The Price of Connectivity for Vertex Cover: Perfect, Near-Perfect and Critical Graphs

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Abstract. In this paper we investigate the ratio of the connected vertex cover number to the vertex cover number in graphs, called the *Price of Connectivity* (PoC). For general graphs, this ratio is strictly bounded by 2. We prove that for every (P_5, C_5, C_4)-free graph the ratio equals 1. We prove also that for every (P_5, C_4)-free graph the ratio is bounded by 4/3 and that for every $(P_7, C_6, \Delta_1, \Delta_2)$-free graph the ratio is bounded by 3/2, where Δ_1 and Δ_2 are two particular graphs. These results directly yields forbidden induced subgraphs characterizations of those graphs for which the PoC of every induced subgraph is bounded by t, for $t \in \{1, 4/3, 3/2\}$. Furthermore, we investigate PoC-critical graphs, namely, graphs whose PoC is strictly greater than the PoC of any proper induced subgraph.

Keywords: vertex cover, connected vertex cover, forbidden induced subgraphs.

1 Introduction

A *vertex cover* of a graph G is a vertex subset X such that every edge of G has at least one endpoint in X. The size of a minimum vertex cover of G, denoted by $\tau(G)$, is called the *vertex cover number* of G. The vertex cover problem is one of the 21 NP-hard problem identified by Karp in 1972, and has since been intensively studied in the literature.

A well-known variant of the notion of vertex cover is that of *connected vertex cover*, defined as a vertex cover X such that the induced subgraph $G[X]$ is connected. (If G is not connected we ask that $G[X]$ has the same number of component as G.) The minimum size of such a set, denoted by $\tau_c(G)$, is the *connected vertex cover number* of G. A connected vertex cover of size $\tau_c(G)$ is called a *minimum connected vertex cover*.

Our contribution is to study the interdependence of $\tau(G)$ and $\tau_c(G)$ in some hereditary classes of graphs or more precisely the ratio $\tau_c(G)/\tau(G)$, that we call *Price of Connectivity* (similarly to the Price of Anarchy as studied in algorithmic game theory).

⋆ Parts of this research have been carried out during the visit of Oliver Schaudt to Université Libre de Bruxelles.

Let us first note that every vertex cover X of a connected graph G such that $G[X]$ has c components can be turned into a connected vertex cover of G by adding at most $c-1$ vertices. This directly yields the following observation.

Observation 1. For every graph G it holds that $\tau_c(G) \leqslant 2\tau(G) - 1$.

As an immediate consequence of Observation 1, the following holds for every graph G (with at least one edge):

$$1 \leqslant \tau_c(G)/\tau(G) < 2. \tag{1}$$

In other words, the Price of Connectivity (PoC) lies in the interval $[1, 2)$. We denote by P_k the path on k vertices and by C_k the cycle on k vertices. Note that the upper bound in (1) is asymptotically sharp in the class of paths and in the class of cycles, in the sense that $\lim\limits_{k\to\infty} \tau_c(P_k)/\tau(P_k) = 2 = \lim\limits_{k\to\infty} \tau_c(C_k)/\tau(C_k)$.

We investigate graph classes for which the PoC is bounded by a constant t with $t \in [1, 2)$. Those classes will be defined by forbidden induced subgraphs. For this we use the following standard notation. If G and H are two graphs we say that G *contains* H if G has an induced subgraph isomorphic to H. We say that G is H*-free* if G does not contain H. Furthermore, we say that G is $(H_1, \ldots, H_\ell)$-free if G does not contain H_i for any $i \in \{1, \ldots, \ell\}$.

The Price of Connectivity (as defined here) has been introduced by Cardinal and Levy [2, 4], who showed that it was bounded by $2/(1+\varepsilon)$ in graphs with average degree εn, where n denotes the number of vertices. Other ratios were previously studied. In a companion paper to the present paper, Camby and Schaudt [1] consider the Price of Connectivity for dominating set. Recently, Schaudt [5] studied the ratio between the connected domination number and the total domination number. Fulman [3] and Zverovich [7] investigated the ratio between the independence number and the upper domination number.

2 Our results

Due to length constraints, all proofs are omitted and can be found in the long version of the paper.

2.1 PoC-Perfect Graphs

We first consider a hereditary class of graphs G for which $\tau_c(G) = \tau(G)$, referred to as *PoC-Perfect graphs*. A similar result had been found by Zverovich [6] for dominating set. There, the corresponding class is that of (P_5, C_5)-free graphs.

Theorem 1. *The following assertions are equivalent for every graph G.*

(i) *For every induced subgraph H of G it holds that $\tau_c(H) = \tau(H)$.*
(ii) *G is (P_5, C_5, C_4)-free.*
(iii) *G is chordal and P_5-free.*

2.2 PoC-Near-Perfect Graphs

First, we observe that (1) is asymptotically sharp in the class of connected (P_8, C_8)-free graphs (see Fig. 1).

Observation 2. There is a family $\{G_k : k \in \mathbb{N}\}$ of (P_8, C_8)-free graphs on $3k+1$ vertices such that $\lim_{k\to\infty} \tau_c(G_k)/\tau(G_k) = 2$.

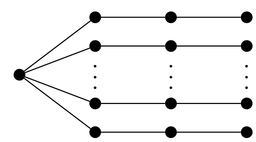

Fig. 1. G_k is obtained from k copies of P_4 with an common endvertex.

Let $t \in [1, 2)$. A graph G is said to be *PoC-near-perfect* with threshold t if every induced subgraph H of G satisfies $\tau_c(H) \leqslant t \cdot \tau(H)$. This defines a hereditary class of graphs for every choice of t. Theorem 1 gives a forbidden induced subgraphs characterization of this class for $t = 1$. Our second result gives such a characterization for $t = 4/3$. Note that $\tau_c(C_5)/\tau(C_5) = 4/3$ and $\tau_c(P_5)/\tau(P_5) = \tau_c(C_4)/\tau(C_4) = 3/2$, hence $t = 4/3$ is the next interesting threshold after $t = 1$.

Theorem 2. *The following assertions are equivalent for every graph G :*

(i) For every induced subgraph H of G it holds that $\tau_c(H) \leqslant \frac{4}{3} \cdot \tau(H)$.
(ii) G is (P_5, C_4)-free.

By Theorem 2, $t = 3/2$ is the next interesting threshold after $t = 4/3$. Our third results states that the list of forbidden induced subgraphs for threshold $t = 3/2$ is $(C_6, P_7, \Delta_1, \Delta_2)$, where Δ_1 is the 1-join of two C_4's, and Δ_2 is obtained from Δ_1 by removing any edge incident to the vertex of degree 4 (see Fig. 2).

Theorem 3. *The following assertions are equivalent for every graph G :*

(i) For every induced subgraph H of G it holds that $\tau_c(H) \leqslant \frac{3}{2} \cdot \tau(H)$.
(ii) G is $(P_7, C_6, \Delta_1, \Delta_2)$-free.

Fig. 2. An illustration of graphs Δ_1 (on the left) and Δ_2 (on the right).

2.3 PoC-critical Graphs

We now turn our attention to *critical graphs*, that is, graphs G for which the PoC of any proper induced subgraph H of G is strictly smaller than the PoC of G. These are exactly the graphs that can appear in a forbidden induced subgraphs characterization of the PoC-near-perfect graphs for some value threshold $t \in [1, 2)$. A perhaps more tractable class of graphs are the *strongly critical* graphs, defined as the graphs G for which every proper (not necessarily induced) subgraph H of G has a PoC that is strictly smaller than the PoC of G. It is clear that every strongly critical graph is critical, but the converse is not true. For instance, C_5 is critical, but not strongly critical.

PoC-Critical Chordal Graphs Let T be a tree. We call T *special* (see Fig. 3) if it is obtained from another tree (vertices indicated by filled circles) by subdividing each edge exactly once (subdivision vertices are indicated by hollow circles) and then attaching a pendant vertex (indicated by squares) to every leaf of the resulting graph.

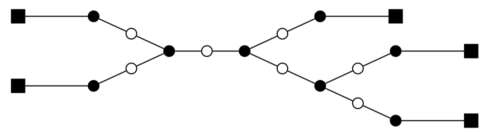

Fig. 3. Example of a special tree.

Our next result characterizes the class of (strongly) critical chordal graphs.

Theorem 4. *For a chordal graph G, the following assertions are equivalent :*

(i) G is critical.
(ii) G is strongly critical.
(iii) G is a special tree.

PoC-Strongly-Critical Graphs Our final result yields structural constraints on the class of strongly critical graphs.

Theorem 5. *Let G be a strongly critical graph.*

(i) Every minimum vertex cover of G is stable. In particular, G is bipartite.
(ii) If G has a cut-vertex, then G is a special tree.

References

1. E. Camby, O. Schaudt, The price of connectivity for minimum dominating sets. Submitted.
2. J. Cardinal, E. Levy, Connected vertex covers in dense graphs, Theor. Comput. Sci. 411 (2010), pp. 2581–2590.
3. J. Fulman, A note on the characterization of domination perfect graphs, J. Graph Theory 17 (1993) pp. 47–51.
4. E. Levy. (2009). Approximation Algorithms for Covering Problems in Dense Graphs. Ph.D. thesis. Université libre de Bruxelles, Brussels.
5. O. Schaudt, On graphs for which the connected domination number is at most the total domination number. to appear in Discrete Appl. Math. (2012).
6. I.E. Zverovich, Perfect connected-dominant graphs. Discuss. Math. Graph Theory 23 (2003), pp. 159–162.
7. I.E. Zverovich, V.E. Zverovich, A semi-induced subgraph characterization of upper domination perfect graphs, J. Graph Theory 31 (1999), pp. 29–49.