Connected Dominating Set in Graphs Without Long Paths And Cycles

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The ratio of the connected domination number, γ_c, and the domination number, γ, is strictly bounded from above by 3. It was shown by Zverovich that for every connected (P_5, C_5)-free graph, $\gamma_c = \gamma$.

We investigate the interdependence of γ and γ_c in the class of (P_k, C_k)-free graphs, for $k \geq 6$. We prove that for every connected (P_6, C_6)-free graph, $\gamma_c \leq \gamma + 1$ holds, and there is a family of (P_6, C_6)-free graphs with arbitrarily large values of γ attaining this bound. Moreover, for every connected (P_8, C_8)-free graph, $\gamma_c/\gamma \leq 2$, and there is a family of (P_7, C_7)-free graphs with arbitrarily large values of γ attaining this bound. In the class of (P_9, C_9)-free graphs, the general bound $\gamma_c/\gamma < 3$ is asymptotically sharp.

1 Introduction

A full paper version of this extended abstract, including proofs and more details, is available online at http://www.zaik.uni-koeln.de/~schaudt/DS-PoC.pdf.

A *dominating set* of a graph G is a vertex subset X such that every vertex not in X has a neighbor in X. The minimum size of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is called a *minimum dominating set*.

Dominating sets have been intensively studied in the literature. The main interest in dominating sets is due to their relevance on both theoretical and practical side. Moreover, there are interesting variants of domination and many of them are well-studied. A good introduction into the topic is given by Haynes, Hedetniemi and Slater [7].

A *connected dominating set* of a graph G is a dominating set X whose induced subgraph, henceforth denoted $G[X]$, is connected. The minimum size of such a set of a connected graph G, the *connected domination number* of G, is denoted by $\gamma_c(G)$. A connected dominating set of size $\gamma_c(G)$ is called a *minimum connected dominating set*. A connected dominating set such that every proper subset is not a connected dominating set is called *minimal connected dominating set*. Among the applications of connected dominating sets is the routing of messages in mobile ad-hoc networks. Blum, Ding, Thaeler and Cheng [1] explain the usefulness of connected dominating sets in this context.

A first impression of the relation of γ_c and γ is given by Duchet and Meyniel [4].

Observation 1.1 (Duchet and Meyniel [4]). *For every connected graph it holds that* $_c \quad 3 \quad -2$.

As an immediate consequence of Observation 1.1,

$$_c/ \quad < 3. \tag{1}$$

Loosely speaking, the price of connectivity for minimum dominating sets, $_c/$, is strictly bounded by 3.

Let P_k be the induced path on k vertices and let C_k be the induced cycle on k vertices. It is easy to see that

$$\lim_{k} \quad _c(P_k)/ \quad (P_k) = 3 = \lim_{k} \quad _c(C_k)/ \quad (C_k). \tag{2}$$

Hence, the upper bound (1) is asymptotically sharp in the class of paths and in the class of cycles.

The price of connectivity has been introduced by Cardinal and Levy [3, 9] for the vertex cover problem. They showed that it was bounded by $2/(1+ \quad)$ in graphs with average degree n, where n is the number of vertices. In a companion paper to the present one, the price of connectivity for vertex cover is studied by Camby, Cardinal, Fiorini and Schaudt [2]. In a similar spirit, Schaudt [11] studied the ratio between the connected domination number and the total domination number. Fulman [5] and Zverovich [13] investigated the ratio between the independence number and the upper domination number. Many results in this area concern graph classes defined by forbidden induced subgraphs. This line of research stems from the classical theory of perfect graphs, for which the clique number and the chromatic number are equal in every induced subgraph [6].

Motivated by (2), we study the interdependence of $_c$ and in graph classes defined by forbidden induced paths and cycles. For this we use the following standard notation. If G and H are two graphs, we say that G is *H-free* if H does not appear as an induced subgraph of G. Furthermore, if G is H_1-free and H_2-free for some graphs H_1 and H_2, we say that G is *(H_1, H_2)-free*. Our starting point is the following result by Zverovich [12].

Theorem 1.2 (Zverovich [12]). *The following assertions are equivalent for every graph G.*

1. *For every connected induced subgraph of G it holds that* $_c = \quad$.
2. *G is (P_5, C_5)-free.*

We aim for similar bounds in the class of (P_k, C_k)-free connected graphs for $k \quad 6$. The properties of connected dominating sets in P_k-free graphs have been studied before, e.g. by Liu, Peng and Zhao [10] and later van 't Hof and Paulusma [8].

Apart from the previous work, this research has an algorithmic motivation. The proofs of our results are constructive in the sense that it is possible to draw polynomial time algorithms from them. These algorithms can be used to build, given a dominating set of size k, a connected dominating set of size at most $f(k)$, for the suitable function f provided by the respective theorem. We do not explicitely give the algorithms, but leave it as a possible future application of our results.

2 Our Results

Our first result establishes the upper bound $_c \quad +1$ in the class of connected (P_6, C_6)-free graphs.

Theorem 2.1. *For every connected* (P_6, C_6)*-free graph it holds that* $_c \quad + 1$.

To see that the bound given by Theorem 2.1 is best possible, consider the following family of connected (P_6, C_6)-free graphs. For each $k \quad \mathbb{N}$, let F_k be the graph obtained from a k disjoint copies of C_4, by picking one vertex from every copy and identifying these picked vertices to a single vertex x, and afterwards attaching a path of length 2 to x (see Fig. 1). It is easy to see that, for all k, $_c(F_k) = \quad (F_k) + 1$. Moreover, the graph F_k does not have a minimum connected dominating set that contains a minimum dominating set as subset.

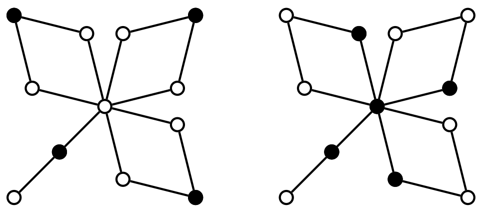

Figure 1: The black vertices indicate a minimum dominating set (resp. a minimum connected dominating set) of F_3.

Theorem 2.2. *For every connected* (P_8, C_8)*-free graph it holds that* $_c/ \quad 2$.

The bound provided by Theorem 2.2 is attained by an infinite number of connected (P_7, C_7)-free graphs, given by the following construction. For every $k \quad \mathbb{N}$, let H_k be the graph defined as follows (cf. Fig. 2). Start with k paths P^1, P^2, ..., P^k on three vertices each. For every $1 \quad i \quad k$, choose an end-vertex v_i of P^i. Let H_k be the graph obtained from the disjoint union of all P^i, $1 \quad i \quad k$, by adding all possible edges between the vertices v_i, $1 \quad i \quad k$. So, $H_k[\{v_i : 1 \quad i \quad k\}]$ is a complete graph. It is easily seen that, for all $k \quad \mathbb{N}$, $_c(H_k)/ \quad (H_k) = 2$.

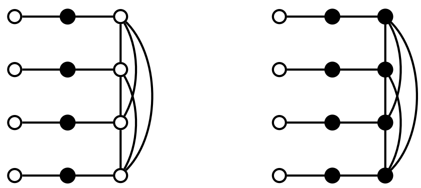

Figure 2: The black vertices indicate a minimum dominating set (resp. a minimum connected dominating set) of H_4.

A similar construction shows that (1) is asymptotically sharp in the class of connected (P_9, C_9)-free graphs, in the sense that there is a family $\{G_k : k \quad \mathbb{N}\}$ of (P_9, C_9)-free graphs such that $\lim_k \quad _c(G_k)/ \quad (G_k) = 3$. For every $k \quad \mathbb{N}$ let G_k be the graph obtained by attaching a pendant vertex to every pendant vertex of H_k. It is easy to check that for every $k \quad 2$, $(G_k) = k + 1$ and $_c(G_k) = 3k$. Furthermore, G_k is (P_9, C_9)-free.

3 A Conjecture

We close this abstract with a conjecture that came up during our research. As Theorem 2.2 shows, $_c \quad 2$ holds in every connected (P_8, C_8)-free graph. However, $_c(P_8)/ \quad (P_8) = 2 = \quad _c(C_8)/ \quad (C_8)$, i.e., both P_8 and C_8 do not violate the bound given by Theorem 2.2.

Conjecture 3.1. *For every connected* (P_9, C_9, H)*-free graph,* $_c$ 2 *(see Fig. 3 for* H*).*

Note that P_9, C_9, and H violate $_c$ 2 . Hence, if true, Conjecture 3.1 would give a characterization of the largest graph class that is closed under connected induced subgraphs where $_c$ 2 holds.

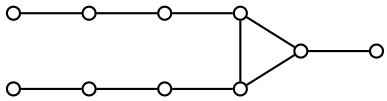

Figure 3: The graph H from Conjecture 3.1.

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