Journal of Nonparametric Statistics
Vol. 00, No. 00, January 2008, 1–27

RESEARCH ARTICLE

Le Cam optimal tests for symmetry against Ferreira and Steel's general skewed distributions

Christophe Ley[a] and Davy Paindaveine[a*]

[a] *Université Libre de Bruxelles, E.C.A.R.E.S., Institut de Recherche en Statistique, and Département de Mathématique, Campus Plaine, CP210, Boulevard du Triomphe, B-1050, Brussels, Belgium*

(*sent December 2008, revised March 2009*)

When testing symmetry of a univariate density, (parametric classes of) densities skewed by means of the general probability transform introduced in [7] are appealing alternatives. This paper first proposes parametric tests of symmetry (about a specified centre) that are locally and asymptotically optimal (in the Le Cam sense) against such alternatives. To improve on these parametric tests, which are valid under well-specified density types only, we turn them into semiparametric tests, either by using a standard studentization approach or by resorting to the invariance principle. The second approach leads to robust yet efficient signed-rank tests, which include the celebrated sign and Wilcoxon tests as special cases, and turn out to be Le Cam optimal irrespective of the underlying original symmetric density. Optimality, however, is only achieved under well-specified "skewing mechanisms", and we therefore evaluate the overall performances of our tests by deriving their asymptotic relative efficiencies with respect to the classical test of skewness. A Monte-Carlo study confirms the asymptotic results.

Keywords: Rank-based inference; tests of symmetry; asymmetry models; location tests; local asymptotic normality

AMS Subject Classification: 62G10; 62G35

1. Introduction

1.1. *Testing for symmetry*

Symmetry is one of the most important and fundamental structural assumptions in statistics, playing a major role, for instance, in the identifiability of location or intercept under nonparametric conditions; see, e.g., [2], [23], and [24]. This explains the huge variety of existing tests for the null hypothesis of symmetry in an i.i.d. sample $X_1, \ldots, X_n$—hypothesis under which there exists some real value θ such that the common cumulative distribution function (cdf) of the X_i's is θ-symmetric; throughout, a cdf F (resp., a probability density function (pdf) f) is said to be θ-symmetric iff $F(\theta - x) = 1 - F(\theta + x)$ a.e. in x (resp., iff $f(\theta - x) = f(\theta + x)$ a.e. in x). Essentially, the tests for symmetry available in the literature belong to two distinct classes.

(a) The first class contains tests achieving consistency under *any* alternative, and are usually of a Kolmogorov-Smirnov or Cramér-von Mises type; see,

*Corresponding author. Email: dpaindav@ulb.ac.be

ISSN: 1048-5252 print/ISSN 1029-0311 online

DOI: 10.1080/1048525YYxxxxxxxx
http://www.informaworld.com

e.g., [3], [12], [17], [18], and [22]. The price to pay, however, for universal consistency is in terms of convergence rates, which are nonparametric, implying that such procedures typically require a large number of observations.

(b) Procedures in the second class usually rather focus on some *favored* alternatives, against which they (i) achieve (semi)parametric consistency rates and (ii) sometimes even are (semi)parametrically optimal; see, e.g., [4], [5], and [14]. While such tests cannot be universally consistent, their main disadvantage remains their important lack of flexibility: typically, the choice of the favored alternatives is very restricted, and, to some extent, quite arbitrary.

The tests we propose in this paper belong to the second class, hence do not achieve universal consistency, but improve on existing similar procedures by giving practitioners the freedom to choose virtually *any* favored skewed alternative. This nice feature is achieved by deriving tests that are designed to behave well (see below) against essentially *arbitrary*—yet fixed—classes of skewed alternatives, more precisely against asymmetric distributions generated via the *general skewing scheme* recently proposed in [7].

1.2. *General skewing mechanisms*

Ferreira and Steel presented in [7] a general mechanism allowing to skew any univariate symmetric distribution. Their idea is to introduce skewness by means of a probability transform that weights the quantile space. More specifically, the θ-symmetric cdf F is turned into a cdf of the form

$$x \mapsto F^L(x) = L(F(x)), \tag{1}$$

where L is a cdf over $[0, 1]$ that is *not* $\frac{1}{2}$-symmetric; it is easy to show that F^L is indeed θ-symmetric iff L is $\frac{1}{2}$-symmetric. Hence, (1) provides a convenient way to skew any θ-symmetric distribution (note that the terminology "skew" here stands for asymmetry with respect to θ, and that a θ-skewed distribution could very well be θ'-symmetric for $\theta' \neq \theta$). If F (resp., L) admits the pdf f (resp., ℓ), the pdf associated with F^L takes the form

$$x \mapsto f^L(x) = \ell(F(x))\, f(x), \tag{2}$$

which, by construction, is a weighted version of the original pdf f. If one restricts to distributions that admit an almost everywhere positive pdf over the real line, any form of skewness can be achieved by means of such a probability transform, since any distribution can clearly be mapped to any other distribution via (1)-(2). This *surjectivity* property thus implies that any type of asymmetric densities over the real line can be obtained via this method, including, e.g., the skew-normal distribution of [1] or the inverse scale factors densities of [6].

These considerations lead to defining a *skewing mechanism* as a collection $\mathcal{L} = \{L\}$ of cdfs over $[0, 1]$ containing no other $\frac{1}{2}$-symmetric cdf than the identity function I—which of course leaves any density f untouched ($f^I = f$). For any symmetric density f, the resulting collection of densities $\{f^L : L \in \mathcal{L} \setminus \{I\}\}$ then is a family of skewed versions of f. A major advantage of this construction lies in the possibility of choosing the skewing mechanism $\mathcal{L}$ independently of the pdf f to be skewed; see [7] for a discussion. The choice of $\mathcal{L}$, which of course crucially determines the type of skewness that is achieved, can be made by imposing that the resulting skewed distributions retain some properties of the initial symmetric density, such as, e.g., fixing the mode or the median of the symmetric density, leaving one of its tails untouched, maintaining the existence of moments, etc. For instance, it is clear that the median of the skewed versions of any symmetric density f is fixed under the action of $\mathcal{L} = \{L\}$ iff $L(\frac{1}{2}) = \frac{1}{2}$ for all $L \in \mathcal{L}$. Note that the latter requirement prevents location shifts, hence leads to "pure" skewed altenatives.

1.3. *Our methodology*

Restricting, for the sake of simplicity, to the problem of testing for symmetry about a specified centre θ, we intend to develop testing procedures that perform well against alternatives obtained by means of an *arbitrary* prespecified skewing mechanism $\mathcal{L} = \{L\}$. As mentioned above, practitioners then would be given the *flexibility* of choosing freely $\mathcal{L}$—equivalently, the favored skewed alternatives—according to their needs or to the modeling assumptions that they are ready to adopt. Again, the surjectivity property stated in the previous section implies that *all* alternatives to the null of θ-symmetry enter this framework, hence explains why we regard this approach as "flexible" and "general".

1.3.1. ULAN and Le Cam optimality of parametric tests of symmetry

This flexibility, of course, should not be obtained at the expense of efficiency and/or robustness of the resulting tests. Aiming first at efficiency, we restrict to *parametric* skewing mechanisms (PSMs) $\mathcal{L} = \{L_\delta\}$ indexed by some real (skewness) parameter δ, and introduce scale-asymmetry models $\mathcal{P}^{\mathcal{L}(n)}$ in which this skewness δ, the value of a scale parameter σ, and a standardized (θ-symmetric) density f_1 remain unspecified. We then show that, under extremely mild regularity assumptions on f_1, the fixed-f_1 parametric submodels $\mathcal{P}^{\mathcal{L}(n)}_{f_1}$ are *uniformly locally and asymptotically normal.* This ULAN property is the key result that allows to derive Le Cam optimal tests of symmetry in such parametric models.

1.3.2. Robustness and semiparametric tests of symmetry

Such tests, however, are in general valid—in the sense that they asymptotically meet the nominal level constraint—at standardized density f_1 only, hence are of little practical interest, as it appears highly unrealistic to assume that the underlying f_1 is known. We solve that problem by introducing (i) *studentized* versions and (ii) *signed-rank* versions of the optimal f_1-parametric tests. While the former are obtained by means of standard studentization arguments, the latter follow from the rich group invariance structure of the null hypothesis of symmetry.

Both resulting f_1-*semiparametric* tests of symmetry inherit the optimality properties, at f_1, from their parametric counterparts, while remaining valid under a much broader class of densities. It is remarkable that the signed-rank tests are even Le Cam optimal *uniformly in* f_1. In both cases, optimality, however, is achieved under the prespecified PSM only. To investigate the overall performances of the proposed tests, we compute their asymptotic relative efficiencies (AREs) with respect to a standard benchmark procedure, namely the classical test of skewness. These AREs show that our tests exhibit very good performances, even when based on a PSM that is not well-specified. This is confirmed in small samples through a Monte-Carlo study.

Quite interestingly, two particular cases of the proposed signed-rank tests of symmetry are the celebrated sign and Wilcoxon signed-rank tests. Le Cam optimality of both tests—each against a specific PSM, but uniformly in the underlying density type f_1—follows from our general results. Since the corresponding local alternatives are not (pure) location alternatives, such optimality results are of a different nature than the well-known optimality of the sign test (resp., the Wilcoxon test) against Laplace (resp., logistic) location alternatives.

1.4. *Outline of the paper*

The paper is organized as follows. In Section 2, we define the notation used throughout, list the assumptions needed on PSMs, and describe the resulting scale-asymmetry models. Section 3 states that these models are ULAN in the vicinity of symmetry, and presents the resulting optimal parametric tests of symmetry. Semiparametric versions of these tests are developed in Section 4.1 (studentized tests) and Section 4.2 (signed-rank tests). In Section 5, we introduce several examples of PSMs, and derive, in each case, the resulting optimal parametric and semiparametric tests of symmetry. The asymptotic relative efficiencies of our tests with respect to the classical test of skewness are computed in Section 6. A Monte-Carlo study then investigates the small-sample performances of the proposed tests in Section 7. Eventually, Section 8 gives some final comments, and an appendix collects technical proofs.

2. Notation and assumptions

As announced in the Introduction, we intend to develop tests for *symmetry about a specified centre*—we simply write *symmetry* in the sequel, and will throughout let this specified centre be the origin of the real line, which is of course without any loss of generality since testing for symmetry about any fixed θ can be obtained by replacing the observations X_i with $X_i - \theta$. More precisely, restricting to the absolutely continuous case, we want to test the null that the underlying density f belongs to

$$\mathcal{F} := \Big\{ f : f(x) > 0 \text{ a.e.},\ f(-x) = f(x)\ \forall x,\ \textstyle\int_{-\infty}^{\infty} f(x)\,dx = 1 \Big\}.$$

Even in a parametric context, assuming that the underlying density is fully specified is not reasonable, and it should rather be assumed that only the underlying *density type*—that is, the density up to a scale factor, σ say—is specified. This motivates rewriting $\mathcal{F}$ as $\mathcal{F} = \{f_{1\sigma} : \sigma \in \mathbb{R}_0^+, f_1 \in \mathcal{F}_1\}$, where we let $f_{1\sigma}(x) := \frac{1}{\sigma} f_1\left(\frac{x}{\sigma}\right)$ and

$$\mathcal{F}_1 := \Big\{ f_1 \in \mathcal{F} : \textstyle\int_{-\infty}^{1} f_1(x)\,dx = 0.75 \Big\}.$$

Clearly, if X has pdf $f_{1\sigma}$, then $\sigma = \text{Med}[|X|]$, so that σ can be interpreted as a *robust* scale measure, in the sense that, unlike the usual standard deviation, it avoids any moment assumption. Denoting by $\mathrm{P}^{(n)}_{\sigma;f_1}$ ($f_1 \in \mathcal{F}_1$) the hypothesis under which $X_1, \ldots, X_n$ are i.i.d. with common density $f_{1\sigma}$, we will discriminate between the null hypothesis $\mathcal{H}^{(n)}_{0,f_1} := \cup_\sigma \{\mathrm{P}^{(n)}_{\sigma;f_1}\}$ of symmetry with specified density type f_1 (throughout, we write $\cup_\sigma$ instead of $\cup_{\sigma \in \mathbb{R}_0^+}$) and the nonparametric null hypothesis $\mathcal{H}^{(n)}_0 := \cup_{f_1 \in \mathcal{F}_1} \mathcal{H}^{(n)}_{0,f_1}$.

Asymmetric alternatives will be defined in terms of the skewing mechanisms described in the Introduction. Since we want to rely on Le Cam's theory of asymptotic experiments and develop tests for symmetry that are locally and asymptotically optimal, we will consider *parametric* skewing mechanisms, that is, skewing mechanisms that are indexed by a real parameter δ (that plays the role of a skewness parameter).

DEFINITION 1. *A parametric skewing mechanism (PSM) is a collection* $\mathcal{L} = \{L_\delta : \delta \in \mathcal{D} \subset \mathbb{R}\}$ *of cdfs over* $[0,1]$ *such that (i)* $\mathcal{D}$ *is an open and connected set containing the value* $\delta = 0$*, (ii)* $L_0 = I$*, and (iii) the only value of* $\delta \in \mathcal{D}$ *for which* L_δ *is* $\frac{1}{2}$*-symmetric is* $\delta = 0$*.*

Now, for any PSM $\mathcal{L} = \{L_\delta : \delta \in \mathcal{D}\}$, denote by $\mathrm{P}^{\mathcal{L}(n)}_{\sigma,\delta;f_1}$, with $\sigma \in \mathbb{R}^+_0$, $\delta \in \mathcal{D}$, and $f_1 \in \mathcal{F}_1$, the joint distribution of an n-tuple of i.i.d. observations $X_1, \ldots, X_n$ with common pdf

$$x \mapsto \tfrac{1}{\sigma} f_1^{L_\delta}(\tfrac{x}{\sigma}) = \tfrac{1}{\sigma} \ell_\delta(F_1(\tfrac{x}{\sigma}))\, f_1(\tfrac{x}{\sigma}),$$

where ℓ_δ stands for the pdf associated with L_δ; see (2). For consistency, we simply write $\mathrm{P}^{(n)}_{\sigma;f_1}$ instead of $\mathrm{P}^{\mathcal{L}(n)}_{\sigma,0;f_1}$—since it is assumed throughout that $L_0 = I$, no reference to the PSM is indeed needed there. Any couple $(f_1, \mathcal{L})$ then induces the parametric scale-asymmetry model

$$\mathcal{P}^{\mathcal{L}(n)}_{f_1} := \left\{ \mathrm{P}^{\mathcal{L}(n)}_{\sigma,\delta;f_1} : \sigma \in \mathbb{R}^+_0, \delta \in \mathcal{D} \right\}, \tag{3}$$

whereas any PSM $\mathcal{L}$ defines the nonparametric scale-asymmetry model $\mathcal{P}^{\mathcal{L}(n)} := \bigcup_{f_1 \in \mathcal{F}_1} \mathcal{P}^{\mathcal{L}(n)}_{f_1}$.

Depending on the nature of the proposed tests, PSMs will have to satisfy various mild assumptions. For the sake of convenience, we group those assumptions into

ASSUMPTION A. *(i) The PSM* $\mathcal{L} = \{L_\delta : \delta \in \mathcal{D}\}$ *is independent of the initial symmetric density to be skewed; (ii) for any* $\delta \in \mathcal{D}$ *such that* $-\delta \in \mathcal{D}$*, we have*

$$L_{-\delta}(u) = 1 - L_\delta(1-u), \quad \forall u \in [0,1]; \tag{4}$$

(iii) the cdf L_δ*, for any* $\delta \in \mathcal{D}$*, admits a pdf* ℓ_δ*, and the mapping* $\delta \mapsto \ell_\delta^{1/2}(\cdot)$ *is differentiable in quadratic mean at* $\delta = 0$*, with quadratic mean derivative* $\partial_\delta \ell_\delta^{1/2}(\cdot)|_{\delta=0} =: \frac{1}{2} J^{\mathcal{L}}(\cdot)$*; (iv) (resp., (iv)′) the mapping* $u \mapsto J^{\mathcal{L}}(u)$ *is continuous over* $(0,1)$ *(resp., over* $(\frac{1}{2},1)$*) and can be written as the difference of two monotone increasing functions.*

The independence condition in Assumption A(i) is natural (see [7]) and will play an important role in the uniform optimality of the proposed signed-rank tests; see Section 4.2. The *duality assumption* in A(ii) is desirable when interpreting δ as a skewness parameter. Indeed, this assumption ensures that the joint distribution of the X_i's is $\mathrm{P}^{\mathcal{L}(n)}_{\sigma,\delta;f_1}$ iff that of their reflections with respect to the origin is $\mathrm{P}^{\mathcal{L}(n)}_{\sigma,-\delta;f_1}$; in other words, this states that if δ is associated with some skewness to the left, then the corresponding skewness to the right is obtained for the value $-\delta$ of the skewness parameter, and vice versa. The regularity assumptions in A(iii) are needed to derive the *uniform local asymptotic normality* (ULAN) property of the considered scale-asymmetry models (see Theorem 3.1 below), which plays a crucial role in the construction of our optimal tests. Note that Assumption A(iii) is fulfilled whenever the mapping $\delta \mapsto \ell_\delta(u)$ admits a (standard) derivative that is a square-integrable function of $u \in (0,1)$, in which case we simply may write $J^{\mathcal{L}}(u) = \partial_\delta \ell_\delta(u)|_{\delta=0}$.

Finally, the technical conditions in Assumptions A(iv)-(iv)$'$ are only required by the proposed optimal semiparametric tests.

3. ULAN and optimal parametric tests

Let $X_i^{(n)}$, $i = 1, \ldots, n$, be a triangular array of observations, such that the joint distribution of $(X_1^{(n)}, \ldots, X_n^{(n)})$ is in $\mathcal{P}_{f_1}^{\mathcal{L}(n)}$ (see (3)) for some fixed pdf $f_1 \in \mathcal{F}_1$ and some fixed PSM $\mathcal{L}$, and consider the problem of testing the null hypothesis $\mathcal{H}_{0,f_1}^{(n)}$ of symmetry with specified density type f_1.

The optimal tests we derive below are based on the ULAN property, in the vicinity of symmetry, of the parametric model $\mathcal{P}_{f_1}^{\mathcal{L}(n)}$. ULAN however requires some further mild assumptions on f_1. To be able to state the latter assumptions, we introduce the following definitions. Consider the measurable space $(\Omega, \mathcal{B}_\Omega)$, where $\Omega \subset \mathbb{R}$ is an open subset and $\mathcal{B}_\Omega$ is its Borel σ-field. Denote by $L^2(\Omega, \nu)$ the space of square-integrable functions with respect to the Lebesgue measure with weight e^x on $(\Omega, \mathcal{B}_\Omega)$, that is, the space of measurable functions $h : \Omega \to \mathbb{R}$ satisfying $\int_\Omega [h(x)]^2 e^x dx < \infty$. Recall that $g \in L^2(\Omega, \nu)$ admits a *weak derivative* T iff $\int_\Omega g(x)\varphi'(x)dx = -\int_\Omega T(x)\varphi(x)dx$ for all infinitely differentiable (in the classical sense) compactly supported functions φ on Ω. The mapping T is also called the *derivative of* g *in the sense of distributions* in $L^2(\Omega, \nu)$. Furthermore, if T itself is in $L^2(\Omega, \nu)$, then g belongs to $W^{1,2}(\Omega, \nu)$, the Sobolev space of order 1 on $L^2(\Omega, \nu)$.

As we will state below, the parametric family $\mathcal{P}_{f_1}^{\mathcal{L}(n)}$ is ULAN in the vicinity of symmetry provided that f_1 belongs to the collection $\mathcal{F}_1^{\mathrm{ULAN}}$ of densities in $\mathcal{F}_1$ for which the mapping $x \mapsto f_{1;\exp}^{1/2}(x) := f_1^{1/2}(e^x)$ belongs to $W^{1,2}(\mathbb{R}, \nu)$. Letting $\psi_{f_1}(x) := -\frac{2}{x}(f_{1;\exp}^{1/2})'(\log|x|)/f_{1;\exp}^{1/2}(\log|x|)$, where $(f_{1;\exp}^{1/2})'$ stands for the weak derivative of $f_{1;\exp}^{1/2}$ in $L^2(\mathbb{R}, \nu)$, this regularity condition ensures the finiteness of the Fisher information for scale $\mathcal{I}_{f_1} := \int_{-\infty}^{\infty} (x\psi_{f_1}(x) - 1)^2 f_1(x)\, dx$. At first sight, such a condition may appear highly technical and not easy to deal with; however, any function f_1 that (i) is absolutely continuous with a.e.-derivative f_1' and (ii) satisfies $\int_{-\infty}^{\infty} (x\varphi_{f_1}(x) - 1)^2 f_1(x)\, dx < \infty$, with $\varphi_{f_1}(x) := -f_1'(x)/f_1(x)$, belongs to $\mathcal{F}_1^{\mathrm{ULAN}}$, and $\varphi_{f_1}(\cdot) = \psi_{f_1}(\cdot)$. In practice, most densities fulfill the latter requirements.

ULAN of the parametric model $\mathcal{P}_{f_1}^{\mathcal{L}(n)}$ with respect to $\vartheta = (\sigma, \delta)'$, in the vicinity of symmetry, then takes the following form.

Theorem 3.1 : *Let $f_1 \in \mathcal{F}_1^{\mathrm{ULAN}}$ and $\mathcal{L}$ be a PSM satisfying Assumptions A(i)-(iii). Then, for any $\sigma \in \mathbb{R}_0^+$, the family of probability distributions $\mathcal{P}_{f_1}^{\mathcal{L}(n)}$ is ULAN at $\vartheta = (\sigma, 0)'$, with* central sequence

$$\Delta_{f_1}^{\mathcal{L}(n)}(\sigma) := \begin{pmatrix} \Delta_{f_1;1}^{(n)}(\sigma) \\ \Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma) \end{pmatrix} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{pmatrix} \frac{1}{\sigma}\left(\frac{X_i^{(n)}}{\sigma}\, \psi_{f_1}\left(\frac{X_i^{(n)}}{\sigma}\right) - 1\right) \\ J^{\mathcal{L}}\left(F_1\left(\frac{X_i^{(n)}}{\sigma}\right)\right) \end{pmatrix}$$

and diagonal information matrix $\Gamma_{f_1}^{\mathcal{L}}(\sigma)$, *with upper-left entry* $\Gamma_{f_1;11}(\sigma) := \mathcal{I}_{f_1}/\sigma^2$ *and lower-right entry* $\Gamma_{22}^{\mathcal{L}} := \int_0^1 (J^{\mathcal{L}}(u))^2\, du$. *More precisely, for any* $\vartheta^{(n)} = (\sigma^{(n)}, 0)' = \vartheta + O(n^{-1/2})$ *and any bounded sequence* $\tau^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)})' \in \mathbb{R}^2$, *we have*

$$\Lambda^{\mathcal{L}(n)}_{\vartheta^{(n)}+n^{-1/2}\tau^{(n)}/\vartheta^{(n)};f_1} := \log\left(dP^{\mathcal{L}(n)}_{\vartheta^{(n)}+n^{-1/2}\tau^{(n)};f_1}/dP^{\mathcal{L}(n)}_{\vartheta^{(n)};f_1}\right)$$
$$= \tau^{(n)\prime}\Delta^{\mathcal{L}(n)}_{f_1}(\sigma^{(n)}) - \frac{1}{2}\tau^{(n)\prime}\,\Gamma^{\mathcal{L}}_{f_1}(\sigma)\tau^{(n)} + o_{\rm P}(1)$$

and $\Delta^{\mathcal{L}(n)}_{f_1}(\sigma^{(n)}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma^{\mathcal{L}}_{f_1}(\sigma))$*, both under* $P^{(n)}_{\vartheta^{(n)};f_1}$ *as* $n \to \infty$*.*

Actually, as shown in the proof of this theorem, Assumption A(i) is not required for $\mathcal{P}^{\mathcal{L}(n)}_{f_1}$ to be ULAN. This assumption however guarantees that the lower-right entry of $\Gamma^{\mathcal{L}}_{f_1}(\sigma)$ does not depend on the density type f_1, which—jointly with the fact that this quantity does not involve the value of the scale parameter—justifies the notation $\Gamma^{\mathcal{L}}_{22}$ in Theorem 3.1 (similarly, note that $\Gamma_{f_1;11}(\sigma)$ does not depend on $\mathcal{L}$). Moreover, even when Assumption A(ii) fails to hold, $\mathcal{P}^{\mathcal{L}(n)}_{f_1}$ remains ULAN, but then with $\underline{\Delta}^{\mathcal{L}(n)}_{f_1;2}(\sigma) := \frac{1}{\sqrt{n}}\sum_{i=1}^n [J^{\mathcal{L}}(F_1(\frac{X_i^{(n)}}{\sigma})) - \int_0^1 J^{\mathcal{L}}(u)\,du]$ and a possibly non-diagonal information matrix $\Gamma^{\mathcal{L}}_{f_1}(\sigma)$. We point out that Assumption A(ii) implies in fact that

$$J^{\mathcal{L}}(\cdot) = -J^{\mathcal{L}}(1-\cdot), \quad \text{a.e. in } (0,1), \tag{5}$$

which entails that $\underline{\Delta}^{\mathcal{L}(n)}_{f_1;2}(\sigma) = \Delta^{\mathcal{L}(n)}_{f_1;2}(\sigma)$ and that $\Delta^{(n)}_{f_1;1}(\sigma)$ and $\Delta^{\mathcal{L}(n)}_{f_1;2}(\sigma)$ are asymptotically uncorrelated under $P^{(n)}_{\sigma;f_1}$. The diagonality of $\Gamma^{\mathcal{L}}_{f_1}(\sigma)$ is important as it is the structural reason why replacing σ with an adequate estimate $\hat{\sigma}^{(n)}$ has no asymptotic impact on the δ-part of the central sequence $\Delta^{\mathcal{L}(n)}_{f_1}(\sigma)$, in the sense that, for any $\sigma \in \mathbb{R}^+_0$ and any $f_1 \in \mathcal{F}_1^{\rm ULAN}$, $\Delta^{\mathcal{L}(n)}_{f_1;2}(\hat{\sigma}^{(n)}) = \Delta^{\mathcal{L}(n)}_{f_1;2}(\sigma) + o_{\rm P}(1)$ as $n \to \infty$, under $P^{(n)}_{\sigma;f_1}$; see Lemma B.1 in the Appendix. More precisely, this actually requires

Assumption B. *The sequence of estimators* $\hat{\sigma}^{(n)}$ $(n \in \mathbb{N}_0)$ *is (i) root-n consistent (i.e.,* $n^{1/2}(\hat{\sigma}^{(n)} - \sigma) = O_{\rm P}(1)$ *as* $n \to \infty$*, under* $\cup_{g_1 \in \mathcal{F}_1} P^{(n)}_{\sigma;g_1}$*) and (ii) locally asymptotically discrete, meaning that, for all* $\sigma \in \mathbb{R}^+_0$ *and all* $c > 0$*, there exists an* $M = M(c) > 0$ *such that the number of possible values of* $\hat{\sigma}^{(n)}$ *in intervals of the form* $\{t \in \mathbb{R} : n^{1/2}|t - \sigma| \leq c\}$ *is bounded by* M*, uniformly as* $n \to \infty$*.*

It should be noted that Assumption B(ii) is a purely technical requirement, with little practical implications (for fixed sample size, any estimator indeed can be considered part of a locally asymptotically discrete sequence), so that Assumption B essentially only requires consistency of $\hat{\sigma}^{(n)}$ under the null at the standard root-n rate. Of course, an obvious example of estimators $\hat{\sigma}^{(n)}$ satisfying Assumption B is (a discretized version of) the sample median of $|X_1^{(n)}|, |X_2^{(n)}|, \ldots, |X_n^{(n)}|$.

Most importantly, the construction of locally and asymptotically optimal tests for $\mathcal{H}^{(n)}_{0,f_1}$ against two-sided alternatives of the form $\mathcal{H}^{\mathcal{L}(n)}_{1,f_1} := \cup_\sigma \cup_{\delta \neq 0} \{P^{\mathcal{L}(n)}_{\sigma,\delta;f_1}\}$ readily follows from the ULAN structure of $\mathcal{P}^{\mathcal{L}(n)}_{f_1}$ in Theorem 3.1. More precisely, denoting by z_β the upper β-quantile of the standard Gaussian distribution, it follows from Section 11.9 of [16] that the test $\phi^{\mathcal{L}(n)}_{f_1}$ that rejects $\mathcal{H}^{(n)}_{0,f_1}$ in favor of $\mathcal{H}^{\mathcal{L}(n)}_{1,f_1}$ whenever

$$\left|Q^{\mathcal{L}(n)}_{f_1}\right| > z_{\alpha/2}, \quad \text{with } Q^{\mathcal{L}(n)}_{f_1} := \frac{\Delta^{\mathcal{L}(n)}_{f_1;2}(\hat{\sigma}^{(n)})}{\left(\Gamma^{\mathcal{L}}_{22}\right)^{1/2}}, \tag{6}$$

is locally and asymptotically maximin at asymptotic level α. Clearly, optimal parametric tests against one-sided alternatives are derived along the same lines.

4. Semiparametric tests

The optimal tests $\phi_{f_1}^{\mathcal{L}(n)}$ of the previous section present a major drawback: in general, they are valid—in the sense that they meet asymptotically the α-level constraint—at the corresponding density type f_1 only, that is, under the null hypothesis $\mathcal{H}_{0,f_1}^{(n)}$ only. In practice, assuming that the underlying density type f_1 is known is of course highly unrealistic, and it is desirable to define tests that are valid under the nonparametric null hypothesis $\mathcal{H}_0^{(n)} = \cup_{f_1 \in \mathcal{F}_1} \mathcal{H}_{0,f_1}^{(n)}$. In this section, we introduce two classes of tests that enjoy the optimality properties of the parametric tests $\phi_{f_1}^{\mathcal{L}(n)}$ above, but are valid under much broader conditions. The first class is obtained via a studentization argument (Section 4.1), whereas the second class arises naturally from the group invariance structure of $\mathcal{H}_0^{(n)}$ (Section 4.2).

4.1. *Optimal studentized tests*

Under $\mathrm{P}_{\sigma;g_1}^{(n)}$, $\Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma)$, with $f_1 \in \mathcal{F}_1^{\mathrm{ULAN}}$, is asymptotically normal with mean 0 and variance $C_{g_1}^{\mathcal{L}}(f_1) := \int_{-\infty}^{\infty} [J^{\mathcal{L}}(F_1(x))]^2 g_1(x)\, dx$, provided that the latter quantity is finite. It is therefore natural to consider the *studentized* test $\phi_{*;f_1}^{\mathcal{L}(n)}$ that rejects (at asymptotic level α) the null of symmetry $\mathcal{H}_0^{(n)}$ in favor of $\mathcal{H}_1^{\mathcal{L}(n)} := \cup_\sigma \cup_{\delta\neq 0} \cup_{g_1 \in \mathcal{F}_1} \{\mathrm{P}_{\sigma,\delta;g_1}^{\mathcal{L}(n)}\}$ as soon as

$$\left| Q_{*;f_1}^{\mathcal{L}(n)} \right| > z_{\alpha/2}, \quad \text{with } Q_{*;f_1}^{\mathcal{L}(n)} := \frac{\Delta_{f_1;2}^{\mathcal{L}(n)}(\hat{\sigma}^{(n)})}{\left(C^{\mathcal{L}(n)}(f_1)\right)^{1/2}},$$

where $C^{\mathcal{L}(n)}(f_1) := \frac{1}{n}\sum_{i=1}^n [J^{\mathcal{L}}(F_1(X_i^{(n)}/\hat{\sigma}^{(n)}))]^2$ and $\hat{\sigma}^{(n)}$ satisfies Assumption B. The asymptotic properties of such tests, under any $g_1 \in \mathcal{F}_{*;f_1}^{\mathcal{L}} := \{g_1 \in \mathcal{F}_1^{\mathrm{ULAN}} : \int_{-\infty}^{\infty} [J^{\mathcal{L}}(F_1(x))]^2 g_1(x)\, dx < \infty\}$, easily follow from the asymptotic linearity in Lemma B.1 (see the Appendix) and are summarized in the following result.

Theorem 4.1: *Fix $f_1 \in \mathcal{F}_1^{\mathrm{ULAN}}$. Let $\mathcal{L}$ (resp., $\mathcal{L}_U$) be a PSM satisfying Assumptions A(i)-(iv) (resp., A(i)-(iii)), and let Assumption B hold. Then,*
(i) under $\cup_\sigma \cup_{g_1 \in \mathcal{F}_{*;f_1}^{\mathcal{L}}} \{\mathrm{P}_{\sigma;g_1}^{(n)}\}$, $Q_{*;f_1}^{\mathcal{L}(n)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ as $n \to \infty$, so that the sequence of tests $\phi_{*;f_1}^{\mathcal{L}(n)}$ has asymptotic level α under the same hypothesis;
(ii) under $\cup_\sigma \{\mathrm{P}_{\sigma,n^{-1/2}\tau_2;g_1}^{\mathcal{L}_U(n)}\}$ with $g_1 \in \mathcal{F}_{*;f_1}^{\mathcal{L}}$, $Q_{*;f_1}^{\mathcal{L}(n)} \xrightarrow{\mathcal{L}} \mathcal{N}((C_{g_1}^{\mathcal{L}}(f_1))^{-1/2} C_{g_1}^{\mathcal{L},\mathcal{L}_U}(f_1,g_1)\tau_2, 1)$ as $n \to \infty$, where $C_{g_1}^{\mathcal{L},\mathcal{L}_U}(f_1,g_1) := \int_{-\infty}^{\infty} J^{\mathcal{L}}(F_1(x)) J^{\mathcal{L}_U}(G_1(x)) g_1(x)\, dx$;
(iii) under $\cup_\sigma \{\mathrm{P}_{\sigma;f_1}^{(n)}\}$, $Q_{*;f_1}^{\mathcal{L}(n)} = Q_{f_1}^{\mathcal{L}(n)} + o_{\mathrm{P}}(1)$ as $n \to \infty$, so that the sequence of tests $\phi_{*;f_1}^{\mathcal{L}(n)}$ is locally and asymptotically maximin, at asymptotic level α, when testing $\cup_\sigma \cup_{g_1 \in \mathcal{F}_{*;f_1}^{\mathcal{L}}} \{\mathrm{P}_{\sigma;g_1}^{(n)}\}$ against alternatives of the form $\cup_\sigma \cup_{\delta \neq 0} \{\mathrm{P}_{\sigma,\delta;f_1}^{\mathcal{L}(n)}\}$.

Theorem 4.1(i) shows that the studentized tests $\phi_{*;f_1}^{\mathcal{L}(n)}$ are valid under a much larger null hypothesis than $\mathcal{H}_{0,f_1}^{(n)}$, namely under $\cup_\sigma \cup_{g_1 \in \mathcal{F}_{*;f_1}^{\mathcal{L}}} \{\mathrm{P}_{\sigma;g_1}^{(n)}\}$. For the sake of generality, we have also considered above alternatives where the underlying density g_1 is turned into an asymmetric density by means of a PSM $\mathcal{L}_U$ that might be different from the PSM $\mathcal{L}$ used in the construction of the test. Note, however, that optimality is achieved only against local alternatives characterized by $\mathcal{L}_U = \mathcal{L}$ and $g_1 = f_1$.

4.2. *Optimal signed-rank tests*

It is well-known that the nonparametric null hypothesis of symmetry $\mathcal{H}_0^{(n)}$ enjoys a strong group invariance structure. More precisely, $\mathcal{H}_0^{(n)}$ is generated by the group $\mathcal{G}_h^{(n)}, \circ$ of transformations g_h of $\mathbb{R}^n$ defined by $g_h(x_1, \ldots, x_n) := (h(x_1), \ldots, h(x_n))$, where $h : \mathbb{R} \to \mathbb{R}$ is any continuous, odd, and strictly monotone increasing function satisfying $\lim_{x\to\infty} h(x) = \infty$. When the null is invariant under a group of transformations, the *invariance principle* suggests restricting to tests that are measurable with respect to the corresponding maximal invariant. In the present context, the maximal invariant is the vector of signed ranks $(S_1^{(n)} R_1^{(n)}, \ldots, S_n^{(n)} R_n^{(n)})$, where $S_i^{(n)} := \mathrm{Sign}(X_i^{(n)})$ stands for the sign of $X_i^{(n)}$ and $R_i^{(n)}$ denotes the rank of $|X_i^{(n)}|$ among $|X_1^{(n)}|, \ldots, |X_n^{(n)}|$. The invariance structure of $\mathcal{H}_0^{(n)}$ thus naturally brings signed-rank tests into the picture.

Now, since F_1 is the cdf of a symmetric distribution, we have that $F_1(x) = (1 + \mathrm{Sign}(x) F_1^+(|x|))/2$, where F_1^+ stands for the cdf of $|X_1^{(n)}|$ under $\mathrm{P}^{(n)}_{1;f_1}$. This, combined with the symmetry property of $J^{\mathcal{L}}(\cdot)$ in (5), allows for rewriting the δ-part of the central sequence as

$$\Delta^{\mathcal{L}(n)}_{f_1;2}(\sigma) = \frac{1}{\sqrt{n}} \sum_{i=1}^n S_i^{(n)} J^{\mathcal{L}}\big((1 + F_1^+(|\tfrac{X_i^{(n)}}{\sigma}|))/2\big) = \frac{1}{\sqrt{n}} \sum_{i=1}^n S_i^{(n)} J_+^{\mathcal{L}}\big(F_1^+(|\tfrac{X_i^{(n)}}{\sigma}|)\big),$$

where we let $J_+^{\mathcal{L}}(u) := J^{\mathcal{L}}(\frac{1+u}{2})$. Defining

$$\Delta^{\mathcal{L}(n)}_{\dagger;2} := \frac{1}{\sqrt{n}} \sum_{i=1}^n S_i^{(n)} J_+^{\mathcal{L}}\Big(\frac{R_i^{(n)}}{n+1}\Big),$$

Hájek's classical projection result for linear signed-rank statistics (see, e.g., Chapter 3 of [20]) then readily yields the following.

Lemma 4.2: *Let $\mathcal{L}$ be a PSM satisfying Assumptions A(i)-(iv)′. Then, for any $\sigma \in \mathbb{R}_0^+$ and any $g_1 \in \mathcal{F}_1$, $\Delta^{\mathcal{L}(n)}_{\dagger;2} = \Delta^{\mathcal{L}(n)}_{g_1;2}(\sigma) + o_{\mathrm{P}}(1)$, as $n \to \infty$, under $\mathrm{P}^{(n)}_{\sigma;g_1}$.*

The resulting signed-rank test $\phi_\dagger^{\mathcal{L}(n)}$ then rejects $\mathcal{H}_0^{(n)}$ in favor of $\mathcal{H}_1^{\mathcal{L}(n)}$ (at asymptotic level α) as soon as

$$\Big|Q_\dagger^{\mathcal{L}(n)}\Big| > z_{\alpha/2}, \quad \text{with } Q_\dagger^{\mathcal{L}(n)} := \frac{\Delta^{\mathcal{L}(n)}_{\dagger;2}}{(\Gamma_{22}^{\mathcal{L}})^{1/2}}.$$

Unlike the studentized tests of the previous section, these signed-rank tests do not require any estimation of the underlying scale value σ. The following theorem states the asymptotic properties of the tests $\phi_\dagger^{\mathcal{L}(n)}$.

Theorem 4.3: *Let $\mathcal{L}$ (resp., $\mathcal{L}_U$) be a PSM satisfying Assumptions A(i)-(iv)′ (resp., A(i)-(iii)), and define $C^{\mathcal{L},\mathcal{L}_U} := \int_0^1 J^{\mathcal{L}}(u) J^{\mathcal{L}_U}(u)\, du$. Then,*
(i) under $\cup_\sigma \cup_{g_1 \in \mathcal{F}_1} \{\mathrm{P}^{(n)}_{\sigma;g_1}\}$, $Q_\dagger^{\mathcal{L}(n)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ as $n \to \infty$, so that the sequence of

tests $\phi_{\dagger}^{\mathcal{L}(n)}$ *has asymptotic level* α *under the same hypothesis;*
(ii) under $\cup_\sigma \{\mathrm{P}^{\mathcal{L}_U(n)}_{\sigma,n^{-1/2}\tau_2;g_1}\}$ *with* $g_1 \in \mathcal{F}_1$, $Q_{\dagger}^{\mathcal{L}(n)} \xrightarrow{\mathcal{L}} \mathcal{N}((\Gamma_{22}^{\mathcal{L}})^{-1/2} C^{\mathcal{L},\mathcal{L}_U}\tau_2, 1)$ *as* $n \to \infty$*;*
(iii) under $\cup_\sigma \{\mathrm{P}^{(n)}_{\sigma;g_1}\}$ *with* $g_1 \in \mathcal{F}_1$, $Q_{\dagger}^{\mathcal{L}(n)} = \Delta^{\mathcal{L}(n)}_{g_1;2}(\sigma)/\big(\Gamma_{22}^{\mathcal{L}}\big)^{1/2} + o_{\mathrm{P}}(1)$ *as* $n \to \infty$*, so that the sequence of tests* $\phi_{\dagger}^{\mathcal{L}(n)}$ *is locally and asymptotically maximin, at asymptotic level* α*, when testing* $\cup_\sigma \cup_{g_1 \in \mathcal{F}_1} \{\mathrm{P}^{(n)}_{\sigma;g_1}\}$ *against alternatives of the form* $\cup_\sigma \cup_{\delta \neq 0} \cup_{g_1 \in \mathcal{F}_1} \{\mathrm{P}^{\mathcal{L}(n)}_{\sigma,\delta;g_1}\}$.

This result shows that the signed-rank tests improve on the studentized ones in several respects. First of all, the signed-rank tests meet the asymptotic α-level constraint under broader conditions, namely under $\cup_\sigma \cup_{g_1 \in \mathcal{F}_1} \{\mathrm{P}^{(n)}_{\sigma;g_1}\}$ (whereas studentized tests are valid under $\cup_\sigma \cup_{g_1 \in \mathcal{F}^{\mathcal{L}}_{*;f_1}} \{\mathrm{P}^{(n)}_{\sigma;g_1}\}$). Secondly, in sharp contrast with the optimal studentized test $\phi^{\mathcal{L}(n)}_{*;f_1}$, which achieves Le Cam optimality at the target density type $f_1 (\in \mathcal{F}_1^{\mathrm{ULAN}})$ only, the signed-rank tests $\phi_{\dagger}^{\mathcal{L}(n)}$ are optimal at *any* $g_1 \in \mathcal{F}_1$. Such uniform optimality rarely occurs in rank-based inference. Note however that, exactly as for the studentized tests, optimality of the signed-rank tests is not uniform in the PSM: optimality is achieved only if the underlying PSM $\mathcal{L}_U$ coincides with the PSM $\mathcal{L}$ used in the tests.

5. Some particular cases

In this section, we consider four examples of PSMs satisfying Assumption A and derive, in each case, the corresponding parametric, studentized, and signed-rank test statistics introduced above. For the sake of illustration, Figure 1 provides plots of several cdfs belonging to each PSM, along with the corresponding skewed (Gaussian) densities.

As a first example, consider the skewing mechanism $\mathcal{L}_1 := \{L_{1\delta} : \delta \in \mathbb{R}\}$ defined by

$$L_{1\delta}(u) := \begin{cases} u e^{\delta(u-1)} & \text{if } \delta \geq 0 \\ 1 - (1-u)e^{\delta u} & \text{if } \delta < 0. \end{cases}$$

Straightforward calculations reveal that $J^{\mathcal{L}_1}(u) = 2u - 1$ and $\Gamma_{22}^{\mathcal{L}_1} = 1/3$, so that the parametric and the studentized test statistics achieving Le Cam optimality at target density f_1 are given by

$$Q_{f_1}^{\mathcal{L}_1(n)} = \sqrt{\frac{3}{n}} \sum_{i=1}^{n} \big(2F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - 1\big)$$

and

$$Q_{*;f_1}^{\mathcal{L}_1(n)} = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \big(2F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - 1\big)}{\big(\frac{1}{n} \sum_{i=1}^{n} \big(2F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - 1\big)^2\big)^{1/2}},$$

respectively (throughout this section, $\hat{\sigma}^{(n)}$ stands for an arbitrary estimator satisfying Assumption B). The score function $J_+^{\mathcal{L}_1}$ simply reduces to the identity function, which implies that the corresponding signed-rank test coincides with the celebrated

Wilcoxon signed-rank test, based on

$$Q_{\dagger}^{\mathcal{L}_1(n)} = \sqrt{\frac{3}{n}} \sum_{i=1}^{n} S_i^{(n)} \frac{R_i^{(n)}}{n+1}.$$

The second PSM $\mathcal{L}_2 := \{L_{2\delta} : \delta \in [-1, 1]\}$ is defined by

$$L_{2\delta}(u) := \begin{cases} u(1-\delta) & \text{if } 0 \le u \le \frac{1}{2} \\ u + \delta(u-1) & \text{if } \frac{1}{2} < u \le 1 \end{cases}$$

(if $|\delta| > 1$, $L_{2\delta}(\cdot)$ is not monotone increasing over $(0, 1)$, hence fails to be a cdf). Although this PSM is not as smooth as the other examples considered in this section, it satisfies Assumption A, with $|J^{\mathcal{L}_2}(\cdot)| = J_+^{\mathcal{L}_2}(\cdot) = 1$ a.e. in $(0, 1)$. Quite interestingly, this implies that all f_1-parametric tests, as well as their respective studentized and signed-rank counterparts, are based on the statistic

$$Q_{f_1}^{\mathcal{L}_2(n)} = Q_{*;f_1}^{\mathcal{L}_2(n)} = Q_{\dagger}^{\mathcal{L}_2(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} S_i^{(n)},$$

hence coincide with the classical *sign test.*

In most textbooks about nonparametric statistics (see, e.g., [9], [19], and [21]), the Wilcoxon signed-rank test and the sign test are presented both as one-sample location tests and as symmetry tests. Their optimality properties, however, are stated against location alternatives only; more precisely, the Wilcoxon test (resp., the sign test) is reported to be locally most powerful against logistic (resp., Laplace) location alternatives, i.e., alternatives of the form $X_i^{(n)} = \theta + \sigma Z_i^{(n)}$, with $\theta \neq 0$, $\sigma > 0$, and where the $Z_i^{(n)}$'s are i.i.d. with pdf $x \mapsto \exp(x)/(1 + \exp(x))^2$ (resp., pdf $x \mapsto \exp(-|x|)/2$). In view of Theorem 4.3, the Wilcoxon test (resp., the sign test) is also Le Cam optimal against the alternatives $\cup_\sigma \cup_{\delta \neq 0} \cup_{g_1 \in \mathcal{F}_1} \{\mathrm{P}_{\sigma,\delta;g_1}^{\mathcal{L}_j(n)}\}$, with $j = 1$ (resp., $j = 2$), which are not pure location alternatives. Moreover, note that the logistic and Laplace densities do not play any special role here, as optimality is uniform in the underlying density type g_1. We stress that other PSMs actually also lead to the Wilcoxon test, hence provide further asymmetric alternatives against which Wilcoxon is Le Cam optimal. Examples of such PSMs are obtained by defining, for $\delta \geq 0$ (the values for $\delta < 0$ are obtained from the duality assumption in (4)), $L_\delta = u(1 - \arctan(\delta(1-u)))$ or $L_\delta = u(1+\delta)^{u-1}$.

As a further example, which does not lead to a classical signed-rank test of symmetry, consider the PSM $\mathcal{L}_3 := \{L_{3\delta} : \delta \in [-\pi^{-1}, \pi^{-1}]\}$ defined by

$$L_{3\delta}(u) := u - \delta \sin(\pi u).$$

It can easily be checked that $J^{\mathcal{L}_3}(u) = -\pi \cos(\pi u)$ and $\Gamma_{22}^{\mathcal{L}_3} = \pi^2/2$, so that the proposed test statistics are based on trigonometric score functions, that is,

$$Q_{f_1}^{\mathcal{L}_3(n)} = -\sqrt{\frac{2}{n}} \sum_{i=1}^{n} \cos(\pi F_1(X_i^{(n)}/\hat{\sigma}^{(n)})),$$

$$Q_{*;f_1}^{\mathcal{L}_3(n)} = \frac{-\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \cos(\pi F_1(X_i^{(n)}/\hat{\sigma}^{(n)}))}{\left(\frac{1}{n} \sum_{i=1}^{n} \cos^2(\pi F_1(X_i^{(n)}/\hat{\sigma}^{(n)}))\right)^{1/2}},$$

and

$$Q_{\dagger}^{\mathcal{L}_3(n)} = \sqrt{\frac{2}{n}} \sum_{i=1}^{n} S_i^{(n)} \sin\Big(\frac{\pi R_i^{(n)}}{2(n+1)}\Big). \tag{7}$$

The last example we consider is the PSM $\mathcal{L}_4 := \{L_{4\delta} : \delta \in \mathcal{D}\}$, where

$$L_{4\delta}(u) := \begin{cases} ue^{\delta(u-\frac{1}{2})^3} & \text{if } 0 \leq u \leq \frac{1}{2} \\ 1-(1-u)e^{\delta(u-\frac{1}{2})^3} & \text{if } \frac{1}{2} < u \leq 1 \end{cases}$$

and the interval $\mathcal{D}$ is such that $L_{4\delta}(\cdot)$ is monotone increasing. Note that $L_{4\delta}(\frac{1}{2}) = \frac{1}{2}$ for any $\delta \in \mathcal{D}$ so that this skewing mechanism, unlike the previous ones, fixes the median of the original symmetric distribution. Since the argument of the exponential term in $L_{4\delta}$ contains a third-order polynomial in u, we obtain quite naturally that $J^{\mathcal{L}_4}(u) = 4(u-\frac{1}{2})^2[u-\frac{1}{2}-\frac{3}{8}\text{Sign}(u-\frac{1}{2})]$, leading to $\Gamma_{22}^{\mathcal{L}_4} = \frac{3}{2240}$ and to test statistics based on third-order score functions, i.e.

$$Q_{f_1}^{\mathcal{L}_4(n)} = 32\sqrt{\frac{35}{3}}\, n^{-1/2} \sum_{i=1}^{n} (F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - \tfrac{1}{2})^2 [F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - \tfrac{1}{2} - \tfrac{3}{8} S_i^{(n)}],$$

$$Q_{*;f_1}^{\mathcal{L}_4(n)} = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - \frac{1}{2})^2 [F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - \frac{1}{2} - \frac{3}{8} S_i^{(n)}]}{\big(\frac{1}{n} \sum_{i=1}^{n} (F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - \frac{1}{2})^4 [F_1(X_i^{(n)}/\hat{\sigma}^{(n)}) - \frac{1}{2} - \frac{3}{8} S_i^{(n)}]^2\big)^{1/2}},$$

and

$$Q_{\dagger}^{\mathcal{L}_4(n)} = 16\sqrt{\frac{35}{3}} n^{-1/2} \sum_{i=1}^{n} S_i^{(n)} \Big(\frac{R_i^{(n)}}{n+1}\Big)^2 \Big(\frac{R_i^{(n)}}{n+1} - \frac{3}{4}\Big).$$

6. Asymptotic relative efficiencies

We now compare the performances of the various proposed tests by deriving their asymptotic relative efficiencies (AREs) with respect to a benchmark test of symmetry, namely the classical *test of skewness* ($\phi_{\text{skew}}^{(n)}$, say). At asymptotic level α, the latter rejects the null of symmetry $\mathcal{H}_0^{(n)}$ in favor of $\mathcal{H}_1^{\mathcal{L}(n)}$ (for any $\mathcal{L}$) iff

$$\Big|Q_{\text{skew}}^{(n)}\Big| > z_{\alpha/2}, \quad \text{with } Q_{\text{skew}}^{(n)} := \frac{n^{1/2} m_3^{(n)}}{\big(m_6^{(n)}\big)^{1/2}},$$

where $m_\ell^{(n)} := \frac{1}{n}\sum_{i=1}^{n}(X_i^{(n)})^\ell$ stands for the sample moment of order ℓ. Clearly, asymptotic validity of this test requires finite sixth-order moments.

Computing the AREs of the proposed tests with respect to $\phi_{\text{skew}}^{(n)}$ of course requires determining the asymptotic behavior of the latter under the local alternatives considered in this paper. This is achieved in the following result (the proof, which is similar to those of Theorems 4.1 and 4.3, is left to the reader).

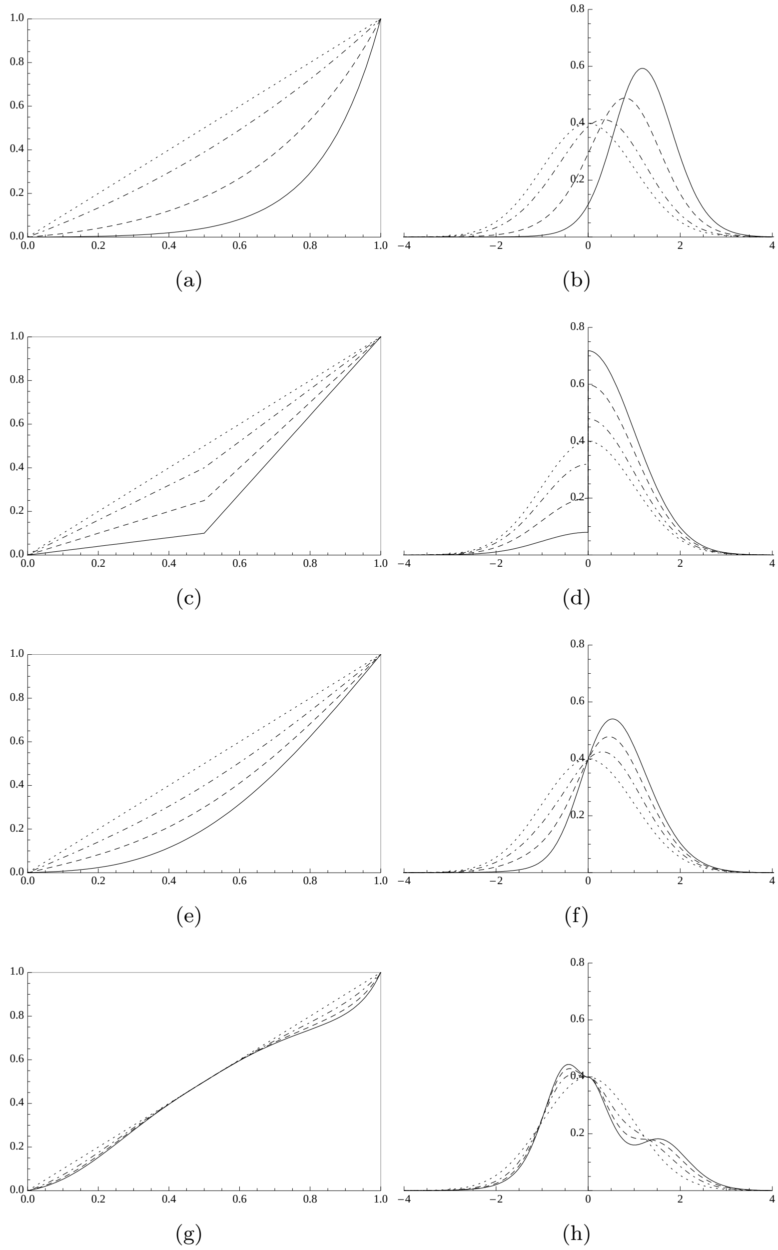

Figure 1. Plots of $L_{j\delta}$, $j = 1, 2, 3$, and 4 ((a), (c), (e), and (g)) and of the resulting skewed versions of the standard Gaussian density ((b), (d), (f), and (h)) for $\delta = 0, 0.5, 2, 5$ in (a)-(b), for $\delta = 0, 0.2, 0.5, 0.8$ in (c)-(d), for $\delta = 0, 0.1, 0.2, 0.3$ in (e)-(f), and for $\delta = 0, 5, 8, 10$ in (g)-(h). Increasing values of δ are successively associated with dotted, dash-dot, dashed, and solid lines.

Proposition 6.1: *Let $\mathcal{L}_U$ be a PSM satisfying Assumptions A(i)-(iii). Define $\mathcal{F}_1^{\text{skew}} := \{g_1 \in \mathcal{F}_1 : \mu_{6;g_1} < \infty\}$, where $\mu_{\ell;g_1} := \int_{-\infty}^{\infty} x^\ell g_1(x)\,dx$, and let $C_{\text{skew}}^{\mathcal{L}_U}(g_1) := \int_{-\infty}^{\infty} x^3 J^{\mathcal{L}_U}(G_1(x)) g_1(x)\,dx$. Then,*
(i) under $\cup_\sigma \cup_{g_1 \in \mathcal{F}_1^{\text{skew}}} \{\mathrm{P}_{\sigma;g_1}^{(n)}\}$, $Q_{\text{skew}}^{(n)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ as $n \to \infty$, so that the sequence of tests $\phi_{\text{skew}}^{(n)}$ has asymptotic level α under the same hypothesis;
(ii) under $\cup_\sigma \{\mathrm{P}_{\sigma,n^{-1/2}\tau_2;g_1}^{\mathcal{L}_U(n)}\}$ with $g_1 \in \mathcal{F}_1^{\text{skew}}$, $Q_{\text{skew}}^{(n)} \xrightarrow{\mathcal{L}} \mathcal{N}((\mu_{6;g_1})^{-1/2} C_{\text{skew}}^{\mathcal{L}_U}(g_1)\tau_2, 1)$ as $n \to \infty$.

The shifts in the asymptotic non-null distributions provided in Theorems 4.1

and 4.3 as well as in Proposition 6.1 allow for computing the desired ARE values, which are simply the squared ratios of those local shifts. As the proposed signed-rank tests do not require any moment assumption, their ARE values with respect to $\phi^{(n)}_{\text{skew}}$ can be considered as being infinite under any PSM $\mathcal{L}_U$ and any g_1 with infinite sixth-order moment $\mu_{6;g_1}$.

Theorem 6.2: *Let $\mathcal{L}$ and $\mathcal{L}_U$ be two PSMs satisfying Assumptions A(i)-(iii). Then, (i) if $f_1 \in \mathcal{F}_1^{\text{ULAN}}$ and if Assumption B holds, the ARE of the studentized test $\phi^{\mathcal{L}(n)}_{*;f_1}$ with respect to $\phi^{(n)}_{\text{skew}}$, for local alternatives of the form $\mathrm{P}^{\mathcal{L}_U(n)}_{\sigma,n^{-1/2}\tau_2;g_1}$, with $\tau_2 \neq 0$ and $g_1 \in \mathcal{F}^{\mathcal{L}}_{*;f_1} \cap \mathcal{F}_1^{\text{skew}}$, is given by*

$$\text{ARE}_{\mathcal{L}_U,g_1}(\phi^{\mathcal{L}(n)}_{*;f_1}/\phi^{(n)}_{\text{skew}}) = \frac{(C^{\mathcal{L},\mathcal{L}_U}_{g_1}(f_1,g_1))^2\mu_{6;g_1}}{(C^{\mathcal{L}_U}_{\text{skew}}(g_1))^2 C^{\mathcal{L}}_{g_1}(f_1)},$$

provided that $\mathcal{L}$ further satisfies Assumption A(iv); (ii) the ARE of the signed-rank test $\phi^{\mathcal{L}(n)}_{\dagger}$ with respect to $\phi^{(n)}_{\text{skew}}$, for local alternatives of the form $\mathrm{P}^{\mathcal{L}_U(n)}_{\sigma,n^{-1/2}\tau_2;g_1}$, with $\tau_2 \neq 0$ and $g_1 \in \mathcal{F}_1^{\text{skew}}$, is given by

$$\text{ARE}_{\mathcal{L}_U,g_1}(\phi^{\mathcal{L}(n)}_{\dagger}/\phi^{(n)}_{\text{skew}}) = \frac{(C^{\mathcal{L},\mathcal{L}_U})^2\mu_{6;g_1}}{(C^{\mathcal{L}_U}_{\text{skew}}(g_1))^2\Gamma^{\mathcal{L}}_{22}},$$

provided that $\mathcal{L}$ further satisfies Assumption A(iv)′.

Table 1 provides numerical values of the AREs, with respect to $\phi^{(n)}_{\text{skew}}$ and under various alternatives, of the Wilcoxon signed-rank test $\phi^{\mathcal{L}_1(n)}_{\dagger}$, the sign test $\phi^{\mathcal{L}_2(n)}_{\dagger}$, the signed-rank test $\phi^{\mathcal{L}_3(n)}_{\dagger}$ based on (7), and of several studentized tests $\phi^{\mathcal{L}_j(n)}_{*;f_1}$ $(j = 1, 3)$; see Section 5. The alternatives considered are those obtained by skewing, via the PSMs $\mathcal{L}_j$ $(j = 1, 2, 3)$, Student densities (g_{t_ν}) with $\nu = 7$ and 10 degrees of freedom, Gaussian densities (g_ϕ), and power-exponential densities (g_{e_η}) with $\eta = 2$ and 5; here, power-exponential densities with parameter η refer to densities of the form $x \mapsto g_{e_\eta}(x) = c_\eta\sigma^{-1}\exp(-a_\eta(x/\sigma)^{2\eta})$, where c_η is a normalization constant, $\eta > 0$ determines the tail weight, and $a_\eta > 0$ is such that $g_{e_\eta} \in \mathcal{F}_1$.

Those ARE values are uniformly high, underlining that the proposed tests strongly dominate the classical test of skewness, with the only exception of the performance of the sign test under $\mathcal{L}_1$-skewed versions of the light-tailed density g_{e_5}. In particular, our tests maintain very good performances when they are based on a PSM that does not correspond to the one generating local alternatives. For some specific tests, however, the latter remark might fail to hold when considering alternatives generated via PSMs that fix the median (for instance, the sign test would not exhibit any power under $\mathcal{L}_4$-alternatives). Eventually, note that those AREs clearly confirm the uniform optimality, under each PSM $\mathcal{L}_j$ $(j = 1, 2, 3)$, of the corresponding signed-rank test $\phi^{\mathcal{L}_j(n)}_{\dagger}$.

7. Simulation results

In order to examine the finite-sample performances of the proposed procedures, we generated $N = 10,000$ independent samples of size $n = 200$ from symmetric Gaussian and Student (with $\nu = 2$, 7, and 10 degrees of freedom) densities, and increasingly skewed (to the right) versions of the same densities (three positive

Table 1. AREs, with respect to $\phi^{(n)}_{\text{skew}}$, under $\mathcal{L}_j$-alternatives ($j = 1, 2, 3$; see Section 5) with t_ν ($\nu = 7$ and 10), Gaussian, and e_η ($\eta = 2$ and 5) densities, of the Wilcoxon signed-rank test $\phi^{\mathcal{L}_1(n)}_{\dagger}$, the sign test $\phi^{\mathcal{L}_2(n)}_{\dagger}$, the signed-rank test $\phi^{\mathcal{L}_3(n)}_{\dagger}$, and various studentized tests $\phi^{\mathcal{L}_j(n)}_{*;f_1}$, $j = 1, 3$.

	Underlying PSM $\mathcal{L}_U$ and density g_1				
	g_{t_7}	$g_{t_{10}}$	g_ϕ	g_{e_2}	g_{e_5}
Test			$\mathcal{L}_1$		
$\phi^{\mathcal{L}_1(n)}_{*;f_{t_7}}$	13.70210	5.18938	2.51315	1.56880	1.26913
$\phi^{\mathcal{L}_1(n)}_{*;f_{t_{10}}}$	13.70200	5.18942	2.51320	1.56871	1.26920
$\phi^{\mathcal{L}_1(n)}_{*;f_\phi}$	13.70030	5.18913	2.51327	1.56848	1.26935
$\phi^{\mathcal{L}_1(n)}_{*;f_{e_2}}$	13.45630	5.10287	2.47841	1.57080	1.26895
$\phi^{\mathcal{L}_1(n)}_{*;f_{e_5}}$	13.43670	5.09584	2.47551	1.56926	1.27022
$\phi^{\mathcal{L}_1(n)}_{\dagger}$	13.70210	5.18942	2.51327	1.57080	1.27022
$\phi^{\mathcal{L}_2(n)}_{\dagger}$	10.27660	3.89206	1.88496	1.17810	0.95266
$\phi^{\mathcal{L}_3(n)}_{*;f_{t_7}}$	13.50390	5.11844	2.48294	1.56755	1.26310
$\phi^{\mathcal{L}_3(n)}_{*;f_{t_{10}}}$	13.49240	5.11435	2.48128	1.56747	1.26301
$\phi^{\mathcal{L}_3(n)}_{*;f_\phi}$	13.46280	5.10380	2.47692	1.56724	1.26277
$\phi^{\mathcal{L}_3(n)}_{*;f_{e_2}}$	12.95680	4.91504	2.38928	1.54807	1.24658
$\phi^{\mathcal{L}_3(n)}_{*;f_{e_5}}$	12.99950	4.93140	2.39740	1.55345	1.25184
$\phi^{\mathcal{L}_3(n)}_{\dagger}$	13.50390	5.11435	2.47692	1.54807	1.25184
			$\mathcal{L}_2$		
$\phi^{\mathcal{L}_1(n)}_{*;f_{t_7}}$	26.02340	9.60127	4.42407	2.48236	1.92523
$\phi^{\mathcal{L}_1(n)}_{*;f_{t_{10}}}$	26.02440	9.60000	4.42185	2.47980	1.92362
$\phi^{\mathcal{L}_1(n)}_{*;f_\phi}$	26.03780	9.60082	4.41786	2.47360	1.91974
$\phi^{\mathcal{L}_1(n)}_{*;f_{e_2}}$	27.67520	10.18720	4.66468	2.49912	1.92270
$\phi^{\mathcal{L}_1(n)}_{*;f_{e_5}}$	27.35630	10.06660	4.60579	2.44426	1.88239
$\phi^{\mathcal{L}_1(n)}_{\dagger}$	26.02340	9.60000	4.41786	2.49912	1.88239
$\phi^{\mathcal{L}_2(n)}_{\dagger}$	34.69780	12.80000	5.89049	3.33216	2.50985
$\phi^{\mathcal{L}_3(n)}_{*;f_{t_7}}$	28.12500	10.36080	4.75367	2.59295	1.99023
$\phi^{\mathcal{L}_3(n)}_{*;f_{t_{10}}}$	28.16560	10.37530	4.75968	2.59397	1.99068
$\phi^{\mathcal{L}_3(n)}_{*;f_\phi}$	28.26490	10.41080	4.77465	2.59669	1.99191
$\phi^{\mathcal{L}_3(n)}_{*;f_{e_2}}$	29.74780	10.95650	5.02325	2.70095	2.05567
$\phi^{\mathcal{L}_3(n)}_{*;f_{e_5}}$	29.52880	10.87480	4.98454	2.67214	2.03441
$\phi^{\mathcal{L}_3(n)}_{\dagger}$	28.12500	10.37530	4.77465	2.70095	2.03441
			$\mathcal{L}_3$		
$\phi^{\mathcal{L}_1(n)}_{*;f_{t_7}}$	17.93350	6.67356	3.12851	1.83980	1.44737
$\phi^{\mathcal{L}_1(n)}_{*;f_{t_{10}}}$	17.93820	6.67485	3.12858	1.83913	1.44707
$\phi^{\mathcal{L}_1(n)}_{*;f_\phi}$	17.95230	6.67898	3.12919	1.83749	1.44631
$\phi^{\mathcal{L}_1(n)}_{*;f_{e_2}}$	18.17070	6.76371	3.17095	1.85060	1.44925
$\phi^{\mathcal{L}_1(n)}_{*;f_{e_5}}$	18.13480	6.75011	3.16424	1.83594	1.43982
$\phi^{\mathcal{L}_1(n)}_{\dagger}$	17.93350	6.67485	3.12919	1.85060	1.43982
$\phi^{\mathcal{L}_2(n)}_{\dagger}$	14.74970	5.48985	2.57365	1.52206	1.18421
$\phi^{\mathcal{L}_3(n)}_{*;f_{t_7}}$	18.19670	6.77275	3.17475	1.86590	1.45756
$\phi^{\mathcal{L}_3(n)}_{*;f_{t_{10}}}$	18.19650	6.77283	3.17493	1.86617	1.45764
$\phi^{\mathcal{L}_3(n)}_{*;f_\phi}$	18.19400	6.77235	3.17512	1.86684	1.45784
$\phi^{\mathcal{L}_3(n)}_{*;f_{e_2}}$	17.94200	6.68371	3.13930	1.87776	1.46050
$\phi^{\mathcal{L}_3(n)}_{*;f_{e_5}}$	17.97390	6.69547	3.14466	1.87718	1.46096
$\phi^{\mathcal{L}_3(n)}_{\dagger}$	18.19670	6.77283	3.17512	1.87776	1.46096

Table 2. Rejection frequencies (out of $N = 10,000$ replications), under various symmetric and $\mathcal{L}_1$-skewed Gaussian and Student (with $\nu = 2$, 7, and 10 degrees of freedom) densities, of the Wilcoxon signed-rank test $\phi_{\dagger}^{\mathcal{L}_1(n)}$, the sign test $\phi_{\dagger}^{\mathcal{L}_2(n)}$, the signed-rank test $\phi_{\dagger}^{\mathcal{L}_3(n)}$, various studentized tests $\phi_{*;f_1}^{\mathcal{L}_j(n)}$ $(j = 1, 3)$, the classical test of skewness $\phi_{\rm skew}^{(n)}$, the Laplace, Wilcoxon, and normal-score versions of the signed-rank tests $\phi_{\rm Ca,L}^{(n)}$, $\phi_{\rm Ca,W}^{(n)}$, and $\phi_{\rm Ca,N}^{(n)}$ and the runs test $\phi_{\rm runs}^{(n)}$.

	g_{t_2}				g_{t_7}			
Test	$\delta = 0$	$\delta = .15$	$\delta = .30$	$\delta = .45$	$\delta = 0$	$\delta = .15$	$\delta = .30$	$\delta = .45$
$\phi_{*;f_{t_2}}^{\mathcal{L}_1(n)}$	0.0505	0.2153	0.6194	0.9127	0.0508	0.2219	0.6284	0.9110
$\phi_{*;f_{t_7}}^{\mathcal{L}_1(n)}$	0.0506	0.2143	0.6188	0.9123	0.0520	0.2221	0.6289	0.9132
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_1(n)}$	0.0509	0.2138	0.6185	0.9123	0.0520	0.2224	0.6291	0.9132
$\phi_{*;f_{\phi}}^{\mathcal{L}_1(n)}$	0.0504	0.2133	0.6174	0.9124	0.0520	0.2223	0.6301	0.9134
$\phi_{\dagger}^{\mathcal{L}_1(n)}$	0.0505	0.2146	0.6185	0.9123	0.0527	0.2209	0.6266	0.9120
$\phi_{\dagger}^{\mathcal{L}_2(n)}$	0.0620	0.1943	0.5239	0.8331	0.0570	0.1894	0.5326	0.8371
$\phi_{*;f_{t_2}}^{\mathcal{L}_3(n)}$	0.0504	0.2138	0.6147	0.9091	0.0501	0.2191	0.6210	0.9073
$\phi_{*;f_{t_7}}^{\mathcal{L}_3(n)}$	0.0500	0.2144	0.6154	0.9099	0.0506	0.2204	0.6220	0.9070
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_3(n)}$	0.0497	0.2142	0.6155	0.9102	0.0503	0.2207	0.6224	0.9070
$\phi_{*;f_{\phi}}^{\mathcal{L}_3(n)}$	0.0498	0.2144	0.6155	0.9105	0.0506	0.2206	0.6222	0.9072
$\phi_{\dagger}^{\mathcal{L}_3(n)}$	0.0505	0.2139	0.6148	0.9097	0.0512	0.2201	0.6195	0.9071
$\phi_{\rm skew}^{(n)}$	0.0095	0.0155	0.0351	0.0698	0.0340	0.0787	0.2057	0.3912
$\phi_{\rm Ca,L}^{(n)}$	0.0518	0.1974	0.5594	0.8699	0.0482	0.1934	0.5654	0.8680
$\phi_{\rm Ca,W}^{(n)}$	0.0505	0.1837	0.5270	0.8409	0.0475	0.1811	0.5321	0.8387
$\phi_{\rm Ca,N}^{(n)}$	0.0516	0.1665	0.4802	0.7931	0.0475	0.1637	0.4827	0.7922
$\phi_{\rm runs}^{(n)}$	0.0458	0.0511	0.0957	0.1857	0.0438	0.0534	0.0904	0.1793

	$g_{t_{10}}$				g_{ϕ}			
Test	$\delta = 0$	$\delta = .15$	$\delta = .30$	$\delta = .45$	$\delta = 0$	$\delta = .15$	$\delta = .30$	$\delta = .45$
$\phi_{*;f_{t_2}}^{\mathcal{L}_1(n)}$	0.0548	0.2279	0.6293	0.9065	0.0520	0.2166	0.6230	0.9077
$\phi_{*;f_{t_7}}^{\mathcal{L}_1(n)}$	0.0549	0.2272	0.6286	0.9074	0.0522	0.2164	0.6243	0.9087
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_1(n)}$	0.0549	0.2272	0.6280	0.9071	0.0523	0.2167	0.6247	0.9086
$\phi_{*;f_{\phi}}^{\mathcal{L}_1(n)}$	0.0551	0.2272	0.6281	0.9071	0.0522	0.2170	0.6243	0.9090
$\phi_{\dagger}^{\mathcal{L}_1(n)}$	0.0545	0.2267	0.6265	0.9068	0.0515	0.2147	0.6219	0.9081
$\phi_{\dagger}^{\mathcal{L}_2(n)}$	0.0598	0.1925	0.5257	0.8266	0.0578	0.1862	0.5245	0.8247
$\phi_{*;f_{t_2}}^{\mathcal{L}_3(n)}$	0.0548	0.2219	0.6214	0.9010	0.0513	0.2128	0.6140	0.9014
$\phi_{*;f_{t_7}}^{\mathcal{L}_3(n)}$	0.0545	0.2212	0.6228	0.9014	0.0516	0.2138	0.6152	0.9026
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_3(n)}$	0.0544	0.2213	0.6234	0.9015	0.0516	0.2137	0.6150	0.9029
$\phi_{*;f_{\phi}}^{\mathcal{L}_3(n)}$	0.0544	0.2216	0.6233	0.9021	0.0512	0.2136	0.6149	0.9033
$\phi_{\dagger}^{\mathcal{L}_3(n)}$	0.0544	0.2228	0.6214	0.9028	0.0515	0.2138	0.6136	0.9033
$\phi_{\rm skew}^{(n)}$	0.0433	0.0951	0.2425	0.4484	0.0434	0.1219	0.3256	0.5799
$\phi_{\rm Ca,L}^{(n)}$	0.0544	0.2045	0.5652	0.8611	0.0509	0.1932	0.5689	0.8639
$\phi_{\rm Ca,W}^{(n)}$	0.0549	0.1915	0.5335	0.8287	0.0484	0.1795	0.5338	0.8393
$\phi_{\rm Ca,N}^{(n)}$	0.0553	0.1721	0.4861	0.7811	0.0473	0.1647	0.4815	0.7860
$\phi_{\rm runs}^{(n)}$	0.0438	0.0553	0.0928	0.1846	0.0438	0.0554	0.0982	0.1778

values of the skewness parameter δ were used in each case). Skewing was achieved through the PSMs $\mathcal{L}_j$ $(j = 1, 2, 3)$ defined in Section 5. For each resulting sample, we performed the following tests of symmetry under two-sided form at asymptotic level $\alpha = 5\%$: the Wilcoxon signed-rank test $\phi_{\dagger}^{\mathcal{L}_1(n)}$, the sign test $\phi_{\dagger}^{\mathcal{L}_2(n)}$, the signed-rank test $\phi_{\dagger}^{\mathcal{L}_3(n)}$, various studentized tests $\phi_{*;f_1}^{\mathcal{L}_j(n)}$ $(j = 1, 3)$, the classical test of skewness $\phi_{\rm skew}^{(n)}$, the Laplace, Wilcoxon, and normal-score versions of the signed-rank tests $\phi_{\rm Ca,L}^{(n)}$, $\phi_{\rm Ca,W}^{(n)}$, and $\phi_{\rm Ca,N}^{(n)}$ proposed in [4], and the runs test $\phi_{\rm runs}^{(n)}$, introduced in [17]. Rejection frequencies are reported in Tables 2, 3, and 4.

Table 3. Rejection frequencies (out of $N = 10,000$ replications), under various symmetric and $\mathcal{L}_2$-skewed Gaussian and Student (with $\nu = 2$, 7, and 10 degrees of freedom) densities, of the Wilcoxon signed-rank test $\phi_{\dagger}^{\mathcal{L}_1(n)}$, the sign test $\phi_{\dagger}^{\mathcal{L}_2(n)}$, the signed-rank test $\phi_{\dagger}^{\mathcal{L}_3(n)}$, various studentized tests $\phi_{*;f_1}^{\mathcal{L}_j(n)}$ $(j = 1, 3)$, the classical test of skewness $\phi_{\mathrm{skew}}^{(n)}$, the Laplace, Wilcoxon, and normal-score versions of the signed-rank tests $\phi_{\mathrm{Ca,L}}^{(n)}$, $\phi_{\mathrm{Ca,W}}^{(n)}$, and $\phi_{\mathrm{Ca,N}}^{(n)}$ and the runs test $\phi_{\mathrm{runs}}^{(n)}$.

	g_{t_2}				g_{t_7}			
Test	$\delta = 0$	$\delta = .10$	$\delta = .19$	$\delta = .27$	$\delta = 0$	$\delta = .10$	$\delta = .19$	$\delta = .27$
$\phi_{*;f_{t_2}}^{\mathcal{L}_1(n)}$	0.0502	0.2273	0.6457	0.9232	0.0493	0.2352	0.6603	0.9259
$\phi_{*;f_{t_7}}^{\mathcal{L}_1(n)}$	0.0503	0.2238	0.6360	0.9165	0.0493	0.2302	0.6500	0.9194
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_1(n)}$	0.0503	0.2231	0.6348	0.9162	0.0496	0.2297	0.6494	0.9190
$\phi_{*;f_{\phi}}^{\mathcal{L}_1(n)}$	0.0499	0.2222	0.6317	0.9147	0.0494	0.2286	0.6462	0.9165
$\phi_{\dagger}^{\mathcal{L}_1(n)}$	0.0498	0.2254	0.6442	0.9219	0.0489	0.2296	0.6487	0.9189
$\phi_{\dagger}^{\mathcal{L}_2(n)}$	0.0579	0.3077	0.7782	0.9754	0.0550	0.3103	0.7813	0.9740
$\phi_{*;f_{t_2}}^{\mathcal{L}_3(n)}$	0.0508	0.2415	0.6805	0.9403	0.0488	0.2482	0.6824	0.9410
$\phi_{*;f_{t_7}}^{\mathcal{L}_3(n)}$	0.0510	0.2396	0.6763	0.9376	0.0491	0.2458	0.6789	0.9393
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_3(n)}$	0.0510	0.2393	0.6761	0.9376	0.0493	0.2454	0.6784	0.9390
$\phi_{*;f_{\phi}}^{\mathcal{L}_3(n)}$	0.0510	0.2391	0.6744	0.9371	0.0494	0.2451	0.6775	0.9379
$\phi_{\dagger}^{\mathcal{L}_3(n)}$	0.0516	0.2434	0.6813	0.9402	0.0493	0.2466	0.6801	0.9386
$\phi_{\mathrm{skew}}^{(n)}$	0.0096	0.0130	0.0246	0.0369	0.0364	0.0585	0.1352	0.2336
$\phi_{\mathrm{Ca,L}}^{(n)}$	0.0476	0.1707	0.4938	0.7990	0.0490	0.1732	0.4980	0.7919
$\phi_{\mathrm{Ca,W}}^{(n)}$	0.0480	0.1496	0.4195	0.7162	0.0515	0.1502	0.4259	0.7088
$\phi_{\mathrm{Ca,N}}^{(n)}$	0.0476	0.1317	0.3622	0.6358	0.0512	0.1326	0.3690	0.6343
$\phi_{\mathrm{runs}}^{(n)}$	0.0435	0.0583	0.1251	0.2631	0.0464	0.0616	0.1306	0.2589

	$g_{t_{10}}$				g_{ϕ}			
Test	$\delta = 0$	$\delta = .10$	$\delta = .19$	$\delta = .27$	$\delta = 0$	$\delta = .10$	$\delta = .19$	$\delta = .27$
$\phi_{*;f_{t_2}}^{\mathcal{L}_1(n)}$	0.0496	0.2410	0.6616	0.9249	0.0564	0.2400	0.6710	0.9309
$\phi_{*;f_{t_7}}^{\mathcal{L}_1(n)}$	0.0491	0.2344	0.6498	0.9183	0.0557	0.2359	0.6591	0.9248
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_1(n)}$	0.0491	0.2344	0.6486	0.9177	0.0559	0.2353	0.6573	0.9239
$\phi_{*;f_{\phi}}^{\mathcal{L}_1(n)}$	0.0493	0.2333	0.6458	0.9155	0.0553	0.2326	0.6549	0.9225
$\phi_{\dagger}^{\mathcal{L}_1(n)}$	0.0489	0.2326	0.6481	0.9178	0.0547	0.2317	0.6538	0.9225
$\phi_{\dagger}^{\mathcal{L}_2(n)}$	0.0561	0.3070	0.7821	0.9759	0.0580	0.3143	0.7914	0.9746
$\phi_{*;f_{t_2}}^{\mathcal{L}_3(n)}$	0.0508	0.2511	0.6851	0.9380	0.0567	0.2529	0.6987	0.9408
$\phi_{*;f_{t_7}}^{\mathcal{L}_3(n)}$	0.0506	0.2497	0.6819	0.9351	0.0564	0.2488	0.6937	0.9387
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_3(n)}$	0.0506	0.2491	0.6814	0.9348	0.0564	0.2486	0.6929	0.9385
$\phi_{*;f_{\phi}}^{\mathcal{L}_3(n)}$	0.0505	0.2486	0.6802	0.9343	0.0564	0.2484	0.6911	0.9378
$\phi_{\dagger}^{\mathcal{L}_3(n)}$	0.0508	0.2492	0.6826	0.9371	0.0559	0.2483	0.6915	0.9373
$\phi_{\mathrm{skew}}^{(n)}$	0.0357	0.0747	0.1659	0.2859	0.0484	0.0861	0.2137	0.3876
$\phi_{\mathrm{Ca,L}}^{(n)}$	0.0514	0.1772	0.5022	0.8033	0.0541	0.1726	0.5042	0.8006
$\phi_{\mathrm{Ca,W}}^{(n)}$	0.0505	0.1535	0.4290	0.7177	0.0514	0.1500	0.4298	0.7208
$\phi_{\mathrm{Ca,N}}^{(n)}$	0.0487	0.1376	0.3703	0.6417	0.0518	0.1313	0.3697	0.6389
$\phi_{\mathrm{runs}}^{(n)}$	0.0440	0.0614	0.1280	0.2665	0.0454	0.0621	0.1287	0.2645

All nonparametric/semiparametric tests meet the 5% nominal level constraint under each symmetric density considered, and seem to be unbiased. In contrast with this, the classical test of skewness is strongly conservative under Student densities with 2 degrees of freedom, which have infinite second-order (hence also sixth-order) moments. This classical test has essentially flat (empirical) power curves under skewed versions of the same densities, irrespective of the considered PSM. This is not the case for the other tests, which maintain significant powers under such heavy-tailed densities.

At densities under which the classical test of skewness is valid, our tests strongly

Table 4. Rejection frequencies (out of $N = 10,000$ replications), under various symmetric and $\mathcal{L}_3$-skewed Gaussian and Student (with $\nu = 2$, 7, and 10 degrees of freedom) densities, of the Wilcoxon signed-rank test $\phi_{\dagger}^{\mathcal{L}_1(n)}$, the sign test $\phi_{\dagger}^{\mathcal{L}_2(n)}$, the signed-rank test $\phi_{\dagger}^{\mathcal{L}_3(n)}$, various studentized tests $\phi_{*;f_1}^{\mathcal{L}_j(n)}$ $(j = 1, 3)$, the classical test of skewness $\phi_{\rm skew}^{(n)}$, the Laplace, Wilcoxon, and normal-score versions of the signed-rank tests $\phi_{\rm Ca,L}^{(n)}$, $\phi_{\rm Ca,W}^{(n)}$, and $\phi_{\rm Ca,N}^{(n)}$ and the runs test $\phi_{\rm runs}^{(n)}$.

	g_{t_2}				g_{t_7}			
Test	$\delta = 0$	$\delta = .04$	$\delta = .07$	$\delta = .11$	$\delta = 0$	$\delta = .04$	$\delta = .07$	$\delta = .11$
$\phi_{*;f_{t_2}}^{\mathcal{L}_1(n)}$	0.0493	0.2419	0.5858	0.9370	0.0525	0.2426	0.5950	0.9380
$\phi_{*;f_{t_7}}^{\mathcal{L}_1(n)}$	0.0490	0.2408	0.5831	0.9359	0.0523	0.2416	0.5934	0.9372
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_1(n)}$	0.0485	0.2409	0.5826	0.9359	0.0527	0.2417	0.5929	0.9366
$\phi_{*;f_\phi}^{\mathcal{L}_1(n)}$	0.0486	0.2400	0.5825	0.9353	0.0527	0.2415	0.5913	0.9363
$\phi_{\dagger}^{\mathcal{L}_1(n)}$	0.0494	0.2416	0.5849	0.9364	0.0528	0.2394	0.5916	0.9364
$\phi_{\dagger}^{\mathcal{L}_2(n)}$	0.0564	0.2188	0.5259	0.8891	0.0560	0.2215	0.5322	0.8957
$\phi_{*;f_{t_2}}^{\mathcal{L}_3(n)}$	0.0484	0.2409	0.5925	0.9405	0.0523	0.2453	0.5979	0.9404
$\phi_{*;f_{t_7}}^{\mathcal{L}_3(n)}$	0.0487	0.2417	0.5925	0.9409	0.0524	0.2450	0.5974	0.9402
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_3(n)}$	0.0486	0.2417	0.5925	0.9409	0.0524	0.2451	0.5971	0.9402
$\phi_{*;f_\phi}^{\mathcal{L}_3(n)}$	0.0484	0.2421	0.5922	0.9405	0.0524	0.2446	0.5966	0.9401
$\phi_{\dagger}^{\mathcal{L}_3(n)}$	0.0486	0.2418	0.5925	0.9396	0.0518	0.2444	0.5959	0.9398
$\phi_{\rm skew}^{(n)}$	0.0089	0.0142	0.0272	0.0567	0.0337	0.0703	0.1572	0.3423
$\phi_{\rm Ca,L}^{(n)}$	0.0478	0.1980	0.4954	0.8744	0.0495	0.1985	0.4970	0.8791
$\phi_{\rm Ca,W}^{(n)}$	0.0470	0.1823	0.4542	0.8374	0.0502	0.1814	0.4523	0.8430
$\phi_{\rm Ca,N}^{(n)}$	0.0468	0.1635	0.4045	0.7744	0.0493	0.1618	0.3985	0.7822
$\phi_{\rm runs}^{(n)}$	0.0436	0.0558	0.0944	0.2135	0.0430	0.0563	0.0862	0.2018

	$g_{t_{10}}$				g_ϕ			
Test	$\delta = 0$	$\delta = .04$	$\delta = .07$	$\delta = .11$	$\delta = 0$	$\delta = .04$	$\delta = .07$	$\delta = .11$
$\phi_{*;f_{t_2}}^{\mathcal{L}_1(n)}$	0.0507	0.2433	0.5972	0.9407	0.0485	0.2315	0.5963	0.9393
$\phi_{*;f_{t_7}}^{\mathcal{L}_1(n)}$	0.0504	0.2412	0.5925	0.9394	0.0483	0.2314	0.5930	0.9394
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_1(n)}$	0.0506	0.2407	0.5923	0.9393	0.0484	0.2316	0.5931	0.9392
$\phi_{*;f_\phi}^{\mathcal{L}_1(n)}$	0.0507	0.2396	0.5912	0.9395	0.0480	0.2324	0.5915	0.9384
$\phi_{\dagger}^{\mathcal{L}_1(n)}$	0.0503	0.2390	0.5917	0.9393	0.0472	0.2303	0.5905	0.9386
$\phi_{\dagger}^{\mathcal{L}_2(n)}$	0.0573	0.2188	0.5237	0.8896	0.0544	0.2137	0.5337	0.8914
$\phi_{*;f_{t_2}}^{\mathcal{L}_3(n)}$	0.0531	0.2428	0.6019	0.9415	0.0478	0.2333	0.5954	0.9388
$\phi_{*;f_{t_7}}^{\mathcal{L}_3(n)}$	0.0530	0.2434	0.6012	0.9423	0.0477	0.2325	0.5954	0.9387
$\phi_{*;f_{t_{10}}}^{\mathcal{L}_3(n)}$	0.0530	0.2434	0.6010	0.9426	0.0478	0.2325	0.5954	0.9386
$\phi_{*;f_\phi}^{\mathcal{L}_3(n)}$	0.0529	0.2438	0.6009	0.9426	0.0474	0.2326	0.5953	0.9388
$\phi_{\dagger}^{\mathcal{L}_3(n)}$	0.0525	0.2417	0.6010	0.9425	0.0473	0.2315	0.5950	0.9395
$\phi_{\rm skew}^{(n)}$	0.0372	0.0823	0.1873	0.4119	0.0458	0.1090	0.2544	0.5413
$\phi_{\rm Ca,L}^{(n)}$	0.0493	0.1998	0.5008	0.8769	0.0504	0.1937	0.4985	0.8763
$\phi_{\rm Ca,W}^{(n)}$	0.0489	0.1842	0.4557	0.8363	0.0507	0.1776	0.4566	0.8382
$\phi_{\rm Ca,N}^{(n)}$	0.0483	0.1623	0.4012	0.7831	0.0492	0.1614	0.4047	0.7819
$\phi_{\rm runs}^{(n)}$	0.0453	0.0517	0.0886	0.2145	0.0454	0.0575	0.0883	0.2028

dominate this procedure as suggested by the AREs of the previous section. The hierarchy between tests associated with a common PSM is not always compatible with the rankings of the AREs, which is mainly due to the tiny differences in the latter. On the contrary, the ARE hierarchy between tests associated with different PSMs is perfectly reflected in our simulations. In particular, for any $j = 1, 2, 3$, the tests based on the PSM $\mathcal{L}_j$ appear to be the best ones under densities skewed by means of $\mathcal{L}_j$.

Finally, the proposed tests almost always do better than their signed-rank competitors $\phi^{(n)}_{\text{Ca,L}}$, $\phi^{(n)}_{\text{Ca,W}}$, and $\phi^{(n)}_{\text{Ca,N}}$, and clearly outperform the runs test, which was to be expected since the latter, as a universally consistent test of symmetry (see [11]), cannot compete with our tests against such alternatives.

8. Final comments

This paper considers the problem of testing symmetry about a specified centre θ. Although this is a classical problem, testing for symmetry about an unspecified centre is more natural in various statistical setups (e.g., when testing symmetry of conditional distributions in a regression context). The present work then may be regarded as a first step towards a general theory of optimal testing for symmetry—the next step consisting in treating the unspecified-θ case. This short section briefly discusses how to achieve this next step.

First, we need considering models that explicitly include location, i.e., location-scale-asymmetry models under which observations admit the common pdf

$$x \mapsto \tfrac{1}{\sigma} f_1^{L_\delta}(\tfrac{x-\theta}{\sigma}) = \tfrac{1}{\sigma}\ell_\delta(F_1(\tfrac{x-\theta}{\sigma}))\, f_1(\tfrac{x-\theta}{\sigma}).$$

In these new models, it is of course crucial to prevent any possible confounding between θ and δ, which can be achieved by restricting to skewing mechanisms that fix the median of the original symmetric distribution.

The scale-asymmetry ULAN property in Theorem 3.1 then has to be extended to this more general model, in which the resulting central sequence decomposes into

$$\Delta^{\mathcal{L}(n)}_{f_1}(\theta,\sigma) := \begin{pmatrix} \Delta^{(n)}_{f_1;1}(\theta,\sigma) \\ \Delta^{(n)}_{f_1;2}(\theta,\sigma) \\ \Delta^{\mathcal{L}(n)}_{f_1;3}(\theta,\sigma) \end{pmatrix} := \frac{1}{\sqrt{n}}\sum_{i=1}^n \begin{pmatrix} \frac{1}{\sigma}\psi_{f_1}\big(\frac{X_i^{(n)}-\theta}{\sigma}\big) \\ \frac{1}{\sigma}\big(\frac{(X_i^{(n)}-\theta)}{\sigma}\,\psi_{f_1}\big(\frac{X_i^{(n)}-\theta}{\sigma}\big)-1\big) \\ J^{\mathcal{L}}\big(F_1\big(\frac{X_i^{(n)}-\theta}{\sigma}\big)\big) \end{pmatrix},$$

where $\Delta^{(n)}_{f_1;1}(\theta,\sigma)$ stands for the location part of the central sequence. The corresponding information matrix $\Gamma^{\mathcal{L}}_{f_1}(\sigma)$ is here of the form

$$\Gamma^{\mathcal{L}}_{f_1}(\sigma) := \begin{pmatrix} \Gamma_{f_1;11}(\sigma) & 0 & \Gamma^{\mathcal{L}}_{f_1;31}(\sigma) \\ 0 & \Gamma_{f_1;22}(\sigma) & 0 \\ \Gamma^{\mathcal{L}}_{f_1;31}(\sigma) & 0 & \Gamma^{\mathcal{L}}_{f_1;33}(\sigma) \end{pmatrix},$$

with $\Gamma^{\mathcal{L}}_{f_1;31}(\sigma) := \frac{1}{\sigma}\int_{-\infty}^{\infty}\psi_{f_1}(x)J^{\mathcal{L}}(F_1(x))f_1(x)dx$. Since the function $x \mapsto \psi_{f_1}(x)J^{\mathcal{L}}(F_1(x))f_1(x)$ is symmetric with respect to zero, this information matrix, contrarily to the θ-specified case, is (in general) not block-diagonal. This implies that, when testing for symmetry about an unspecified centre, not knowing θ has a positive cost. In such a setup, ULAN and the convergence of local experiments to the Gaussian shift experiment imply that locally and asymptotically optimal (at f_1) parametric tests for symmetry about an unspecified centre should be based on the f_1-efficient central sequence for asymmetry

$$\Delta^{*\mathcal{L}(n)}_{f_1;3}(\theta,\sigma) := \Delta^{\mathcal{L}(n)}_{f_1;3}(\theta,\sigma) - \Gamma^{\mathcal{L}}_{f_1;31}(\sigma)\Gamma^{-1}_{f_1;11}(\sigma)\Delta^{(n)}_{f_1;1}(\theta,\sigma). \tag{8}$$

More precisely, tests of symmetry are obtained by substituting in $\Delta^{*\mathcal{L}(n)}_{f_1;3}(\theta,\sigma)$ appropriately discretized root-n consistent estimators for θ and σ, and by defining test statistics based on the limiting distribution of $\Delta^{*\mathcal{L}(n)}_{f_1;3}(\theta,\sigma)$. Note that the strong dependence of $\Delta^{*\mathcal{L}(n)}_{f_1;3}(\theta,\sigma)$ on f_1 makes clear that signed-rank versions of the resulting parametric tests will no longer be independent of the underlying density type f_1, which implies that uniform (in f_1) optimality of signed-rank tests for symmetry is lost in this θ-unspecified setup.

Note that if one does not restrict to skewing mechanisms fixing the median of the original symmetric distribution, then information about asymmetry under unspecified location might be arbitrarily small due to the possibly dramatic confounding between location and skewness parameters; for instance, along a sequence of density types f_1 converging to the logistic (resp., Laplace) density type, local powers of the θ-unspecified $\mathcal{L}_1$-based ("Wilcoxon-type") signed-rank test (resp., $\mathcal{L}_2$-based ("sign-type") test) will converge to the nominal level α, due to an increasing collinearity between the central sequences for location $\Delta^{(n)}_{f_1;1}(\theta,\sigma)$ and skewness $\Delta^{\mathcal{L}(n)}_{f_1;3}(\theta,\sigma)$.

Acknowledgements

Christophe Ley thanks the Fonds National de la Recherche Scientifique, Communauté française de Belgique, for support via a Mandat d'Aspirant FNRS. Davy Paindaveine, who is member of ECORE, the association between CORE and ECARES, is grateful to the Fonds National de la Recherche Scientifique, Communauté française de Belgique, for a Mandat d'Impulsion Scientifique. The authors would like to thank the editor, the associate editor, and an anonymous referee for their valuable comments that helped them improve significantly the quality of the paper.

Appendix A. Proof of Theorem 3.1

Our proof relies on Lemma 1 of [25]—more precisely, on its extension in [8]. The sufficient conditions for LAN in those results readily follow from standard arguments (hence are left to the reader), once it is shown that $(\sigma,\delta)'\mapsto\frac{1}{\sigma^{1/2}}(f_1^{L_\delta}(\frac{x}{\sigma}))^{1/2}=\frac{1}{\sigma^{1/2}}\ell_\delta^{1/2}\big(F_1\big(\frac{x}{\sigma}\big)\big)f_1^{1/2}\big(\frac{x}{\sigma}\big)$ (see (2)) is quadratic mean differentiable at any $(\sigma,0)'$, which we establish in the following lemma.

Lemma A.1: *Fix $f_1\in\mathcal{F}_1^{\mathrm{ULAN}}$ and let $\mathcal{L}$ be a PSM satisfying Assumptions A(i)-(iii). Define $g_{\sigma,\delta;f_1,\mathcal{L}}(x):=\frac{1}{\sigma}\ell_\delta\big(F_1\big(\frac{x}{\sigma}\big)\big)f_1\big(\frac{x}{\sigma}\big)$,*

$$D_\sigma g^{1/2}_{\sigma,0;f_1,\mathcal{L}}(x):=\frac{1}{2}\sigma^{-3/2}f_1^{1/2}\Big(\frac{x}{\sigma}\Big)\Big(\frac{x}{\sigma}\,\psi_{f_1}\Big(\frac{x}{\sigma}\Big)-1\Big),$$

and

$$D_\delta g^{1/2}_{\sigma,\delta;f_1,\mathcal{L}}(x)|_{\delta=0}:=\sigma^{-1/2}f_1^{1/2}\Big(\frac{x}{\sigma}\Big)\partial_\delta\ell_\delta^{1/2}\Big(F_1\Big(\frac{x}{\sigma}\Big)\Big)\Big|_{\delta=0}.$$

Then, for any $\sigma\in\mathbb{R}_0^+$ and $\delta\in\mathbb{R}$, we have that, as $(r,s)\to(0,0)$,

(i) $\int_{-\infty}^{\infty}\{g^{1/2}_{\sigma+s,r;f_1,\mathcal{L}}(x)-g^{1/2}_{\sigma+s,0;f_1,\mathcal{L}}(x)-rD_\delta g^{1/2}_{\sigma+s,\delta;f_1,\mathcal{L}}(x)|_{\delta=0}\}^2dx=o(r^2)$,

(ii) $\int_{-\infty}^{\infty}\{g^{1/2}_{\sigma+s,0;f_1,\mathcal{L}}(x)-g^{1/2}_{\sigma,0;f_1,\mathcal{L}}(x)-sD_\sigma g^{1/2}_{\sigma,0;f_1,\mathcal{L}}(x)\}^2dx=o(s^2)$,

(iii) $\int_{-\infty}^{\infty}\{D_\delta g^{1/2}_{\sigma+s,\delta;f_1,\mathcal{L}}(x)|_{\delta=0} - D_\delta g^{1/2}_{\sigma,\delta;f_1,\mathcal{L}}(x)|_{\delta=0}\}^2 dx = o(1)$, *and*

(iv) $\int_{-\infty}^{\infty}\{g^{1/2}_{\sigma+s,r;f_1,\mathcal{L}}(x) - g^{1/2}_{\sigma,0;f_1,\mathcal{L}}(x) - \binom{s}{r}'\begin{pmatrix} D_\sigma g^{1/2}_{\sigma,0;f_1,\mathcal{L}}(x) \\ D_\delta g^{1/2}_{\sigma,\delta;f_1,\mathcal{L}}(x)|_{\delta=0}\end{pmatrix}\}^2 dx = o\Big(\Big\|\binom{s}{r}\Big\|^2\Big).$

Proof of Lemma A.1. In this proof, all $o(\cdot)$ and $O(\cdot)$ quantities are taken as their arguments converge to zero.

(i) Rewriting the integral under the form

$$(\sigma+s)^{-1}\int_{-\infty}^{\infty} f_1\Big(\frac{x}{\sigma+s}\Big)\Big[\ell_r^{1/2}\Big(F_1\Big(\frac{x}{\sigma+s}\Big)\Big) - 1 - r\,\partial_\delta \ell_\delta^{1/2}\Big(F_1\Big(\frac{x}{\sigma+s}\Big)\Big)\Big|_{\delta=0}\Big]^2 dx$$

and substituting u for $F_1(\frac{x}{\sigma+s})$ yields $\int_0^1[\ell_r^{1/2}(u) - 1 - r\,\partial_\delta\ell_\delta^{1/2}(u)|_{\delta=0}]^2 du$, a quantity that is $o(r^2)$ in view of Assumption A(iii).

(ii) Letting $y = \frac{x}{\sigma}$, the left-hand side of (ii) takes the form

$$\int_{-\infty}^{\infty}\Big[\frac{1}{(1+\frac{s}{\sigma})^{1/2}}\, f_1^{1/2}\Big(\frac{y}{1+\frac{s}{\sigma}}\Big) - f_1^{1/2}(y) - \frac{s}{2\sigma}\, f_1^{1/2}(y)(y\psi_{f_1}(y)-1)\Big]^2 dy \le C(T_1+T_2+T_3),$$

where C is some positive constant,

$$T_1 := \int_{-\infty}^{\infty}\Big[\frac{1}{(1+\frac{s}{\sigma})^{1/2}} - 1 + \frac{s}{2\sigma}\Big]^2 f_1\Big(\frac{y}{1+\frac{s}{\sigma}}\Big)\,dy,$$

$$T_2 := \frac{s^2}{4\sigma^2}\int_{-\infty}^{\infty}\Big[f_1^{1/2}\Big(\frac{y}{1+\frac{s}{\sigma}}\Big) - f_1^{1/2}(y)\Big]^2 dy,$$

and

$$T_3 := \int_{-\infty}^{\infty}\Big[f_1^{1/2}\Big(\frac{y}{1+\frac{s}{\sigma}}\Big) - f_1^{1/2}(y) - \frac{s}{2\sigma}\, f_1^{1/2}(y)y\psi_{f_1}(y)\Big]^2 dy.$$

Clearly, routine Taylor series arguments directly yield

$$T_1 = \Big(1+\frac{s}{\sigma}\Big)\Big[\frac{1}{(1+\frac{s}{\sigma})^{1/2}} - 1 + \frac{s}{2\sigma}\Big]^2 = o(s^2).$$

Now, using the symmetry of f_1 with respect to zero and substituting z for $\log(y)$ leads to

$$\begin{aligned} T_2 &= \frac{s^2}{2\sigma^2}\int_0^{\infty}\Big[f^{1/2}_{1;\exp}\Big(\log(y) - \log\Big(1+\frac{s}{\sigma}\Big)\Big) - f^{1/2}_{1;\exp}(\log(y))\Big]^2 dy \\ &= \frac{s^2}{2\sigma^2}\int_{-\infty}^{\infty}\Big[f^{1/2}_{1;\exp}\Big(z - \log\Big(1+\frac{s}{\sigma}\Big)\Big) - f^{1/2}_{1;\exp}(z)\Big]^2 e^z\,dz\,; \qquad \text{(A1)} \end{aligned}$$

since $f^{1/2}_{1;\exp} \in L^2(\mathbb{R},\nu)$, quadratic mean continuity implies that the integral in (A1) is $o(1)$, which implies that $T_2 = o(s^2)$. As for T_3, performing similar manipulations as for T_2 and taking into account the fact that $\psi_{f_1}(\cdot)$ is an antisymmetric function yields

$$\begin{aligned}
T_3 &= 2\int_0^\infty \Big[f_{1;\exp}^{1/2}\Big(\log(y)-\log\Big(1+\frac{s}{\sigma}\Big)\Big) - f_{1;\exp}^{1/2}(\log(y)) - \frac{s}{2\sigma}\, f_{1;\exp}^{1/2}(\log(y)) y\psi_{f_1}(y)\Big]^2 dy \\
&= 2\int_0^\infty \Big[f_{1;\exp}^{1/2}\Big(\log(y)-\log\Big(1+\frac{s}{\sigma}\Big)\Big) - f_{1;\exp}^{1/2}(\log(y)) + \frac{s}{\sigma}(f_{1;\exp}^{1/2})'(\log(y))\Big]^2 dy \\
&= 2\int_{-\infty}^\infty \Big[f_{1;\exp}^{1/2}\Big(z-\log\Big(1+\frac{s}{\sigma}\Big)\Big) - f_{1;\exp}^{1/2}(z) + \frac{s}{\sigma}(f_{1;\exp}^{1/2})'(z)\Big]^2 e^z\, dz \\
&\leq 4(T_{3a}+T_{3b}),
\end{aligned}$$

where

$$T_{3a} := \int_{-\infty}^\infty \Big[f_{1;\exp}^{1/2}\Big(z-\log\Big(1+\frac{s}{\sigma}\Big)\Big) - f_{1;\exp}^{1/2}(z) + \log\Big(1+\frac{s}{\sigma}\Big)(f_{1;\exp}^{1/2})'(z)\Big]^2 e^z\, dz$$

and

$$T_{3b} := \Big(\frac{s}{\sigma} - \log\Big(1+\frac{s}{\sigma}\Big)\Big)^2 \int_{-\infty}^\infty [(f_{1;\exp}^{1/2})'(z)]^2 e^z\, dz.$$

Lemma A.2 in [10] and the fact that $\log\big(1+\frac{s}{\sigma}\big) = O(s)$ imply that $T_{3a} = o(s^2)$. By assumption, $(f_{1;\exp}^{1/2})'$ belongs to $L^2(\mathbb{R},\nu)$, so that the fact that $\frac{s}{\sigma} - \log\big(1+\frac{s}{\sigma}\big) = o(s)$ yields that T_{3b} (hence, also T_3) is $o(s^2)$. The claim in (ii) follows.

(iii) Split the left-hand side of (iii) into two integrals, one over $\mathbb{R}^-$ and the other over $\mathbb{R}^+$, and consider at first the latter integral. Defining $F_{1;\exp}^+(x) := F_1(e^x)$, trivial manipulations show that

$$\begin{aligned}
&\int_0^\infty \{D_\delta g^{1/2}_{\sigma+s,\delta;f_1,\mathcal{L}}(x)|_{\delta=0} - D_\delta g^{1/2}_{\sigma,\delta;f_1,\mathcal{L}}(x)|_{\delta=0}\}^2\, dx \\
= &\int_0^\infty \Big\{\Big(1+\frac{s}{\sigma}\Big)^{-1/2} f_{1;\exp}^{1/2}\Big(\log(y)-\log\Big(1+\frac{s}{\sigma}\Big)\Big)\, \partial_\delta \ell_\delta^{1/2}\Big(F_{1;\exp}^+\Big(\log(y)-\log\Big(1+\frac{s}{\sigma}\Big)\Big)\Big)\Big|_{\delta=0} \\
&\qquad - f_{1;\exp}^{1/2}(\log(y))\, \partial_\delta \ell_\delta^{1/2}(F_{1;\exp}^+(\log(y)))|_{\delta=0}\Big\}^2 dy.
\end{aligned}$$

Substituting z for $\log(y)$ leads to

$$\begin{aligned}
\int_{-\infty}^\infty &\Big\{e^{\frac12(z-\log(1+\frac{s}{\sigma}))}\, f_{1;\exp}^{1/2}\Big(z-\log\Big(1+\frac{s}{\sigma}\Big)\Big)\, \partial_\delta \ell_\delta^{1/2}\Big(F_{1;\exp}^+\Big(z-\log\Big(1+\frac{s}{\sigma}\Big)\Big)\Big)\Big|_{\delta=0} \\
&\qquad - e^{\frac{z}{2}}\, f_{1;\exp}^{1/2}(z)\, \partial_\delta \ell_\delta^{1/2}(F_{1;\exp}^+(z))|_{\delta=0}\Big\}^2 dz. \qquad \text{(A2)}
\end{aligned}$$

Assumption A(iii) implies that $z \mapsto e^{\frac{z}{2}} f_{1;\exp}^{1/2}(z)\partial_\delta \ell_\delta^{1/2}(F_{1;\exp}^+(z))|_{\delta=0}$ is square-integrable over the real line; quadratic mean continuity thus implies that (A2) is $o(1)$ as $s \to 0$. Now, if one writes $F_1(z) = F_{1;\exp}^-(\log(-z))$, with $F_{1;\exp}^-(x) := F_1(-e^x)$, instead of $F_1(z) = F_{1;\exp}^+(\log(z))$ and uses the symmetry of f_1, the same reasoning yields that the integral over $\mathbb{R}^-$ is also $o(1)$. The result follows.

(iv) The left-hand side in (iv) is bounded by $C(S_1 + S_2 + r^2 S_3)$, where

$$S_1 = \int_{-\infty}^{\infty} \{g^{1/2}_{\sigma+s,r;f_1,\mathcal{L}}(x) - g^{1/2}_{\sigma+s,0;f_1,\mathcal{L}}(x) - r\, D_\delta g^{1/2}_{\sigma+s,\delta;f_1,\mathcal{L}}(x)|_{\delta=0}\}^2\, dx,$$

$$S_2 = \int_{-\infty}^{\infty} \{g^{1/2}_{\sigma+s,0;f_1,\mathcal{L}}(x) - g^{1/2}_{\sigma,0;f_1,\mathcal{L}}(x) - s\, D_\sigma g^{1/2}_{\sigma,0;f_1,\mathcal{L}}(x)\}^2\, dx,$$

and

$$S_3 = \int_{-\infty}^{\infty} \{D_\delta g^{1/2}_{\sigma+s,\delta;f_1,\mathcal{L}}(x)|_{\delta=0} - D_\delta g^{1/2}_{\sigma,\delta;f_1,\mathcal{L}}(x)|_{\delta=0}\}^2\, dx.$$

The result then follows from (i), (ii), and (iii). □

We stress that, as announced in Section 3, the proof of Lemma A.1—hence also the uniform local asymptotic normality of the family $\mathcal{P}^{\mathcal{L}(n)}_{f_1}$—actually does not require Assumptions A(i)-(ii).

Appendix B. Asymptotic linearity

The following *asymptotic linearity* result is needed to study the asymptotic behavior of the optimal studentized tests introduced in Section 4.1.

Lemma B.1: *Fix $f_1 \in \mathcal{F}_1$ and $g_1 \in \mathcal{F}^{\mathcal{L}}_{*;f_1}$, and let $\mathcal{L}$ be a PSM satisfying Assumptions A(i)-(iv). Then, for any $\sigma \in \mathbb{R}^+_0$ and any $s \in \mathbb{R}$, we have that, under $\mathrm{P}^{(n)}_{\sigma;g_1}$, (i) $\Delta^{\mathcal{L}(n)}_{f_1;2}(\sigma + n^{-1/2}s) = \Delta^{\mathcal{L}(n)}_{f_1;2}(\sigma) + o_\mathrm{P}(1)$ and (ii) $\frac{1}{n}\sum_{i=1}^n [J^{\mathcal{L}}(F_1(X^{(n)}_i/(\sigma + n^{-1/2}s)))]^2 = \frac{1}{n}\sum_{i=1}^n [J^{\mathcal{L}}(F_1(X^{(n)}_i/\sigma))]^2 + o_\mathrm{P}(1)$, as $n \to \infty$. Moreover, (iii) if $\hat\sigma^{(n)}$ satisfies Assumption B, then both $\Delta^{\mathcal{L}(n)}_{f_1;2}(\hat\sigma^{(n)}) - \Delta^{\mathcal{L}(n)}_{f_1;2}(\sigma)$ and $C^{\mathcal{L}(n)}(f_1) - C^{\mathcal{L}}_{g_1}(f_1)$ are $o_\mathrm{P}(1)$ as $n \to \infty$, under $\mathrm{P}^{(n)}_{\sigma;g_1}$.*

Proof of Lemma B.1(i). Throughout this proof, we write Z_i, $Z_{i;n}$, S_i, and $S_{i;n}$ for $X^{(n)}_i/\sigma$, $X^{(n)}_i/(\sigma + n^{-1/2}s)$, $\mathrm{Sign}(Z_i)$, and $\mathrm{Sign}(Z_{i;n})$, respectively, and let $J_{f_1;g_1}(u) := J^{\mathcal{L}}(F_1(G^{-1}_{1+}(u)))$, where G_{1+} stands for the cdf of $|X^{(n)}_i|$ under $\mathrm{P}^{(n)}_{1;g_1}$. Since $J^{\mathcal{L}}(F_1(z)) = \mathrm{Sign}(z) J_{f_1;g_1}(G_{1+}(|z|))$ for all real number z, we actually have to prove that, under $\mathrm{P}^{(n)}_{\sigma;g_1}$, as $n \to \infty$,

$$D^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n [S_{i;n} J_{f_1;g_1}(G_{1+}(|Z_{i;n}|)) - S_i J_{f_1;g_1}(G_{1+}(|Z_i|))] = o_\mathrm{P}(1). \qquad \text{(B1)}$$

To do so, truncate (for any $m \in \mathbb{N}_0$) the score function $J_{f_1;g_1}$ into $J^{(m)}_{f_1;g_1}$, where

$$J^{(m)}_{f_1;g_1}(u) := \begin{cases} 0 & \text{if } u \le \frac{1}{m} \\ J_{f_1;g_1}(\frac{2}{m}) m(u - \frac{1}{m}) & \text{if } \frac{1}{m} < u \le \frac{2}{m} \\ J_{f_1;g_1}(u) & \text{if } \frac{2}{m} < u \le 1 - \frac{2}{m} \\ J_{f_1;g_1}(1 - \frac{2}{m}) m((1 - \frac{1}{m}) - u) & \text{if } 1 - \frac{2}{m} < u \le 1 - \frac{1}{m} \\ 0 & \text{if } u > 1 - \frac{1}{m}. \end{cases}$$

Assumption A(iv) implies that $J^{(m)}_{f_1;g_1}$ is then continuous (hence, bounded) on $[0,1]$; moreover, it can be assumed without loss of generality that $J_{f_1;g_1}$ is a monotone increasing function (rather than the difference of two monotone increasing functions), hence $|J^{(m)}_{f_1;g_1}|$ is bounded by $|J_{f_1;g_1}|$ uniformly in m and u (at least for m sufficiently large). Now, decompose $D^{(n)}$ into $V^{(n,m)} - W_1^{(n,m)} + W_2^{(n,m)}$, where

$$V^{(n,m)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n \big[S_{i;n} J^{(m)}_{f_1;g_1}(G_{1+}(|Z_{i;n}|)) - S_i J^{(m)}_{f_1;g_1}(G_{1+}(|Z_i|))\big],$$

$$W_1^{(n,m)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n S_i \big[J_{f_1;g_1}(G_{1+}(|Z_i|)) - J^{(m)}_{f_1;g_1}(G_{1+}(|Z_i|))\big],$$

and

$$W_2^{(n,m)} := \frac{1}{\sqrt{n}} \sum_{i=1}^n S_{i;n} \big[J_{f_1;g_1}(G_{1+}(|Z_{i;n}|)) - J^{(m)}_{f_1;g_1}(G_{1+}(|Z_{i;n}|))\big].$$

To establish (B1), it is clearly sufficient to prove

Lemma B.2: *With the same notation as above, (a) for any fixed m, $V^{(n,m)}$ is $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{\sigma;g_1}$, as $n \to \infty$; (b) $W_1^{(n,m)}$ is $o_{\mathrm{P}}(1)$ as $m \to \infty$, uniformly in n, under $\mathrm{P}^{(n)}_{\sigma;g_1}$; (c) $W_2^{(n,m)}$ is $o_{\mathrm{P}}(1)$ as $m \to \infty$, uniformly in n (for n sufficiently large), under $\mathrm{P}^{(n)}_{\sigma;g_1}$.*

Proof of Lemma B.2. (a) Since, for any n, the random variables $S_{i;n} J^{(m)}_{f_1;g_1}(G_{1+}(|Z_{i;n}|)) - S_i J^{(m)}_{f_1;g_1}(G_{1+}(|Z_i|))$, $i = 1, \ldots, n$, are i.i.d. with mean 0, we have that (E_0 stands for expectation under $\mathrm{P}^{(n)}_{\sigma;g_1}$)

$$\begin{aligned}\mathrm{E}_0[(V^{(n,m)})^2] &= \mathrm{E}_0[(S_{1;n} J^{(m)}_{f_1;g_1}(G_{1+}(|Z_{1;n}|)) - S_1 J^{(m)}_{f_1;g_1}(G_{1+}(|Z_1|)))^2] \\ &\leq 2\,\mathrm{E}_0[(V_1^{(n,m)})^2] + 2\,\mathrm{E}_0[(V_2^{(n,m)})^2],\end{aligned}$$

with

$$V_1^{(n,m)} := (S_{1;n} - S_1) J^{(m)}_{f_1;g_1}(G_{1+}(|Z_{1;n}|))$$

and

$$V_2^{(n,m)} := J^{(m)}_{f_1;g_1}(G_{1+}(|Z_{1;n}|)) - J^{(m)}_{f_1;g_1}(G_{1+}(|Z_1|)).$$

Clearly, for n sufficiently large, we have that $S_{1;n} = S_1$ a.e., hence also that $\mathrm{E}_0[(V_1^{(n,m)})^2] = 0$. As for $V_2^{(n,m)}$, first note that $||Z_{i;n}| - |Z_i|| \leq |Z_{i;n} - Z_i| = o_{\mathrm{P}}(1)$ as $n \to \infty$. The continuity of $J^{(m)}_{f_1;g_1} \circ G_{1+}$ then implies that $V_2^{(n,m)}$ converges to zero in probability as $n \to \infty$, hence also in quadratic mean (in view of the boundedness of $J^{(m)}_{f_1;g_1}$).

(b) Under $\mathrm{P}^{(n)}_{\sigma;g_1}$, one easily obtains that

$$\mathrm{E}_0[(W_1^{(n,m)})^2] = \int_0^1 \left(J_{f_1;g_1}(u) - J^{(m)}_{f_1;g_1}(u)\right)^2 du.$$

For any $u \in (0,1)$, the mapping $J^{(m)}_{f_1;g_1}$ converges to $J_{f_1;g_1}$ as $m \to \infty$ and the integrand is bounded (uniformly in m) by $4(J_{f_1;g_1}(u))^2$, which is integrable on $(0,1)$ (with integral $4C^{\mathcal{L}}_{g_1}(f_1)$), since g_1 is assumed to belong to $\mathcal{F}^{\mathcal{L}}_{*;f_1}$. Thus the Lebesgue dominated convergence theorem allows to conclude that $\mathrm{E}_0[(W_1^{(n,m)})^2] = o(1)$ as $m \to \infty$, uniformly in n.

(c) The claim is exactly the same as in (b), with $Z_{i;n}$ replacing Z_i (hence also with $S_{i;n} = \mathrm{Sign}(Z_{i;n})$ replacing $S_i = \mathrm{Sign}(Z_i)$). Consequently, (c) holds under $\mathrm{P}^{(n)}_{\sigma+n^{-1/2}s;g_1}$. That it also holds under $\mathrm{P}^{(n)}_{\sigma;g_1}$ follows from Lemma 3.5 in [13]. □

Proof of Lemma B.1(ii). For the sake of simplicity, let us write $J_{i;n,+} := J^{\mathcal{L}}(F_1(X_i^{(n)}/(\sigma+n^{-1/2}s)))$ and $J_{i;n} := J^{\mathcal{L}}(F_1(X_i^{(n)}/\sigma))$, $i = 1,\ldots,n$. Then, denoting again by E_0 the expectation under $\mathrm{P}^{(n)}_{\sigma;g_1}$, similar manipulations as in the proof of Lemma A.1(iii) entail that

$$\begin{aligned}&\mathrm{E}_0[(J_{1;n,+} - J_{1;n})^2]\\ &= 2\int_0^\infty \left\{J^{\mathcal{L}}\left(F^+_{1;\exp}\left(\log(y) - \log\left(1+\frac{s}{\sigma\sqrt{n}}\right)\right)\right) - J^{\mathcal{L}}(F^+_{1;\exp}(\log(y)))\right\}^2 g_{1;\exp}(\log(y))\,dy,\end{aligned}$$

where we wrote $g_{1;\exp}(x) := g_1(e^x)$ and $F^+_{1;\exp}(x) := F_1(e^x)$. Substituting z for $\log(y)$ yields

$$\mathrm{E}_0[(J_{1;n,+}-J_{1;n})^2] = 2\int_{-\infty}^\infty \left\{J^{\mathcal{L}}\left(F^+_{1;\exp}\left(z-\log\left(1+\frac{s}{\sigma\sqrt{n}}\right)\right)\right)-J^{\mathcal{L}}(F^+_{1;\exp}(z))\right\}^2 g_{1;\exp}(z)\,e^z\,dz,$$

which, from quadratic mean continuity, is $o(1)$ as $n \to \infty$. Note indeed that $g_1 \in \mathcal{F}^{\mathcal{L}}_{*;f_1}$ implies that

$$\mathrm{E}_0\left[J^2_{1;n}\right] = 2\int_{-\infty}^\infty \left[J^{\mathcal{L}}(F^+_{1;\exp}(z))\right]^2 g_{1;\exp}(z)\,e^z\,dz < \infty. \tag{B2}$$

We conclude that

$$\mathrm{E}_0\left[\left(J_{1;n,+} - J_{1;n}\right)^2\right] = o(1) \quad \text{as } n \to \infty. \tag{B3}$$

Clearly, (B2)-(B3) entail that $\mathrm{E}_0\left[J^2_{1;n,+}\right] = O(1)$ as $n \to \infty$. Therefore,

$$\begin{aligned}\mathrm{E}_0^2[|\tfrac{1}{n}\textstyle\sum_{i=1}^n(J^2_{i;n,+} - J^2_{i;n})|] &\leq \mathrm{E}_0\left[\left(J_{1;n,+} - J_{1;n}\right)^2\right] \times \mathrm{E}_0\left[\left(J_{1;n,+} + J_{1;n}\right)^2\right]\\ &\leq 2\mathrm{E}_0\left[\left(J_{1;n,+} - J_{1;n}\right)^2\right] \times \left(\mathrm{E}_0\left[J^2_{1;n,+}\right] + \mathrm{E}_0\left[J^2_{1;n}\right]\right) = o(1)\end{aligned}$$

as $n \to \infty$. This establishes the result, since convergence in the L^1 sense implies convergence in probability. □

Proof of Lemma B.1(iii). The result directly follows from Lemma 4.4 in [15], since the latter shows that Assumption B allows to replace the nonrandom quantity $\sigma + n^{-1/2}s$ with the random one $\hat{\sigma}^{(n)}$ in Parts (i) and (ii) of the Lemma. □

Appendix C. Proofs of Lemma 4.2 and of Theorems 4.1-4.3

Proof of Lemma 4.2. First note that Assumptions A(iii)-(iv)$'$ imply that the score function $u \mapsto J_+^{\mathcal{L}}(u)$ is a continuous and square-integrable function over $(0,1)$ that can be written as the difference of two monotone increasing functions. Hence, since the signed ranks of the $X_i^{(n)}/\sigma$'s, under $\mathrm{P}_{\sigma;g_1}^{(n)}$, with $\sigma \in \mathbb{R}_0^+$ and $g_1 \in \mathcal{F}_1$, are those of n i.i.d. random variables with common cdf G_1, Hájek's classical projection theorem for linear signed-rank statistics (see, e.g., Chapter 3 in [20]) entails that $\Delta_{\dagger;2}^{\mathcal{L}(n)} - \Delta_{g_1;2}^{\mathcal{L}(n)}(\sigma)$ converges to zero in quadratic mean as $n \to \infty$ under $\mathrm{P}_{\sigma;g_1}^{(n)}$, which establishes the result. □

Proof of Theorem 4.1. Fix $\sigma \in \mathbb{R}_0^+$ and $g_1 \in \mathcal{F}_{*;f_1}^{\mathcal{L}}$. Lemma B.1(iii) entails that

$$Q_{*;f_1}^{\mathcal{L}(n)} = \frac{\Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma)}{(C_{g_1}^{\mathcal{L}}(f_1))^{1/2}} + o_{\mathrm{P}}(1) \tag{C1}$$

as $n \to \infty$, under $\mathrm{P}_{\sigma;g_1}^{(n)}$. Part (i) of the result then follows from the fact that $\Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma)$, under $\mathrm{P}_{\sigma;g_1}^{(n)}$, is asymptotically normal with mean 0 and variance $C_{g_1}^{\mathcal{L}}(f_1)$.

Now, under $\mathrm{P}_{\sigma,n^{-1/2}\tau_2;g_1}^{\mathcal{L}_U(n)}$, the asymptotic normality of $\Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma)$ with mean $C_{g_1}^{\mathcal{L},\mathcal{L}_U}(f_1,g_1)\tau_2$ and variance $C_{g_1}^{\mathcal{L}}(f_1)$ is obtained as usual, by establishing the joint normality of $\Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma)$ and $\log\big(d\mathrm{P}_{\sigma,n^{-1/2}\tau_2^{(n)};g_1}^{\mathcal{L}_U(n)}/d\mathrm{P}_{\sigma;g_1}^{(n)}\big)$ under $\mathrm{P}_{\sigma;g_1}^{(n)}$ and then applying Le Cam's third Lemma. Since (C1), from contiguity, also holds under $\mathrm{P}_{\sigma,n^{-1/2}\tau_2;g_1}^{\mathcal{L}_U(n)}$, this yields Part (ii) of the Theorem. Finally, applying Lemma B.1(iii) (with $g_1 = f_1$) to the statistic $Q_{f_1}^{\mathcal{L}(n)}$ defined in (6) and noting that $\Gamma_{22}^{\mathcal{L}} = C_{f_1}^{\mathcal{L}}(f_1)$ yields that

$$Q_{f_1}^{\mathcal{L}(n)} = \frac{\Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma)}{\big(\Gamma_{22}^{\mathcal{L}}\big)^{1/2}} + o_{\mathrm{P}}(1) = \frac{\Delta_{f_1;2}^{\mathcal{L}(n)}(\sigma)}{(C_{f_1}^{\mathcal{L}}(f_1))^{1/2}} + o_{\mathrm{P}}(1),$$

as $n \to \infty$, under $\mathrm{P}_{\sigma;f_1}^{(n)}$. Jointly with the $g_1 = f_1$ version of (C1), this establishes Part (iii) of the result. □

Proof of Theorem 4.3. Fix $\sigma \in \mathbb{R}_0^+$ and $g_1 \in \mathcal{F}_1$. Note that the fact that

$$Q_{\dagger}^{\mathcal{L}(n)} = \frac{\Delta_{g_1;2}^{\mathcal{L}(n)}(\sigma)}{\big(\Gamma_{22}^{\mathcal{L}}\big)^{1/2}} + o_{\mathrm{P}}(1) \tag{C2}$$

as $n \to \infty$, under $\mathrm{P}_{\sigma;g_1}^{(n)}$—hence, also Part (iii) of the result—is a direct corollary of Lemma 4.2. Part (i) also follows from (C2) since $\Delta_{g_1;2}^{\mathcal{L}(n)}(\sigma)$, under $\mathrm{P}_{\sigma;g_1}^{(n)}$, is clearly asymptotically normal with mean 0 and variance $\Gamma_{22}^{\mathcal{L}}$.

Now, under $\mathrm{P}_{\sigma,n^{-1/2}\tau_2;g_1}^{\mathcal{L}_U(n)}$, the asymptotic normality of $\Delta_{g_1;2}^{\mathcal{L}(n)}(\sigma)$ with mean $C^{\mathcal{L},\mathcal{L}_U}\tau_2$ and variance $\Gamma_{22}^{\mathcal{L}}$ is obtained as in the proof of Theorem 4.1, by establishing the joint normality of $\Delta_{g_1;2}^{\mathcal{L}(n)}(\sigma)$ and $\log\left(d\mathrm{P}^{\mathcal{L}_U(n)}_{\sigma,n^{-1/2}\tau_2^{(n)};g_1}/d\mathrm{P}^{(n)}_{\sigma;g_1}\right)$ under $\mathrm{P}^{(n)}_{\sigma;g_1}$ and then applying Le Cam's third Lemma. Since (C2), from contiguity, also holds under $\mathrm{P}^{\mathcal{L}_U(n)}_{\sigma,n^{-1/2}\tau_2;g_1}$, this yields Part (ii) of the Theorem. □

References

[1] A. Azzalini, *A class of distributions which includes the normal ones*, Scand. J. Stat. 12 (1985), pp. 171–178.

[2] R. Beran, *Asymptotically efficient adaptive rank estimates in location models*, Ann. Statist. 2 (1974), pp. 63–74.

[3] C.C. Butler, *A test for symmetry using the sample distribution function*, Ann. Math. Statist. 40 (1969), pp. 2209–2210.

[4] D. Cassart, M. Hallin and D. Paindaveine, *Optimal detection of Fechner-asymmetry*, J. Statist. Plann. Inference 138 (2008), pp. 2499–2525.

[5] D. Cassart, M. Hallin and D. Paindaveine, *A class of optimal tests for symmetry based on Edgeworth approximations*, submitted.

[6] C. Fernández and M.F.J. Steel, *On Bayesian modeling of fat tails and skewness*, J. Amer. Statist. Assoc. 93 (1998), pp. 359–371.

[7] J.T.A.S. Ferreira and M.F.J. Steel, *A constructive representation of univariate skewed distributions*, J. Amer. Statist. Assoc. 101 (2006), pp. 823–829.

[8] B. Garel and M. Hallin, *Local asymptotic normality of multivariate ARMA processes with a linear trend*, Ann. Inst. Statist. Math. 47 (1995), pp. 551–579.

[9] J. Hájek, Z. Sidák and P.K. Sen, *Theory of rank tests*, Academic Press, San Diego, California, 1999.

[10] M. Hallin and D. Paindaveine, *Semiparametrically efficient rank-based inference for shape: I. Optimal rank-based tests for sphericity*, Ann. Statist. 34 (2006), pp. 2707–2756.

[11] N. Henze, *On the consistency of a test for symmetry based on a runs statistic*, J. Nonparametr. Stat. 3 (1993), pp. 195–199.

[12] D.L. Hill and P.V. Rao, *Tests of symmetry based on Cramér-von Mises statistics*, Biometrika 64 (1977), pp. 489–494.

[13] J. Jurečková, *Asymptotic linearity of a rank statistic in regression parameter*, Ann. Math. Statist. 40 (1969), pp. 1889–1900.

[14] T.J. Kozubowski and A.K. Panorska, *Testing symmetry under a skew Laplace model*, J. Statist. Plann. Inference 120 (2004), pp. 41–63.

[15] J-P. Kreiss, *On adaptive estimation in stationary ARMA processes*, Ann. Statist. 15 (1987), pp. 112–133.

[16] L. Le Cam, *Asymptotic methods in statistical decision theory*, Springer-Verlag, New York, 1986.

[17] T.P. McWilliams, *A distribution-free test for symmetry based on a runs statistic*, J. Amer. Statist. Assoc. 85 (1990), pp. 1130–1133.

[18] R. Modarres and J.L. Gastwirth, *A modified runs test for symmetry*, Statist. Probab. Lett. 31 (1996), pp. 107–112.

[19] J.W. Pratt and J.D. Gibbons, *Concepts of nonparametric theory*, Springer-Verlag, New York, 1981.

[20] M.L. Puri and P.K. Sen, *Nonparametric methods in general linear models*, J. Wiley, New York, 1985.

[21] R.H. Randles and D.A. Wolfe, *Introduction to the theory of nonparametric statistics*, J. Wiley, New York, 1979.

[22] E.D. Rothman and M. Woodrofe, *A Cramér-von Mises type statistic for testing symmetry*, Ann. Math. Statist. 43 (1972), pp. 2035–2038.

[23] C. Stein, *Efficient nonparametric testing and estimation*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability 1 (1956), pp. 187–195, Univ. California Press, Berkeley.

[24] C. Stone, *Adaptive maximum likelihood estimation of a location parameter*, Ann. Statist. 3 (1975), pp. 267–284.

[25] A.R. Swensen, *The asymptotic distribution of the likelihood ratio for autoregressive time series with a regression trend*, J. Multivariate Anal. 16 (1985), pp. 54–70.