one out of the eight indices $t-i$, $\tilde{t}-i$, t, $\tilde{t}$, $s-i$, $\tilde{s}-i$, s, and $\tilde{s}$ is distinct of the seven other ones. Let $g : \mathbf{X} \mapsto g(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be even in all its arguments. Then, letting*

$$C_{t,\tilde{t};i} := \cos(\pi p_{t,\tilde{t}}) \cos(\pi p_{t-i,\tilde{t}-i}) - \mathbf{U}'_t(\mathbf{I}_k)\mathbf{U}_{\tilde{t}}(\mathbf{I}_k)\, \mathbf{U}'_{t-i}(\mathbf{I}_k)\mathbf{U}_{\tilde{t}-i}(\mathbf{I}_k), \tag{19}$$

we have $\mathrm{E}[g(\mathbf{X})\, C_{s,\tilde{s};i}\, C_{t,\tilde{t};i}] = 0$ under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{I}_k, .)$, provided that this expectation exists. Similarly, defining

$$D_{t,\tilde{t};i} := \mathbf{U}'_t(\mathbf{I}_k)\mathbf{U}_{\tilde{t}}(\mathbf{I}_k)\, \mathbf{U}'_{t-i}(\mathbf{I}_k)\mathbf{U}_{\tilde{t}-i}(\mathbf{I}_k),$$

$\mathrm{E}[g(\mathbf{X})\, D_{t,\tilde{t};i}] = 0$ *for $t \neq \tilde{t}$ under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{I}_k, .)$, provided that this expectation exists*

Proof. Define $b_t := \mathrm{sgn}(X_{t1})$ and $\mathbf{Y}_t := b_t \mathbf{X}_t$, where $\mathrm{sgn}(z) := I[z > 0] - I[z < 0]$ stands for the sign function and X_{t1} denotes the first component of $\mathbf{X}_t$. Under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{I}_k, .)$, the b_t's are i.i.d. Bernoulli with $P[b_t = \pm 1] = 1/2$, and are independent of the $\mathbf{Y}_t$'s. Denote by $p^Y_{t,\tilde{t}}$ the interdirection associated with $(\mathbf{Y}_t, \mathbf{Y}_{\tilde{t}})$ within the $\mathbf{Y}$-sample; also write $\mathbf{U}_t$ and $\mathbf{U}^Y_t$ for $\mathbf{U}_t(\mathbf{I}_k) = \mathbf{X}_t/\|\mathbf{X}_t\|$ and $\mathbf{Y}_t/\|\mathbf{Y}_t\|$, respectively. Finally, let $C^Y_{t,\tilde{t};i} := \cos(\pi p^Y_{t,\tilde{t}}) \cos(\pi p^Y_{t-i,\tilde{t}-i}) - (\mathbf{U}^Y_t)'\mathbf{U}^Y_{\tilde{t}}\, (\mathbf{U}^Y_{t-i})'\mathbf{U}^Y_{\tilde{t}-i}$. Without loss of generality, suppose that t is distinct of $t-i$, $\tilde{t}-i$, $\tilde{t}$, $s-i$, $\tilde{s}-i$, s, and $\tilde{s}$. Then, using the fact that $g(\mathbf{X})$ is a function of the $\mathbf{Y}_m$'s,

$$\begin{aligned} &\mathrm{E}\left[g(\mathbf{X})\, C_{s,\tilde{s};i}\, C_{t,\tilde{t};i} \,\middle|\, \mathbf{X}_1, \ldots, \mathbf{X}_{t-1}, \mathbf{Y}_t, \mathbf{X}_{t+1}, \ldots, \mathbf{X}_n\right] \\ &\qquad = g(\mathbf{X})\, b_s\, b_{\tilde{s}}\, b_{s-i}\, b_{\tilde{s}-i}\, C^Y_{s,\tilde{s};i}\; b_{\tilde{t}}\, b_{t-i}\, b_{\tilde{t}-i}\, C^Y_{t,\tilde{t};i}\, \mathrm{E}\left[b_t \,\middle|\, \mathbf{Y}_t\right] = 0, \end{aligned}$$

in view of the symmetry properties (12) of interdirections. The second assertion is proved in the same way.

□

We are now ready to prove Lemma 3.

Proof of Lemma 3. Without loss of generality, we may assume that $\mathbf{\Sigma} = \mathbf{I}_k$. In this proof, we will write d_t, R_t and $\mathbf{U}_t$ for $d_t(\mathbf{I}_k)$, $R_t(\mathbf{I}_k)$ and $\mathbf{U}_t(\mathbf{I}_k)$, respectively. Decompose $Q_K^{(n)} - \tilde{Q}_{K;\mathbf{I}_k,f}^{(n)}$ into

$$\frac{k^2}{\mathrm{E}[K_1^2(U)]\mathrm{E}[K_2^2(U)]}(\sum_{i=1}^{\pi} T_{1;i}^{(n)} + T_2^{(n)}),$$

where

$$T_{1;i}^{(n)} := (n-i)^{-1} \sum_{t,\tilde{t}=i+1}^{n} K_1\Big(\frac{\hat{R}_t}{n+1}\Big) K_1\Big(\frac{\hat{R}_{\tilde{t}}}{n+1}\Big) K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big) K_2\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big) \times \Big(\cos(\pi p_{t,\tilde{t}})\cos(\pi p_{t-i,\tilde{t}-i}) - \mathbf{U}_t'\mathbf{U}_{\tilde{t}}\,\mathbf{U}_{t-i}'\mathbf{U}_{\tilde{t}-i}\Big),$$

and

$$T_2^{(n)} := \sum_{i=1}^{\pi} (n-i)^{-1} \sum_{t,\tilde{t}=i+1}^{n} \Bigg(K_1\Big(\frac{\hat{R}_t}{n+1}\Big) K_1\Big(\frac{\hat{R}_{\tilde{t}}}{n+1}\Big) K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big) K_2\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big) - K_1(\tilde{F}_k(d_t))\, K_1(\tilde{F}_k(d_{\tilde{t}})) K_2(\tilde{F}_k(d_{t-i})) K_2(\tilde{F}_k(d_{\tilde{t}-i}))\Bigg) \mathbf{U}_t'\mathbf{U}_{\tilde{t}}\,\mathbf{U}_{t-i}'\mathbf{U}_{\tilde{t}-i}.$$

Let us show that, under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{I}_k, f)$ (throughout this proof, all convergences and mathematical expectations are taken under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{I}_k, f)$), there exists $s > 0$ such that $T_{1;i}^{(n)}$ and $T_2^{(n)} \stackrel{L^s}{\to} 0$ for all i as $n \to \infty$. Slutzky's classical argument then concludes the proof.

Let us start with $T_2^{(n)}$. Define

$$\begin{aligned}
\mathbf{T}_{K;f}^{(n)} &:= ((\mathbf{T}_{K;f}^{(n)})_1', \ldots, (\mathbf{T}_{K;f}^{(n)})_\pi')', \\
\mathbf{S}_K^{(n)} &:= ((\mathbf{S}_K^{(n)})_1', \ldots, (\mathbf{S}_K^{(n)})_\pi')', \\
\hat{\mathbf{S}}_K^{(n)} &:= ((\hat{\mathbf{S}}_K^{(n)})_1', \ldots, (\hat{\mathbf{S}}_K^{(n)})_\pi')',
\end{aligned}$$

where

$$\begin{aligned}
(\mathbf{T}_{K;f}^{(n)})_i &:= (n-i)^{-1/2} \sum_{t=i+1}^{n} \mathrm{vec}\,\Big(K_1(\tilde{F}_k(d_t))K_2(\tilde{F}_k(d_{t-i}))\,\mathbf{U}_t\mathbf{U}_{t-i}'\Big), \\
(\mathbf{S}_K^{(n)})_i &:= (n-i)^{-1/2} \sum_{t=i+1}^{n} \mathrm{vec}\,\Big(K_1\Big(\frac{R_t}{n+1}\Big)K_2\Big(\frac{R_{t-i}}{n+1}\Big)\,\mathbf{U}_t\mathbf{U}_{t-i}'\Big),
\end{aligned}$$

and

$$(\hat{\mathbf{S}}_K^{(n)})_i := (n-i)^{-1/2} \sum_{t=i+1}^{n} \mathrm{vec}\,\Big(K_1\Big(\frac{\hat{R}_t}{n+1}\Big)K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big)\,\mathbf{U}_t\mathbf{U}_{t-i}'\Big).$$

Note that

$$\Big\|\mathbf{S}_K^{(n)} - \mathbf{T}_{K;f}^{(n)}\Big\|_{L^2}^2 = \sum_{i=1}^{\pi}\sum_{t=i+1}^{n} (c_{t;i}^{(n)})^2 \mathrm{E}\Bigg[\Big(K_1\Big(\frac{R_t}{n+1}\Big)K_2\Big(\frac{R_{t-i}}{n+1}\Big) - K_1(\tilde{F}_k(d_t))K_2(\tilde{F}_k(d_{t-i}))\Big)^2\Bigg],$$

where $c_{t;i}^{(n)} = (n-i)^{-1/2}$ for all $t = i+1, \ldots, n$. Proposition 2.1 in Hallin and Puri (1991) thus implies that $\|\mathbf{S}_K^{(n)} - \mathbf{T}_{K;f}^{(n)}\|_{L^2} = o(1)$ as $n \to \infty$—incidentally, the same result also implies that, for all $i = 1, \ldots, \pi$ and for all $t = i+1, \ldots, n$,

$$\mathrm{E}\left[\left(K_1\Big(\frac{R_t}{n+1}\Big)K_2\Big(\frac{R_{t-i}}{n+1}\Big) - K_1(\tilde{F}_k(d_t))K_2(\tilde{F}_k(d_{t-i}))\right)^2\right] = o(1) \tag{20}$$

as $n \to \infty$. Using Lemma 4, we obtain similarly

$$\left\|\hat{\mathbf{S}}_K^{(n)} - \mathbf{S}_K^{(n)}\right\|_{L^2}^2 = \sum_{i=1}^{\pi}\sum_{t=i+1}^{n}(n-i)^{-1}\mathrm{E}\left[\left(K_1\Big(\frac{\hat{R}_t}{n+1}\Big)K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big) - K_1\Big(\frac{R_t}{n+1}\Big)K_2\Big(\frac{R_{t-i}}{n+1}\Big)\right)^2\right].$$

Consequently, $\left\|\hat{\mathbf{S}}_K^{(n)} - \mathbf{S}_K^{(n)}\right\|_{L^2}$ is $o(1)$ if

$$K_1\Big(\frac{\hat{R}_t}{n+1}\Big)K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big) - K_1\Big(\frac{R_t}{n+1}\Big)K_2\Big(\frac{R_{t-i}}{n+1}\Big) \stackrel{L^2}{\to} 0, \quad \text{as } n \to \infty. \tag{21}$$

Lemma 2 establishes the same convergence as in (21), but in probability. We have seen above that $K_1(R_t/(n+1))K_2(R_{t-i}/(n+1)) - K_1(\tilde{F}_k(d_t))K_2(\tilde{F}_k(d_{t-i}))$ tends to zero in quadratic mean, so that $[K_1(R_t/(n+1))K_2(R_{t-i}/(n+1))]^2$ is uniformly integrable. In view of Assumption (A4), the same conclusion holds for $[K_1(\hat{R}_t/(n+1))K_2(\hat{R}_{t-i}/(n+1))]^2$, and (21) follows. Consequently, $\|\hat{\mathbf{S}}_K^{(n)} - \mathbf{T}_{K;f}^{(n)}\|_{L^2} = o(1)$ as $n \to \infty$.

On the other hand, $\|\mathbf{T}_{K;f}^{(n)}\|_{L^2}^2 = \pi\,\mathrm{E}[K_1^2(U)]\,\mathrm{E}[K_2^2(U)]$ for all n, so that the sequence $\|\hat{\mathbf{S}}_K^{(n)}\|_{L^2}$ is bounded. Finally, in view of Cauchy-Schwarz,

$$\begin{aligned}\|T_2^{(n)}\|_{L^1} &= \|(\hat{\mathbf{S}}_K^{(n)})'\hat{\mathbf{S}}_K^{(n)} - (\mathbf{T}_{K;f}^{(n)})'\mathbf{T}_{K;f}^{(n)}\|_{L^1}\\ &\leq \|\hat{\mathbf{S}}_K^{(n)} + \mathbf{T}_{K;f}^{(n)}\|_{L^2}\|\hat{\mathbf{S}}_K^{(n)} - \mathbf{T}_{K;f}^{(n)}\|_{L^2} = o(1) \quad \text{as } n \to \infty.\end{aligned}$$

Turning to $T_{1;i}^{(n)}$, write $T_{1;i}^{(n)} = (n-i)^{-1}\sum_{t,\tilde{t}=i+1}^{n} r_{t,\tilde{t};i}$, where

$$r_{t,\tilde{t};i} := K_1\Big(\frac{\hat{R}_t}{n+1}\Big)K_1\Big(\frac{\hat{R}_{\tilde{t}}}{n+1}\Big)K_2\Big(\frac{\hat{R}_{t-i}}{n+1}\Big)K_2\Big(\frac{\hat{R}_{\tilde{t}-i}}{n+1}\Big)\,C_{t,\tilde{t};i}$$

and $C_{t,\tilde{t};i}$ is defined in (19). It follows from Lemma 4 that

$$\begin{aligned}\|(n-i)T_{1;i}^{(n)}\|_{L^2}^2 &= \mathrm{E}\left[\left(\sum_{t,\tilde{t}=i+1}^{n} r_{t,\tilde{t};i}\right)^2\right] = 4\,\mathrm{E}\left[\left(\sum_{\substack{t,\tilde{t}=i+1\\ t<\tilde{t}}}^{n} r_{t,\tilde{t};i}^2\right) + \left(\sum_{\substack{s,\tilde{s}=i+1\\ s<\tilde{s}}}^{n}\ \sum_{\substack{t,\tilde{t}=i+1\\ t<\tilde{t},(s,\tilde{s})\neq(t,\tilde{t})}}^{n} r_{s,\tilde{s};i}\, r_{t,\tilde{t};i}\right)\right]\\ &= 4\left[\left(\frac{(n-i)(n-i-1)}{2} - (n-2i)\right)\mathrm{E}[r_{i+1,3i+1;i}^2] + (n-2i)\,\mathrm{E}[r_{i+1,2i+1;i}^2]\right],\end{aligned}$$

so that it is sufficient to prove that

$$\mathrm{E}[r_{i+1,3i+1;i}^2] = o(1) \quad \text{and} \quad \mathrm{E}[r_{i+1,2i+1;i}^2] = o(n) \tag{22}$$

as $n \to \infty$. Writing $K_{\ell;t}$ for $K_\ell(\frac{\hat{R}_t}{n+1})$ ($\ell = 1,2$), Hölder's inequality yields

$$\mathrm{E}[r^2_{i+1,3i+1;i}] \leq (\mathrm{E}\big[|K_{1;i+1}K_{1;3i+1}K_{2;1}K_{2;2i+1}|^{2+\delta}\big])^{2/(2+\delta)}\,(\mathrm{E}\big[|C_{i+1,3i+1;i}|^{2(2+\delta)/\delta}\big])^{\delta/(2+\delta)},$$

and

$$\mathrm{E}[r^2_{i+1,2i+1;i}] \leq (\mathrm{E}\big[|K_{1;i+1}K_{1;2i+1}K_{2;1}K_{2;i+1}|^{2+\delta}\big])^{2/(2+\delta)}\,(\mathrm{E}\big[|C_{i+1,2i+1;i}|^{2(2+\delta)/\delta}\big])^{\delta/(2+\delta)},$$

where $\delta > 0$ is as in Assumption (A3). Now, Lemma 1 and the boundedness of $C_{i+1,3i+1;i}$ yield that $\mathrm{E}\big[|C_{i+1,3i+1;i}|^{2(2+\delta)/\delta}\big] = o(1)$ as $n \to \infty$. On the other hand, since the $\hat{R}_t$'s are the ranks of an exchangeable n-tuple (see Assumption (A4)), we obtain that

$$\begin{aligned}
&\frac{n(n-1)(n-2)(n-3)}{(n+1)^4}\,\mathrm{E}\big[|K_{1;i+1}K_{1;3i+1}K_{2;1}K_{2;2i+1}|^{2+\delta}\big]\\
&= \frac{1}{(n+1)^4}\sum_{\substack{j_1,j_2,j_3,j_4=1\\ \text{all}\neq}}^{n}\left|K_1\left(\frac{j_1}{n+1}\right)K_1\left(\frac{j_2}{n+1}\right)K_2\left(\frac{j_3}{n+1}\right)K_2\left(\frac{j_4}{n+1}\right)\right|^{2+\delta}\\
&\leq \left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left|K_1\left(\frac{j}{n+1}\right)\right|^{2+\delta}\right)^2\left(\frac{1}{n+1}\sum_{j=1}^{n+1}\left|K_2\left(\frac{j}{n+1}\right)\right|^{2+\delta}\right)^2 = O(1) \qquad (23)
\end{aligned}$$

as $n \to \infty$. Indeed, the two sums in the upper bound (23) are Riemann sums, for $\int_0^1 |K_1(u)|^{2+\delta}\,du$ and $\int_0^1 |K_2(u)|^{2+\delta}\,du$, respectively, and these two integrals are finite from Assumption (A3). Consequently, $\mathrm{E}\big[|K_{1;i+1}K_{1;3i+1}K_{2;1}K_{2;2i+1}|^{2+\delta}\big] = O(1)$ as $n \to \infty$. Working along the same lines, one can show that $\mathrm{E}\big[|C_{i+1,2i+1;i}|^{2(2+\delta)/\delta}\big] = o(1)$ and $\mathrm{E}\big[|K_{1;i+1}K_{1;2i+1}K_{2;1}K_{2;i+1}|^{2+\delta}\big] = O(n)$ as $n \to \infty$; (22) follows.

□

Proof of Proposition 3. From Lemma 3, we have, under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{\Sigma}, f)$,

$$Q^{(n)}_K = (\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f})'\,(\mathbf{\Gamma}_{K;\mathbf{\Sigma},f})^{-1}\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f} + o^{(n)}_{\mathrm{P}}(1) = \tilde{Q}^{(n)}_{K;\mathbf{\Sigma},f} + o^{(n)}_{\mathrm{P}}(1),$$

where

$$\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f} := ((\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f})'_1, \ldots, (\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f})'_\pi)',$$

$$(\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f})_i := (n-i)^{-1/2}\mathrm{vec}\left(\sum_{t=i+1}^{n} K_1(\tilde{F}_k(d_t(\mathbf{\Sigma})))K_2(\tilde{F}_k(d_{t-i}(\mathbf{\Sigma})))\,\mathbf{\Sigma}^{-1/2}\mathbf{U}_t(\mathbf{\Sigma})\mathbf{U}'_{t-i}(\mathbf{\Sigma})\mathbf{\Sigma}^{1/2}\right),$$

and

$$\mathbf{\Gamma}_{K;\mathbf{\Sigma},f} := \frac{1}{k^2}\,\mathrm{E}[K_1^2(U)]\mathrm{E}[K_2^2(U)]\;\mathbf{I}_\pi \otimes (\mathbf{\Sigma} \otimes \mathbf{\Sigma}^{-1}).$$

The proof of the first part of Proposition 3 follows, since $\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f}$ under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{\Sigma}, f)$ is asymptotically $\mathcal{N}_{k^2\pi}(\mathbf{0}, \mathbf{\Gamma}_{K;\mathbf{\Sigma},f})$.

It is also easy to see that, still under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{\Sigma}, f)$, $\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f}$ and the local log-likelihood $L^{(n)}_{n^{-\frac{1}{2}}\boldsymbol{\tau}/\mathbf{0};\mathbf{\Sigma},f}$ are jointly multinormal, with asymptotic covariance

$$\frac{1}{k^2}\,D_k(K_2; f)\,C_k(K_1; f)\,[\mathbf{I}_\pi \otimes (\mathbf{\Sigma} \otimes \mathbf{\Sigma}^{-1})]\,\mathbf{M}\,\boldsymbol{\tau};$$

Le Cam's third Lemma thus implies that $\mathbf{T}^{(n)}_{K;\mathbf{\Sigma},f}$ under $\mathcal{H}^{(n)}(n^{-\frac{1}{2}}\boldsymbol{\tau},\mathbf{\Sigma},f)$ is asymptotically $\mathcal{N}_{k^2\pi}(\frac{1}{k^2}\,D_k(K_2;f)\,C_k(K_1;f)\,[\mathbf{I}_\pi\otimes(\mathbf{\Sigma}\otimes\mathbf{\Sigma}^{-1})]\,\mathbf{M}\boldsymbol{\tau},\mathbf{\Gamma}_{K;\mathbf{\Sigma},f})$. This establishes the second part of Proposition 3. □

6.2 Appendix B : Pitman non-admissibility of correlogram-based methods and lower bounds for the efficiency of Spearman procedures.

Proof of Proposition 6. The asymptotic relative efficiency of the van der Waerden test, with respect to the Gaussian procedure, under radial density f, is

$$\text{ARE}^{(\text{ser})}_{k,f}(\phi_{vdW}/\phi_{\mathcal{N}})=\frac{1}{k^4}\,D_k^2(\phi,f)\,E^2[\tilde{\Phi}_k^{-1}(U)\,J_{k,f}(U)],$$

where, letting $\phi(r):=\exp(-r^2/2)$, $\tilde{\Phi}_k$ stands for the distribution function associated with $\tilde{\phi}_k(r):=(\mu_{k-1;\phi})^{-1}r^{k-1}\phi(r)\,I_{[r>0]}$. Without loss of generality, we restrict ourselves to the radial densities f satisfying $D_k(\phi,f)=\text{E}[\tilde{\Phi}_k^{-1}(U)\,\tilde{F}_k^{-1}(U)]=k$. Indeed, writing $f_a(r):=f(ar)$, $a>0$, we have $\tilde{F}_{ak}^{-1}(u)=a^{-1}\tilde{F}_k^{-1}(u)$ and $\varphi_{f_a}(r)=a\,\varphi_f(ar)$, so that $D_k(\phi,f_a)=a^{-1}D_k(\phi,f)$ and $\text{ARE}^{(\text{ser})}_{k,f_a}(\phi_{vdW}/\phi_{\mathcal{N}})=\text{ARE}^{(\text{ser})}_{k,f}(\phi_{vdW}/\phi_{\mathcal{N}})$.

Thus, we only have to show that, for any $k\in\mathbb{N}_0$ and any f such that $D_k(\phi,f)=k$,

$$H_k(f):=\text{E}[\tilde{\Phi}_k^{-1}(U)\,J_{k,f}(U)]\geq k,$$

with equality at $f=\phi$ only. This variational problem takes a simpler form after the following change of notation. First rewrite the functional H as

$$\begin{aligned}
H_k(f) &= \int_0^\infty \tilde{\Phi}_k^{-1}(\tilde{F}_k(r))\,\varphi_f(r)\,\tilde{f}_k(r)\,dr\\
&= \frac{1}{\mu_{k-1;f}}\int_0^\infty \tilde{\Phi}_k^{-1}(\tilde{F}_k(r))\,(-f'(r))\,r^{k-1}\,dr\\
&= \int_0^\infty \Big[\frac{1}{\tilde{\phi}_k(\tilde{\Phi}_k^{-1}(\tilde{F}_k(r)))}\,\tilde{f}_k(r)+\frac{k-1}{r}\tilde{\Phi}_k^{-1}(\tilde{F}_k(r))\Big]\tilde{f}_k(r)\,dr.
\end{aligned}$$

For any radial density f satisfying Assumption (A1), the function $R: z\mapsto \tilde{F}_k^{-1}\circ\tilde{\Phi}_k(z)$ and its inverse $R^{-1}: r\mapsto\tilde{\Phi}_k^{-1}\circ\tilde{F}_k(r)$ are continuous monotone increasing transformations, mapping $\mathbb{R}_0^+$ onto itself, and satisfying $\lim_{z\downarrow 0}R(z)=\lim_{r\downarrow 0}R^{-1}(r)=0$ and $\lim_{z\to\infty}R(z)=\lim_{r\to\infty}R^{-1}(r)=\infty$. Similarly, any continuous monotone increasing transformation R of $\mathbb{R}_0^+$ such that

$$\lim_{z\downarrow 0}R(z)=0 \qquad\text{and}\qquad \lim_{z\to\infty}R(z)=\infty \tag{24}$$

characterizes a nonvanishing radial density f over $\mathbb{R}_0^+$ via the relation $R=\tilde{F}_k^{-1}\circ\tilde{\Phi}_k$. The variational problem just described thus consists in minimizing

$$\begin{aligned}
H_k(R) &= \int_0^\infty \Big[\frac{1}{\tilde{\phi}_k(z)}\,\frac{\tilde{\phi}_k(z)}{R'(z)}+\frac{k-1}{R(z)}z\Big]\tilde{\phi}_k(z)\,dz \qquad (25)\\
&= \int_0^\infty \Big[\frac{1}{R'(z)}+\frac{k-1}{R(z)}z\Big]\tilde{\phi}_k(z)\,dz,
\end{aligned}$$

with respect to $R : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ continuous and monotone increasing, under the constraints (24), since $\tilde{f}_k(r) = \frac{d}{dr}\tilde{F}_k(r) = \tilde{\phi}_k(z)/(\frac{d}{dz}R)$, and $\tilde{f}_k(r)\, dr = d\tilde{F}_k(r) = \tilde{\phi}_k(z)\, dz$. The constraint $D_k(\phi, f) = k$ now takes the form

$$D_k(\phi, R) = \int_0^\infty z\, R(z)\, \tilde{\phi}_k(z)\, dz = k. \tag{26}$$

This problem is very similar to its one sample location counterpart, which is solved in Hallin and Paindaveine (2002). While both problems share the same functional $H_k(R)$, the situation here is a lot simpler, due to the fact that the constraint (26) is linear in R (while the associated constraint in the location case is quadratic in R). This allows for the following simple solution.

Let $\mathcal{R}$ be the class of monotone increasing and continuous functions $R : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that (24) holds and $D_k(\phi, R) = k$. Then the following lemma clearly follows from the convexity of $\mathcal{R}$ and $H_k(R)$.

Lemma 5 *Let R_1 belong to the class $\mathcal{R}$. Then R_1 is the unique solution of the minimization problem under study if and only if $H_k'(0) := \frac{d}{dw}(H_k((1-w)R_1 + wR_2))|_{w=0} \geq 0$, for any $R_2 \in \mathcal{R}$.*

Now, it is easy to check that

$$\begin{aligned} H_k'(0) &= \int_0^\infty \Big[-\frac{R_2'(z) - R_1'(z)}{(R_1'(z))^2} - \frac{(k-1)z(R_2(z) - R_1(z))}{(R_1(z))^2} \Big] \tilde{\phi}_k(z)\, dz \\ &= \int_0^\infty (R_2(z) - R_1(z)) \Big[\frac{\tilde{\phi}_k'(z)}{(R_1'(z))^2} - \frac{2\, \tilde{\phi}_k(z) R_1''(z)}{(R_1'(z))^3} - \frac{(k-1)z\tilde{\phi}_k(z)}{(R_1(z))^2} \Big]\, dz, \end{aligned}$$

so that, if $R_1(z) := z$ for all $z > 0$,

$$\begin{aligned} H_k'(0) &= \int_0^\infty (R_2(z) - z) \Big[\tilde{\phi}_k'(z) - \frac{(k-1)\tilde{\phi}_k(z)}{z} \Big]\, dz \\ &= \frac{1}{\mu_{k-1;\phi}} \int_0^\infty (R_2(z) - z)\, z^{k-1} \phi'(z)\, dz \\ &= \frac{1}{\mu_{k-1;\phi}} \int_0^\infty (R_2(z) - z)\, z^{k-1} \Big[- z\phi(z) \Big]\, dz = k - D_k(\phi, R_2), \end{aligned}$$

which equals zero if R_2 belongs to $\mathcal{R}$. Lemma 5 therefore establishes the result. □

We now turn to the proof of the multivariate extension of the Hallin and Tribel (2000) result.

Proof of Proposition 7. First note that, from Proposition 5,

$$\mathrm{ARE}_{k,f}^{(\mathrm{ser})}(\phi_{SP}/\phi_{\mathcal{N}}) = \frac{9}{k^2}\, \mathrm{E}^2[U\tilde{F}_k^{-1}(U)]\, \mathrm{E}^2[UJ_{k,f}(U)].$$

As in the proof of Proposition 6, it is clear (by considering $f_a(r) := f(ar)$, $a, r > 0$) that we may assume that $\mathrm{E}[U\tilde{F}_k^{-1}(U)] = 1$. Therefore, the problem reduces to the variational problem

$$\inf_{f \in \mathcal{C}} \mathrm{E}[UJ_{k,f}(U)], \qquad \text{with } \mathcal{C} := \Big\{ f \,\Big|\, \mathrm{E}[U\tilde{F}_k^{-1}(U)] = 1 \Big\}. \tag{27}$$

Integrating by parts, we obtain

$$\mathrm{E}[UJ_{k,f}(U)] = \int_0^\infty \tilde{F}_k(r)\,\varphi_f(r)\,\tilde{f}_k(r)\,dr = \int_0^\infty \left[(\tilde{f}_k(r))^2 + \frac{k-1}{r}\,\tilde{F}_k(r)\,\tilde{f}_k(r)\right] dr,$$

so that (27) in turn is equivalent to

$$\inf_{\tilde{f}\in\tilde{\mathcal{C}}} \int_0^\infty \left[(\tilde{f}_k(r))^2 + \frac{k-1}{r}\,\tilde{F}_k(r)\,\tilde{f}_k(r)\right] dr, \tag{28}$$

where $\tilde{\mathcal{C}}$ is the set of all $\tilde{f}$ defined on $\mathbb{R}_0^+$ such that

$$\int_0^\infty \tilde{f}_k(r)\,dr = \int_0^\infty r\,\tilde{f}_k(r)\,\tilde{F}_k(r)\,dr = 1. \tag{29}$$

Substituting $y, \dot{y}$, and t for $\tilde{F}_k, \tilde{f}_k$, and r, respectively, the Euler-Lagrange equation associated with the variational problem (28)-(29) takes the form

$$2t^2\ddot{y} - (k-1-\lambda_2 t^2)y = 0, \tag{30}$$

where λ_2 stands for the Lagrange multiplier associated with the second constraint in (29). Letting $y = t^{1/2}u$, equation (30) reduces to the Bessel Equation

$$t^2\ddot{u} + t\dot{u} + \left(\frac{\lambda_2}{2}\,t^2 - \frac{2k-1}{4}\right) u = 0,$$

so that, denoting by $J_r(.)$ and $Y_r(.)$ the first and second type Bessel functions of order r respectively, the general solution of (30) is given by

$$y(t) = \alpha\, t^{1/2} J_{r_k}(\omega t) + \beta\, t^{1/2} Y_{r_k}(\omega t),$$

where $r_k := \sqrt{2k-1}/2$ and $\omega := \sqrt{\lambda_2/2}$. Since $y(0+) = 0$, it is clear that $\beta = 0$. On the other hand, $\dot{y} \geq 0$ implies that $\dot{y}$ is compactly supported in $\mathbb{R}_0^+$, with support $[0, a]$, say.

It follows from the constraints (29) and the continuity of $\dot{y}$ that the extremals of the variational problem under study are the solutions of (30) that satisfy

$$y(a) = 1,\ \ \dot{y}(a) = 0 \ \text{ and } \int_0^a t\,y(t)\dot{y}(t)\,dt = \frac{a}{2} - \frac{1}{2}\int_0^a (y(t))^2\,dt = 1. \tag{31}$$

By using the identities $xJ_r'(x) = rJ_r(x) - xJ_{r+1}(x)$ and $xJ_r'(x) = -rJ_r(x) + xJ_{r-1}(x)$, it is easily checked that the constraints (31) take the form

$$\alpha\, a^{1/2} J_{r_k} = 1, \tag{32}$$

$$\alpha\, a^{-1/2}\left[\left(r_k + \frac{1}{2}\right) J_{r_k} - (\omega a) J_{r_k+1}\right] = 0 \tag{33}$$

$$\alpha^2\left[\frac{a^2}{2}(J_{r_k})^2 + \frac{a^2}{2}(J_{r_k+1})^2 - \frac{a}{\omega}\, r_k\, J_{r_k} J_{r_k+1}\right] = a - 2, \tag{34}$$

where all Bessel functions are evaluated in ωa.

Equations (32) and (33) allow to compute $J_{r_k}(\omega a)$ and $J_{r_k+1}(\omega a)$, with respect to a, α, ω and r_k. Substituting these values in (34) yields $2k-1 = 4\,r_k^2 = 1 + 16\,\omega^2 a - 4\,\omega^2 a^2$, or

$$a = \frac{8(\omega a)^2}{2(\omega a)^2 + k - 1}.$$

Since (33) implies that $\omega a = c(r_k)$ (where $c(r_k)$ is defined in Proposition 7), we obtain

$$\omega = \frac{\omega a}{a} = \frac{2(\omega a)^2 + k - 1}{8(\omega a)} = \frac{2\,c(r_k)^2 + k - 1}{8\,c(r_k)}. \tag{35}$$

To conclude, note that, integrating by parts and using (30),

$$\inf_f \mathrm{E}[U J_{k,f}(U)] = \int_0^\infty \left[(\dot{y}(t))^2 + \frac{k-1}{t}\, y(t)\,\dot{y}(t) \right] dt$$

$$= \int_0^\infty \left[-2\,t\,\dot{y}(t)\ddot{y}(t) + \frac{k-1}{t}\, y(t)\,\dot{y}(t) \right] dt = \int_0^\infty \lambda_2\, t\, y(t)\dot{y}(t)\, dt = \lambda_2 = 2\,\omega^2,$$

so that $\inf_f \mathrm{ARE}^{(\mathrm{ser})}_{k,f}(\phi_{SP}/\phi_{\mathcal{N}}) = \frac{36}{k^2}\,\omega^4$. This completes the proof of Proposition 7. $\square$

Remark. As an immediate corollary, we also obtain that the infimum in Proposition 7 is reached (for fixed k) at the collection of radial densities f for which $\tilde{F}_k$ is in $\{\tilde{F}_{k,\sigma}(r) := \tilde{F}_{k,1}(\sigma^{-1} r)\}$, with

$$\tilde{F}_{k,1}(r) := \sqrt{\frac{\omega r}{c(r_k)}}\; \frac{J_{r_k}(\omega r)}{J_{r_k}(c(r_k))}\; I\left[0 < r \le \frac{c(r_k)}{\omega}\right] +\; I\left[r > \frac{c(r_k)}{\omega}\right],$$

where ω has been obtained in (35). Recall that $r_k := \sqrt{2k-1}/2$. This also justifies the somewhat mysterious definition of $c(r_k)$ in Proposition 7. See Figure 2 for the graphs of the associated densities $\tilde{f}_{k,1}$ for several values of the space dimension k.

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k	$\inf_f \text{ARE}^{(\text{ser})}_{k,f}(\phi_{SP}/\phi_{\mathcal{N}})$	$\inf_f \text{ARE}^{(\text{loc})}_{k,f}(\phi_{W}/\phi_{\mathcal{N}})$
1	0.856	0.864
2	0.913	0.916
3	0.878	0.883
4	0.845	0.853
6	0.797	0.811
10	0.742	0.765
$+\infty$	0.563	0.648

Table 1 Some numerical values, for various values of the space dimension k, of the lower bound for the asymptotic relative efficiency of the Spearman-autocorrelation rank-based procedure ϕ_{SP} and the Wilcoxon-type procedure for location ϕ_W with respect to the corresponding Gaussian test $\phi_{\mathcal{N}}$, in the serial and location case, respectively.

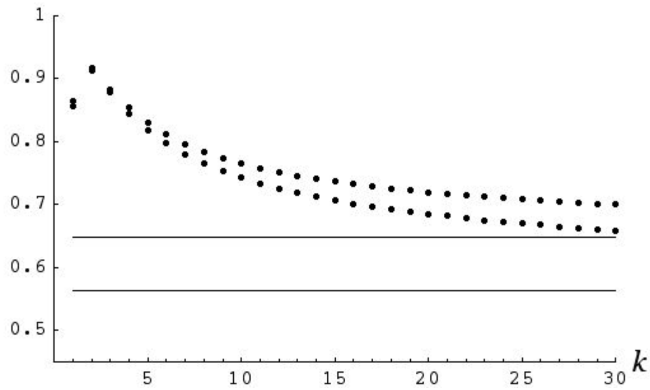

Figure 1 Plot of the values of the lower bound (14) for the asymptotic relative efficiency of the Spearman procedure ϕ_{SP} with respect to the Gaussian test $\phi_{\mathcal{N}}$, for space dimension $k = 1, 2, \ldots, 30$ (lower dotted curve). The upper dotted curve is associated with the corresponding lower bound for the ARE of the multivariate Wilcoxon procedure for location ϕ_W with respect to the Hotelling test. The horizontal lines correspond to the asymptotic values of these lower bounds (.5625 and .648 respectively).

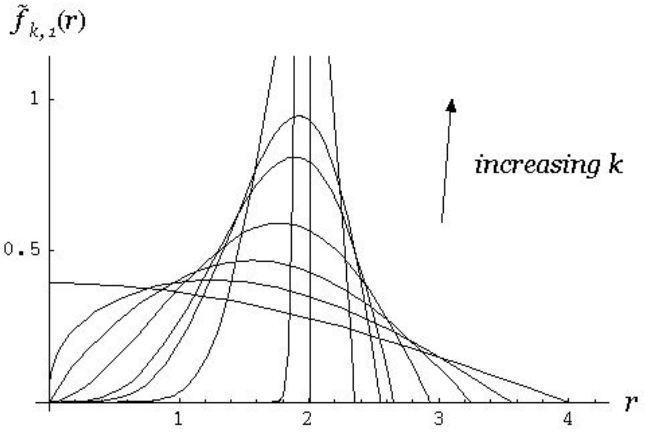

Figure 2 Graphs of the densities $\tilde{f}_{k,1}$ at which the infimum of the AREs of Spearman-autocorrelation type tests with respect to the Gaussian test is reached, for dimensions $k = 1, 2, 4, 10, 30, 50, 200$, and 10^5, respectively.

		degrees of freedom of the underlying t density								
k	$f_\star$	3	4	5	6	8	10	15	20	∞
	t_5	1.828	1.415	1.250	1.162	1.072	1.026	0.972	0.948	0.885
		1.955	1.423	1.250	1.165	1.080	1.038	0.991	0.970	0.915
1	t_8	1.670	1.356	1.228	1.161	1.091	1.056	1.015	0.997	0.952
		1.878	1.393	1.238	1.163	1.091	1.056	1.018	1.001	0.961
	t_{15}	1.532	1.285	1.185	1.132	1.080	1.054	1.026	1.014	0.985
		1.786	1.345	1.207	1.143	1.083	1.055	1.026	1.014	0.987
	$\mathcal{N}$	1.356	1.176	1.106	1.071	1.038	1.024	1.010	1.005	1.000
	(vdW)	1.639	1.257	1.144	1.093	1.048	1.030	1.013	1.007	1.000
	t_5	1.953	1.485	1.296	1.195	1.090	1.036	0.973	0.944	0.868
		2.097	1.495	1.296	1.198	1.100	1.051	0.995	0.969	0.903
2	t_8	1.774	1.419	1.272	1.193	1.111	1.069	1.020	0.999	0.942
		2.015	1.462	1.283	1.196	1.111	1.070	1.024	1.004	0.953
	t_{15}	1.614	1.336	1.221	1.160	1.098	1.067	1.032	1.017	0.981
		1.910	1.407	1.249	1.173	1.102	1.068	1.032	1.018	0.984
	$\mathcal{N}$	1.400	1.204	1.125	1.085	1.047	1.030	1.013	1.007	1.000
	(vdW)	1.729	1.301	1.171	1.112	1.059	1.037	1.016	1.009	1.000
	t_5	2.122	1.584	1.364	1.245	1.120	1.055	0.977	0.942	0.845
		2.289	1.595	1.364	1.248	1.131	1.073	1.004	0.973	0.889
4	t_8	1.918	1.509	1.336	1.242	1.143	1.091	1.030	1.002	0.928
		2.203	1.561	1.350	1.246	1.143	1.092	1.034	1.009	0.943
	t_{15}	1.729	1.411	1.277	1.204	1.128	1.089	1.044	1.024	0.975
		2.084	1.499	1.311	1.220	1.132	1.090	1.044	1.025	0.979
	$\mathcal{N}$	1.458	1.242	1.153	1.106	1.061	1.039	1.018	1.010	1.000
	(vdW)	1.853	1.364	1.212	1.142	1.077	1.049	1.022	1.012	1.000
	t_5	2.231	1.649	1.410	1.280	1.143	1.070	0.983	0.943	0.831
		2.412	1.662	1.410	1.284	1.155	1.090	1.013	0.978	0.881
6	t_8	2.013	1.570	1.381	1.278	1.167	1.108	1.038	1.006	0.918
		2.328	1.628	1.397	1.281	1.167	1.109	1.043	1.014	0.936
	t_{15}	1.806	1.464	1.316	1.236	1.150	1.106	1.054	1.030	0.970
		2.202	1.564	1.356	1.254	1.155	1.107	1.054	1.031	0.975
	$\mathcal{N}$	1.493	1.267	1.172	1.122	1.071	1.047	1.022	1.013	1.000
	(vdW)	1.935	1.408	1.242	1.164	1.092	1.059	1.027	1.016	1.000
	t_5	2.363	1.732	1.471	1.328	1.175	1.094	0.995	0.949	0.814
		2.562	1.746	1.471	1.331	1.189	1.117	1.030	0.989	0.872
10	t_8	2.131	1.648	1.440	1.325	1.200	1.133	1.052	1.014	0.905
		2.482	1.714	1.458	1.329	1.200	1.135	1.058	1.023	0.927
	t_{15}	1.905	1.533	1.370	1.279	1.182	1.130	1.068	1.040	0.963
		2.355	1.649	1.417	1.302	1.188	1.132	1.068	1.040	0.969
	$\mathcal{N}$	1.535	1.299	1.197	1.142	1.086	1.058	1.029	1.017	1.000
	(vdW)	2.041	1.467	1.283	1.195	1.112	1.074	1.035	1.021	1.000

Table 2 AREs of some $\phi_{f_{\nu_\star}}$ tests for randomness (upper line) and for location (lower line), with respect to the corresponding Gaussian tests, under k-dimensional Student (3, 4, 5, 6, 8, 10, 15, and 20 degrees of freedom) and normal densities, respectively, for $k = 1, 2, 4, 6,$ and 10.

degrees of freedom $\nu_\star$ of the underlying t density									
3	4	5	6	8	10	12	15	20	∞
3.000	2.000	1.667	1.500	1.333	1.250	1.200	1.154	1.111	1.000
limiting ARE values									

Table 3 Limiting AREs, as the dimension k of the observation space tends to infinity, of some $\phi_{f_{\nu_\star}}$ tests for randomness with respect to the Gaussian procedure, under the corresponding k-dimensional Student and normal densities, respectively; see (15).

		degrees of freedom of the underlying t density								
k	test	3	4	5	6	8	10	15	20	∞
	S	0.657	0.563	0.519	0.494	0.467	0.453	0.435	0.427	0.405
	L	1.477	1.106	0.954	0.873	0.788	0.745	0.695	0.672	0.613
1	$S^{(\text{loc})}$	1.621	1.125	0.961	0.879	0.798	0.757	0.710	0.690	0.637
	SP	1.299	1.139	1.070	1.032	0.992	0.972	0.948	0.938	0.912
	W	1.900	1.401	1.241	1.164	1.089	1.054	1.014	0.997	0.955
	S	1.000	0.856	0.790	0.752	0.711	0.689	0.662	0.650	0.617
	L	1.777	1.354	1.176	1.080	0.979	0.927	0.866	0.838	0.765
2	$S^{(\text{loc})}$	2.000	1.388	1.185	1.084	0.984	0.934	0.877	0.851	0.785
	SP	1.305	1.152	1.089	1.055	1.022	1.006	0.990	0.983	0.970
	W	1.748	1.317	1.184	1.123	1.066	1.041	1.015	1.005	0.985
	S	1.266	1.084	1.000	0.952	0.900	0.872	0.838	0.823	0.781
	L	1.926	1.498	1.314	1.213	1.105	1.049	0.981	0.951	0.869
4	$S^{(\text{loc})}$	2.250	1.561	1.333	1.220	1.107	1.051	0.986	0.958	0.884
	SP	1.189	1.050	0.994	0.966	0.941	0.930	0.922	0.920	0.924
	W	1.533	1.171	1.064	1.018	0.979	0.964	0.954	0.952	0.961
	S	1.373	1.176	1.085	1.033	0.977	0.946	0.910	0.893	0.847
	L	1.955	1.539	1.359	1.258	1.150	1.093	1.025	0.994	0.910
6	$S^{(\text{loc})}$	2.344	1.626	1.389	1.271	1.153	1.094	1.027	0.997	0.920
	SP	1.115	0.982	0.929	0.903	0.879	0.870	0.865	0.865	0.880
	W	1.422	1.090	0.994	0.953	0.921	0.911	0.908	0.911	0.938
	S	1.467	1.256	1.159	1.104	1.043	1.011	0.972	0.954	0.905
	L	1.950	1.559	1.387	1.290	1.185	1.129	1.061	1.030	0.944
10	$S^{(\text{loc})}$	2.422	1.681	1.436	1.313	1.192	1.131	1.062	1.031	0.951
	SP	1.039	0.909	0.857	0.831	0.808	0.799	0.795	0.797	0.823
	W	1.315	1.007	0.919	0.882	0.855	0.848	0.851	0.857	0.907

Table 4 AREs with respect to the Gaussian procedure of the sign test for randomness (S), the Laplace test for randomness (L), Randles' multivariate sign test for location ($S^{(\text{loc})}$), the Spearman test for randomness (SP), and Peters and Randles' Wilcoxon-type multivariate signed-rank test for location (W), under various k-variate Student and normal densities, ($k = 1, 2, 4, 6, 10$).

		autoregression matrix $m\mathbf{A}$					autoregression matrix $m\mathbf{A}$			
test	density	$\mathbf{0}$	$\mathbf{A}$	$2\mathbf{A}$	$3\mathbf{A}$	density	$\mathbf{0}$	$\mathbf{A}$	$2\mathbf{A}$	$3\mathbf{A}$
$\phi_{\mathcal{N}}$		.0424	.1564	.5480	.9244		.0444	.1472	.5584	.9212
ϕ_{vdW}		.0384	.1596	.5592	.9276		.0424	.1540	.5812	.9360
$\phi_{f_{15}}$		.0404	.1540	.5524	.9228		.0448	.1664	.6100	.9492
ϕ_{f_8}	$\mathcal{N}$	.0432	.1472	.5348	.9088	t_8	.0476	.1716	.6044	.9512
ϕ_{f_5}		.0456	.1340	.4880	.8776		.0488	.1660	.5868	.9456
S		.0436	.1180	.3620	.7316		.0436	.1320	.4172	.8040
SP		.0412	.1504	.5516	.9232		.0452	.1596	.5732	.9368
$\phi_{\mathcal{N}}$		.0440	.1476	.5492	.9208		.0356	.1456	.5352	.8736
ϕ_{vdW}		.0440	.1568	.5640	.9272		.0448	.1964	.7028	.9764
$\phi_{f_{15}}$		.0460	.1544	.5772	.9328		.0468	.2212	.7684	.9876
ϕ_{f_8}	t_{15}	.0452	.1520	.5688	.9312	t_3	.0480	.2360	.7884	.9924
ϕ_{f_5}		.0428	.1404	.5420	.9148		.0460	.2436	.8020	.9948
S		.0436	.1276	.3900	.7648		.0436	.1600	.5420	.9084
SP		.0420	.1540	.5600	.9252		.0488	.1892	.6848	.9720

Table 5 Estimated sizes and powers of the Gaussian test $\phi_{\mathcal{N}}$, ϕ_{f_5}, ϕ_{f_8}, $\phi_{f_{15}}$, ϕ_{vdW}, the sign test for randomness (S), and the Spearman type test (SP), under various values of the autoregression matrix $m\mathbf{A}$ (cf. (18)) and various densities; simulations are based on 2,500 replications.