SUPPLEMENT TO "TESTING UNIFORMITY ON HIGH-DIMENSIONAL SPHERES AGAINST MONOTONE ROTATIONALLY SYMMETRIC ALTERNATIVES"

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In this supplementary article, we first derive the fixed-p asymptotic non-null distribution of the Rayleigh test statistic provided in Equation (3.4). Then we show that, under FvML distributions, the conditions (i)-(iii) of Theorem 5.1 *always* hold, that is, that they hold without any constraint on the concentration κ_n nor on the way the dimension p_n goes to infinity with n.

Throughout, this supplement, $(n.m)$ (resp., (S.$n.m$)) denotes Equation m of Section n from the main manuscript (resp., from this supplement article). Theorem $m.n$ or Lemma S.$m.n$ are used in a similar way.

1. Asymptotic non-null distribution of the Rayleigh test statistic in the fixed-p case. We focus on the fixed-p case ($p_n = p$ for any n) and derive the asymptotic distribution of the Rayleigh test statistic R_n under sequences of contiguous rotationally symmetric alternatives. More specifically, we consider sequences of alternatives of the form $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_n,\kappa_n,f}$, with $\kappa_n = \tau_n/\sqrt{n}$, where the sequence (τ_n) converges to some $\tau \in (0,\infty)$. Since R_n is invariant when a common rotation is applied to $\mathbf{X}_{ni}$, $i = 1,\ldots,n$, we will without any loss of generality assume that $\boldsymbol{\theta}_n$ coincides for any n with the first vector of the canonical basis of $\mathbb{R}^p$ (to avoid introducing an extra notation, we will simply write $\boldsymbol{\theta}$ for this constant value of $\boldsymbol{\theta}_n$). The null of uniformity will still be denoted as $\mathrm{P}^{(n)}_0$.

Adopting the notation u_{ni}, v_{ni}, $\mathbf{S}_{ni}$ introduced in Appendix B of the main manuscript,

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rewrite then the Rayleigh test statistic R_n as

$$R_n = \frac{p}{n} \sum_{i,j=1}^{n} \mathbf{X}'_{ni}\mathbf{X}_{nj} = \frac{p}{n} \sum_{i,j=1}^{n} (u_{ni}\boldsymbol{\theta} + v_{ni}\mathbf{S}_{ni})'(u_{nj}\boldsymbol{\theta} + v_{nj}\mathbf{S}_{nj})$$

$$= \frac{p}{n} \sum_{i,j=1}^{n} (u_{ni}u_{nj} + v_{ni}v_{nj}\mathbf{S}'_{ni}\mathbf{S}_{nj}) = Y_n^2 + \frac{1}{p}\mathbf{Z}'_n\mathbf{A}\mathbf{Z}_n,$$

where we let $\mathbf{A} := p(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}')$,

$$Y_n := \frac{\sqrt{p}}{\sqrt{n}} \sum_{i=1}^{n} u_{ni} \quad \text{and} \quad \mathbf{Z}_n := \frac{\sqrt{p}}{\sqrt{n}} \sum_{i=1}^{n} v_{ni}\mathbf{S}_{ni}.$$

Under $\mathrm{P}_0^{(n)}$, the multivariate CLT, along with the identities

$$\text{(S.1.1)} \qquad \mathrm{E}\big[u_{ni}^2\big] = \frac{1}{p} = 1 - \mathrm{E}\big[v_{ni}^2\big] \quad \text{and} \quad \mathrm{E}\big[\mathbf{S}_{ni}\mathbf{S}'_{ni}\big] = \frac{1}{p-1}\,(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}')$$

(see Lemma B.1(i)), provides

$$\text{(S.1.2)} \qquad \begin{pmatrix} Y_n \\ \mathbf{Z}_n \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}\Bigg(\mathbf{0}, \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \mathbf{\Sigma} \end{pmatrix}\Bigg), \quad \text{with } \mathbf{\Sigma} := \frac{1}{p}\,(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}').$$

Clearly, Y_n^2 is asymptotically χ_1^2 under $\mathrm{P}_0^{(n)}$. By using the identities $\mathbf{\Sigma A \Sigma A \Sigma} = \mathbf{\Sigma A \Sigma}$ and $\mathrm{tr}[\mathbf{\Sigma A}] = p-1$, Theorem 9.2.1 in Rao and Mitra (1971) shows that $(1/p)\mathbf{Z}'_n\mathbf{A}\mathbf{Z}_n$ is asymptotically χ_{p-1}^2 under $\mathrm{P}_0^{(n)}$. Since the joint asymptotic normality result in (S.1.2) ensures asymptotic independence of Y_n^2 and $(1/p)\mathbf{Z}'_n\mathbf{A}\mathbf{Z}_n$, this confirms that R_n is asymptotically χ_p^2 under $\mathrm{P}_0^{(n)}$.

Let us now turn to the sequence of alternatives $\mathrm{P}_{\boldsymbol{\theta},\kappa_n,f}^{(n)}$ considered above. In view of (S.1.1) and Theorem 2.2, Le Cam's third lemma directly shows that

$$\text{(S.1.3)} \qquad \begin{pmatrix} Y_n \\ \mathbf{Z}_n \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}\Bigg(\begin{pmatrix} \tau \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \mathbf{\Sigma} \end{pmatrix}\Bigg)$$

as $n \to \infty$ under $\mathrm{P}_{\boldsymbol{\theta}_n,\kappa_n,f}^{(n)}$. Under the same sequence of alternatives, Y_n^2 is therefore asymptotically $\chi_1^2(\tau)^2)$ (that is, non-central χ_1^2 with non-centrality parameter τ^2) and $\frac{1}{p}\mathbf{Z}'_n\mathbf{A}\mathbf{Z}_n$ is still asymptotically χ_{p-1}^2. Asymptotic independence between Y_n and $\mathbf{Z}_n$ still holds under contiguous alternatives, which shows that

$$R_n \xrightarrow{\mathcal{D}} \chi_1^2(\tau^2) + \chi_{p-1}^2,$$

where the χ^2 terms are independent. This establishes the asymptotic result in (3.4).

2. Universality of the asymptotic non-null distribution of the Rayleigh test in the FvML case. In the rest of this supplement, we prove that the conditions (i)-(iii) of Theorem 5.1 universally hold under FvML distributions, in the sense that, if F_n is FvML for any n, then the theorem does not require any condition on the dependence of p_n and κ_n on n (but for the fact that $p_n \to \infty$ as $n \to \infty$). The proof is quite lengthy and requires several preliminary results.

LEMMA S.2.1. *Write* $e_\ell := e_{\ell;p,\kappa} := \mathrm{E}[(\mathbf{X}'\boldsymbol{\theta})^\ell]$, *where* $\mathbf{X}$ *follows a* p*-dimensional FvML distribution with concentration* $\kappa(>0)$ *and location* $\boldsymbol{\theta}(\in \mathcal{S}^{p-1})$. *Then*

$$e_1 = r, \qquad e_2 = -\frac{p-1}{\kappa} r + 1, \qquad e_3 = \frac{p(p-1)+\kappa^2}{\kappa^2} r - \frac{p-1}{\kappa}$$

and

$$e_4 = -\frac{(p-1)((p+1)p+2\kappa^2)}{\kappa^3} r + \frac{(p-1)(p+1)+\kappa^2}{\kappa^2},$$

where we let $r := r_{p,\kappa} := \mathcal{I}_{\frac{p}{2}}(\kappa)/\mathcal{I}_{\frac{p}{2}-1}(\kappa)$ *(as in the main manuscript,* $\mathcal{I}_\nu(\cdot)$ *stands here for the order-*ν *modified Bessel function of the first kind).*

PROOF OF LEMMA S.2.1. Using integration by parts in the representation result

$$\mathcal{I}_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} (1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds \tag{S.2.4}$$

(see, e.g., (10.32.2) in Olver et al. (2010)) provides

$$\begin{aligned}\int_{-1}^{1} s(1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds &= \frac{z}{2\nu+1} \int_{-1}^{1} (1-s^2)^{(\nu+1)-\frac{1}{2}} \exp(zs)\, ds \\ &= \frac{z\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})\mathcal{I}_{\nu+1}(z)}{(2\nu+1)(z/2)^{\nu+1}},\end{aligned} \tag{S.2.5}$$

which readily leads to

$$e_1 = \frac{(\kappa/2)^{(p-2)/2}}{\sqrt{\pi}\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)} \int_{-1}^{1} s(1-s^2)^{\frac{p-3}{2}} \exp(\kappa s)\, ds = \frac{\Gamma(\frac{p+1}{2})\mathcal{I}_{\frac{p}{2}}(\kappa)}{\frac{p-1}{2}\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)} = \frac{\mathcal{I}_{\frac{p}{2}}(\kappa)}{\mathcal{I}_{\frac{p}{2}-1}(\kappa)}.$$

Turning to e_2, (S.2.4) above yields

$$\begin{aligned}\int_{-1}^{1} s^2(1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds &= \int_{-1}^{1} (1-(1-s^2))(1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds \\ &= \frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})\mathcal{I}_\nu(z)}{(z/2)^\nu} - \frac{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})\mathcal{I}_{\nu+1}(z)}{(z/2)^{\nu+1}}.\end{aligned}$$

Hence,

$$
\begin{aligned}
e_2 &= \frac{(\kappa/2)^{(p-2)/2}}{\sqrt{\pi}\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)} \int_{-1}^{1} s^2(1-s^2)^{\frac{p-3}{2}} \exp(\kappa s)\, ds \\
&= \frac{(\kappa/2)^{(p-2)/2}}{\sqrt{\pi}\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)} \left[\frac{\sqrt{\pi}\,\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)}{(\kappa/2)^{\frac{p}{2}-1}} - \frac{\sqrt{\pi}\,\Gamma(\frac{p+1}{2})\mathcal{I}_{\frac{p}{2}}(\kappa)}{(\kappa/2)^{p/2}} \right] \\
&= 1 - \frac{\Gamma(\frac{p+1}{2})\mathcal{I}_{\frac{p}{2}}(\kappa)}{(\kappa/2)\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)} = 1 - \frac{(p-1)\mathcal{I}_{\frac{p}{2}}(\kappa)}{\kappa\mathcal{I}_{\frac{p}{2}-1}(\kappa)},
\end{aligned}
$$

as was to be showed. The results for

$$
e_\ell = \frac{(\kappa/2)^{(p-2)/2}}{\sqrt{\pi}\,\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)} \int_{-1}^{1} s^2(1-s^2)^{\frac{p-3}{2}} \exp(\kappa s)\, ds, \qquad \ell = 3, 4,
$$

follow similarly by using the expressions obtained by plugging (S.2.4)-(S.2.5) into

$$
\int_{-1}^{1} s^3(1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds = \int_{-1}^{1} s(1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds - \int_{-1}^{1} s(1-s^2)^{\nu+1-\frac{1}{2}} \exp(zs)\, ds
$$

and

$$
\begin{aligned}
\int_{-1}^{1} s^4(1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds &= \int_{-1}^{1} (1-s^2)^{\nu+2-\frac{1}{2}} \exp(zs)\, ds \\
&\quad -2\int_{-1}^{1} (1-s^2)^{\nu+1-\frac{1}{2}} \exp(zs)\, ds + \int_{-1}^{1} (1-s^2)^{\nu-\frac{1}{2}} \exp(zs)\, ds,
\end{aligned}
$$

along with the well-known recurrence relation

$$
\mathcal{I}_{\nu+1}(z) = \mathcal{I}_{\nu-1}(z) - \frac{2\nu}{z}\,\mathcal{I}_\nu(z); \tag{S.2.6}
$$

see (10.29.1) in Olver et al. (2010). □

Note that closed form expressions for $f_\ell := f_{\ell;p,\kappa} = \mathrm{E}[(1-(\mathbf{X}'\boldsymbol{\theta})^2)^{\ell/2}]$, where $\mathbf{X}$ still follows a p-dimensional FvML distribution with concentration $\kappa(>0)$ and location $\boldsymbol{\theta}$, can be obtained much more directly than in Lemma S.2.1, as (S.2.4) readily yields

$$
f_\ell = \frac{(\kappa/2)^{(p-2)/2}}{\sqrt{\pi}\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)} \int_{-1}^{1} (1-s^2)^{\frac{p+\ell-3}{2}} \exp(\kappa s)\, ds = \frac{\Gamma(\frac{p+\ell-1}{2})\mathcal{I}_{\frac{p+\ell}{2}-1}(\kappa)}{(\kappa/2)^{\frac{\ell}{2}}\Gamma(\frac{p-1}{2})\mathcal{I}_{\frac{p}{2}-1}(\kappa)}. \tag{S.2.7}
$$

The main result of this supplementary article is the following theorem, that implies that, under FvML distributions, Theorem 5.1 does not impose any condition on the way p_n should go to infinity as a function of n.

THEOREM S.2.1. *Let us still write* $e_\ell := e_{\ell;p,\kappa} := \mathrm{E}[(\mathbf{X}'\boldsymbol{\theta})^\ell]$ *and* $f_\ell := f_{\ell;p,\kappa} = \mathrm{E}[(1-(\mathbf{X}'\boldsymbol{\theta})^2)^{\ell/2}]$, *where* $\mathbf{X}$ *follows a* p*-dimensional FvML distribution with concentration* $\kappa(>0)$ *and location* $\boldsymbol{\theta}$. *Further let* $\tilde{e}_\ell := \tilde{e}_{\ell;p,\kappa} := \mathrm{E}[(\mathbf{X}'\boldsymbol{\theta} - e_1)^\ell]$. *Then there exist a positive integer* p_0 *and a real constant* C *such that*

$$(i)\ \frac{p\tilde{e}_2^2}{f_2^2} \leq C, \qquad (ii)\ \frac{\tilde{e}_4}{\tilde{e}_2^2} \leq C, \qquad \text{and} \qquad (iii)\ \frac{f_4}{f_2^2} \leq C,$$

for any $p \geq p_0$ *and any* $\kappa > 0$.

The proof requires both following lemmas on the modified Bessel functions ratio $\mathcal{R}_\nu(z) := \mathcal{I}_{\nu+1}(z)/\mathcal{I}_\nu(z)$ (we adopt the same notation as in Hornika and Grün (2013)).

LEMMA S.2.2. *Fix* $\nu > 0$ *and* $z > 0$, *and let* $G_{\alpha,\beta}(t) = t/(\alpha + \sqrt{t^2+\beta^2})$. *Then (i)* $\mathcal{R}_\nu(z) \geq G_{\nu+1,\nu+1}(z)$ *(ii)* $\mathcal{R}_\nu(z) \geq G_{\nu+1/2,\nu+3/2}(z)$ *(iii)* $\mathcal{R}_\nu(z) \leq G_{\nu+1/2,\nu+1/2}(z)$ *(iv)* $\mathcal{R}_\nu(z) \leq G_{\nu,\nu+2}(z)$, *(v)* $\mathcal{R}_\nu(z) \leq G_{\nu,\nu}(z)$, *and (vi)* $\mathcal{R}_\nu(z) \leq G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(z)$.

These bounds, that have been obtained in Amos (1974) ((i)-(v)) and Simpson and Spector (1984) ((vi)), are actually sufficient to establish Theorem S.2.1(i) and (iii). To prove Theorem S.2.1(ii), however, we will need the following reinforcement of the bounds in Lemma S.2.2(ii)-(iii) and an appropriate control of the resulting approximation error; see Paindaveine (2016) for a proof.

LEMMA S.2.3. *Fix* $\nu > 0$ *and* $z \geq 0$, *and let*

$$a_\nu(z) := \frac{(\nu+\frac{3}{2})(\nu+4)+z^2}{(\nu+\frac{3}{2})(\nu+\frac{7}{2})+z^2}, \quad b_\nu(z) := \frac{(\nu+\frac{5}{2})(\nu+2)+z^2}{(\nu+\frac{3}{2})(\nu+\frac{7}{2})+z^2},$$

$$c_\nu(z) := \frac{(\nu+\frac{3}{2})(\nu+\frac{5}{2})+z^2}{(\nu+\frac{3}{2})(\nu+\frac{7}{2})+z^2}, \quad d_\nu(z) := \frac{(\nu+1)(\nu+\frac{3}{2})+z^2}{(\nu+\frac{1}{2})(\nu+\frac{3}{2})+z^2}$$

and

$$e_\nu(z) := \frac{(\nu+\frac{1}{2})(\nu+\frac{5}{2})+z^2}{(\nu+\frac{1}{2})(\nu+\frac{3}{2})+z^2}.$$

Then (i) $L_\nu(z) \leq R_\nu(z) \leq U_\nu(z)$, *with*

$$L_\nu(z) := \frac{z}{a_\nu(z)(\nu+\frac{1}{2}) + \sqrt{\big(b_\nu(z)(\nu+\frac{3}{2})\big)^2 + c_\nu(z)z^2}}$$

and

$$U_\nu(z) := \frac{z}{d_\nu(z)(\nu+\frac{1}{2}) + \sqrt{\big(d_\nu(z)(\nu+\frac{1}{2})\big)^2 + e_\nu(z)z^2}};$$

(ii) there exists $\nu_0 > 0$ *such that*

$$\frac{\nu^7+z^7}{z^3}\,(U_\nu(z) - L_\nu(z)) \leq \frac{3(2\nu+3)^2}{8}$$

for any $\nu \geq \nu_0$ *and any* $z > 0$.

We prove Theorem S.2.1(i)-(iii) separately. Throughout, we still write r for the key quantity $R_{\frac{p}{2}-1}(\kappa) = \mathcal{I}_{\frac{p}{2}}(\kappa)/\mathcal{I}_{\frac{p}{2}-1}(\kappa)$.

Proof of Theorem S.2.1(i). From Lemma S.2.1, we obtain

$$\text{(S.2.8)} \qquad \frac{p\tilde{e}_2^2}{f_2^2} = \frac{p\tilde{e}_2^2}{(1-e_2)^2} = \frac{\kappa^2 p\big(r^2 + \frac{p-1}{\kappa} r - 1\big)^2}{(p-1)^2 r^2} = \frac{\kappa^2 p (g_a(r) g_b(r))^2}{(p-1)^2 r^2},$$

where we let $g_a(x) = G_{-\frac{p-1}{2},\frac{p-1}{2}}(\kappa) + x$ and $g_b(x) = G_{\frac{p-1}{2},\frac{p-1}{2}}(\kappa) - x$. We need to control both $g_a(r)$ and $g_b(r)$, which can be achieved by using Lemma S.2.2.

Starting with $g_a(r)$, Lemma S.2.2(iii) readily yields that

$$g_a(r) \leq G_{-\frac{p-1}{2},\frac{p-1}{2}}(\kappa) + G_{\frac{p-1}{2},\frac{p-1}{2}}(\kappa) = \frac{2}{\kappa}\sqrt{\big(\tfrac{p-1}{2}\big)^2 + \kappa^2}.$$

As for $g_b(r)$, Lemma S.2.2(ii) entails

$$g_b(r) \leq G_{\frac{p-1}{2},\frac{p-1}{2}}(\kappa) - G_{\frac{p-1}{2},\frac{p+1}{2}}(\kappa) = \frac{\kappa\Big(\sqrt{\big(\frac{p+1}{2}\big)^2+\kappa^2} - \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)}{\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p+1}{2}\big)^2+\kappa^2}\Big)}$$

$$= \frac{\kappa p}{\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p+1}{2}\big)^2+\kappa^2}\Big)\Big(\sqrt{\big(\frac{p+1}{2}\big)^2+\kappa^2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)}$$

$$\leq \frac{\kappa p}{2\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)^2\sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}}.$$

Plugging into (S.2.8) provides

$$\frac{p\tilde{e}_2^2}{f_2^2} \leq \frac{\kappa^2 p^3}{(p-1)^2\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)^4 r^2}.$$

Using Lemma S.2.2(ii) again then yields

$$\frac{p\tilde{e}_2^2}{f_2^2} \leq \frac{\kappa^2 p^3}{(p-1)^2\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)^4 (G_{\frac{p-1}{2},\frac{p+1}{2}}(\kappa))^2} = \frac{p^3\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p+1}{2}\big)^2+\kappa^2}\Big)^2}{(p-1)^2\Big(\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}\Big)^4}$$

$$\leq \frac{p^3}{(p-1)^4}\left(\frac{\frac{p-1}{2} + \sqrt{\big(\frac{p+1}{2}\big)^2+\kappa^2}}{\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}}\right)^2 \leq \frac{8}{p-1}\left(\frac{\frac{p-1}{2} + \sqrt{9\big(\frac{p-1}{2}\big)^2+\kappa^2}}{\frac{p-1}{2} + \sqrt{\big(\frac{p-1}{2}\big)^2+\kappa^2}}\right)^2 \leq 24,$$

for any $p \geq 2$ and any $\kappa > 0$, which establishes the result. □

Proof of Theorem S.2.1(iii). From (S.2.7), we obtain

$$\frac{f_4}{f_2^2} = \frac{(p+1)\mathcal{I}_{\frac{p}{2}-1}(\kappa)\mathcal{I}_{\frac{p}{2}+1}(\kappa)}{(p-1)\mathcal{I}^2_{\frac{p}{2}}(\kappa)} \leq \frac{3\mathcal{I}_{\frac{p}{2}-1}(\kappa)\mathcal{I}_{\frac{p}{2}+1}(\kappa)}{\mathcal{I}^2_{\frac{p}{2}}(\kappa)},$$

for any $p \geq 2$ and any $\kappa > 0$. By using (S.2.6), this provides

$$\frac{f_4}{3f_2^2} \leq \frac{\mathcal{I}_{\frac{p}{2}-1}(\kappa)(\mathcal{I}_{\frac{p}{2}-1}(\kappa) - (p/\kappa)\mathcal{I}_{\frac{p}{2}}(\kappa))}{\mathcal{I}^2_{\frac{p}{2}}(\kappa)} = \frac{1}{r^2} - \frac{p}{\kappa r} = \frac{\kappa - pr}{\kappa r^2}$$

(note that Lemma S.2.2(iv) yields $r \leq G_{\frac{p}{2}-1,\frac{p}{2}+1}(\kappa) \leq \kappa/p$). Lemma S.2.2(i) then entails

$$\begin{aligned}
\frac{f_4}{3f_2^2} &\leq \frac{\kappa - pG_{\frac{p-1}{2},\frac{p+1}{2}}(\kappa)}{\kappa(G_{\frac{p-1}{2},\frac{p+1}{2}}(\kappa))^2} = \frac{1}{\kappa^2}\Big(\tfrac{p-1}{2} + \sqrt{\big(\tfrac{p+1}{2}\big)^2 + \kappa^2}\Big)^2\Bigg(1 - \frac{p}{\frac{p-1}{2} + \sqrt{\big(\frac{p+1}{2}\big)^2 + \kappa^2}}\Bigg) \\
&\leq \frac{1}{\kappa^2}\Big(\tfrac{p-1}{2} + \sqrt{\big(\tfrac{p+1}{2}\big)^2 + \kappa^2}\Big)\Big(-\tfrac{p+1}{2} + \sqrt{\big(\tfrac{p+1}{2}\big)^2 + \kappa^2}\Big) \\
&= 1 + \frac{1}{\kappa^2}\Big(\tfrac{p+1}{2} - \sqrt{\big(\tfrac{p+1}{2}\big)^2 + \kappa^2}\Big) = 1 - \frac{1}{\frac{p+1}{2} + \sqrt{\big(\frac{p+1}{2}\big)^2 + \kappa^2}} \leq 1,
\end{aligned}$$

which proves the result. □

PROOF OF THEOREM S.2.1(ii). Plugging the expressions of e_ℓ, $\ell = 1, 2, 3, 4$, from Lemma S.2.1 in

$$\frac{\tilde{e}_{n4}}{\tilde{e}_{n2}^2} = \frac{e_4 - 4e_3e_1 + 6e_2e_1^2 - 4e_1^4 + e_1^4}{(e_2 - e_1^2)^2} = \frac{e_4 - 4e_3e_1 + 6e_2e_1^2 - 3e_1^4}{e_2^2 - 2e_1^2e_2 + e_1^4} \tag{S.2.9}$$

yields (after tedious computations)

$$\frac{\tilde{e}_{n4}}{\tilde{e}_{n2}^2} = \frac{-(p^2 + 2p + 4\kappa^2 - 3)x^2 - \frac{p-1}{\kappa}(p^2 + p + 4\kappa^2)x + (p^2 + 4\kappa^2 - 1)}{\kappa^2\big(x^2 + \frac{p-1}{\kappa}x - 1\big)^2} - 3 =: h(r) - 3,$$

We need to show that $h(r)$ is bounded in (p, κ) for p large enough, which will be done on the basis of the factorization

$$h(r) = \frac{(p^2 + 2p + 4\kappa^2 - 3)f_a(r)f_b(r)}{\kappa^2\big(g_a(r)g_b(r)\big)^2}\,, \tag{S.2.10}$$

where we let

$$f_a(x) := \big(\tfrac{p-1}{2\kappa}\big)\frac{(\frac{p}{2})(\frac{p+1}{2}) + \kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2}) + \kappa^2} + \sqrt{\Big(\big(\tfrac{p-1}{2\kappa}\big)\frac{(\frac{p}{2})(\frac{p+1}{2}) + \kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2}) + \kappa^2}\Big)^2 + \frac{(\frac{p-1}{2})(\frac{p+1}{2}) + \kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2}) + \kappa^2}} + x,$$

$$f_b(x) := -\big(\tfrac{p-1}{2\kappa}\big)\frac{(\frac{p}{2})(\frac{p+1}{2}) + \kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2}) + \kappa^2} + \sqrt{\Big(\big(\tfrac{p-1}{2\kappa}\big)\frac{(\frac{p}{2})(\frac{p+1}{2}) + \kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2}) + \kappa^2}\Big)^2 + \frac{(\frac{p-1}{2})(\frac{p+1}{2}) + \kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2}) + \kappa^2}} - x,$$

and where $g_a(x) = G_{-\frac{p-1}{2},\frac{p-1}{2}}(\kappa) + x$ and $g_b(x) = G_{\frac{p-1}{2},\frac{p-1}{2}}(\kappa) - x$ are the functions already considered in the proof of Theorem S.2.1(i).

We start with $g_a(r)$, which, in view of Lemma S.2.2(i), satisfies

$$\begin{aligned}
g_a(r) &\geq G_{-\frac{p-1}{2},\frac{p-1}{2}}(\kappa) + G_{\frac{p}{2},\frac{p}{2}}(\kappa) \geq \sqrt{1 + \big(\tfrac{p-1}{2\kappa}\big)^2} + \tfrac{p-1}{2\kappa} + \sqrt{1 + \big(\tfrac{p}{2\kappa}\big)^2} - \tfrac{p}{2\kappa} \\
&\geq 2\sqrt{1 + \big(\tfrac{p-1}{2\kappa}\big)^2} - \tfrac{1}{2\kappa} \geq \sqrt{2}\big(1 + \big(\tfrac{p-1}{2\kappa}\big)\big) - \tfrac{1}{2\kappa} \geq \sqrt{2}\big(1 + \big(\tfrac{p-2}{2\kappa}\big)\big) \geq \frac{C(p + \kappa)}{\kappa},
\end{aligned}$$

where C stands for a positive real constant (that may change from line to line in the rest of the proof). Turning to $g_b(r)$, Lemma S.2.2(vi) yields

$$
\begin{aligned}
g_b(r) = G_{\frac{p-1}{2},\frac{p-1}{2}}(\kappa) - G_{\frac{p-1}{2},\sqrt{(p^2-1)/4}}(\kappa) &= \frac{\kappa(\sqrt{\frac{p^2-1}{4}+\kappa^2} - \sqrt{\left(\frac{p-1}{2}\right)^2+\kappa^2})}{(\frac{p-1}{2}+\sqrt{\left(\frac{p-1}{2}\right)^2+\kappa^2})(\frac{p-1}{2}+\sqrt{\frac{p^2-1}{4}+\kappa^2})} \\
&= \frac{\kappa(p-1)}{2(\frac{p-1}{2}+\sqrt{\left(\frac{p-1}{2}\right)^2+\kappa^2})(\frac{p-1}{2}+\sqrt{\frac{p^2-1}{4}+\kappa^2})(\sqrt{\frac{p^2-1}{4}+\kappa^2}+\sqrt{\left(\frac{p-1}{2}\right)^2+\kappa^2})} \\
&\geq \frac{\kappa(p-1)}{4(p+\sqrt{p^2+\kappa^2})^3} \geq \frac{C\kappa p}{p^3+\kappa^3}.
\end{aligned}
$$

Now, by applying Lemma S.2.2(v), we obtain

$$
\begin{aligned}
f_a(r) &\leq \tfrac{p-1}{2\kappa} + \sqrt{(\tfrac{p-1}{2\kappa})^2+1} + r \leq \tfrac{p-1}{2\kappa} + \sqrt{(\tfrac{p-1}{2\kappa})^2+1} + G_{\frac{p}{2}-1,\frac{p}{2}-1}(\kappa) \\
&= \frac{\frac{p-1}{2}+\sqrt{(\frac{p-1}{2})^2+\kappa^2}}{\kappa} + \frac{\sqrt{(\frac{p}{2}-1)^2+\kappa^2}-(\frac{p}{2}-1)}{\kappa} \leq \frac{\frac{1}{2}+2\sqrt{(\frac{p-1}{2})^2+\kappa^2}}{\kappa} \leq \frac{C(p+\kappa)}{\kappa}.
\end{aligned}
$$

Finally, using the notation and results from Lemma S.2.3, we obtain

$$
\begin{aligned}
f_b(r) &= -(\tfrac{p-1}{2\kappa})\frac{(\frac{p}{2})(\frac{p+1}{2})+\kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2})+\kappa^2} + \sqrt{\left((\tfrac{p-1}{2\kappa})\frac{(\frac{p}{2})(\frac{p+1}{2})+\kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2})+\kappa^2}\right)^2 + \frac{(\frac{p-1}{2})(\frac{p+1}{2})+\kappa^2}{(\frac{p-1}{2})(\frac{p+3}{2})+\kappa^2}} - r \\
&= U_{\frac{p}{2}-1} - r \leq U_{\frac{p}{2}-1} - L_{\frac{p}{2}-1} \leq \frac{C\kappa^3(\frac{p+1}{2})^2}{(\frac{p}{2}-1)^7+\kappa^7} \leq \frac{C\kappa^3p^2}{p^7+\kappa^7},
\end{aligned}
$$

for p large enough and any $\kappa > 0$.

Plugging in (S.2.10) the bounds just obtained on $g_a(r)$, $g_b(r)$, $f_a(r)$ and $f_b(r)$ entails

$$
\begin{aligned}
\frac{\tilde{e}_{n4}}{\tilde{e}_{n2}^2} + 3 = h(r) &\leq C\,\frac{p^2+2p+4\kappa^2-3}{\kappa^2} \times \frac{\kappa^2}{(p+\kappa)^2} \times \frac{(p^3+\kappa^3)^2}{\kappa^2p^2} \times \frac{p+\kappa}{\kappa} \times \frac{\kappa^3p^2}{p^7+\kappa^7} \\
&\leq C\,\frac{(p^2+\kappa^2)(p^3+\kappa^3)^2}{(p+\kappa)(p^7+\kappa^7)} \leq C,
\end{aligned}
$$

for p large enough and any $\kappa > 0$, as was to be proved.

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