Research on the Price of Connectivity for the vertex cover problem and the dominating set problem, with the help of the system GraphsInGraphs

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The vertex cover problem and the dominating set problem are two well-known problems in graph theory. Both hold a connected version, which imposes that the vertex subset must induce a connected component. To study the interdependence between the connected version and the original version of a problem, the Price of Connectivity (PoC) was introduced by Cardinal and Levy [7, 10] as the ratio between invariants from the connected version and the original version of the problem.

Some classes of PoC-Near-Perfect graphs, hereditary classes of graphs in which the Price of Connectivity is bounded by a fixed constant, have been already studied [5, 6]. To go further, we present for the vertex cover problem conjectures on these graphs with the help of the computer software GraphsInGraphs [4].

Moreover, Camby, Cardinal, Fiorini and Schaudt [5] introduced, for the vertex cover problem, the notion of critical graphs, graphs that can appear in the list of forbidden induced subgraphs characterization. By definition, the Price of Connectivity of a critical graph is strictly greater than that of any proper induced subgraph. In this paper, we prove that for the vertex cover problem, every critical graph is either isomorphic to a cycle on 5 vertices or bipartite. Moreover, for the dominating set problem, we investigate critical trees and we show that every minimum dominating set of a critical graph is independent.

1 Introduction

A *vertex cover* is a vertex subset X such that every edge of G has at least one endpoint in X. A *connected vertex cover* is a vertex cover X such that the induced subgraph $G[X]$ is connected. A *dominating set* of a graph G is a vertex subset X such that every vertex either is in X or has a neighbor in X. A *connected dominating set* of G is a dominating set X of G that induces a connected subgraph. In both cases, when G is not connected, we require that $G[X]$ has the same number of connected components as G. Table 1 fixes all notations about these notions.

In 1972, Karp identified 21 NP-hard problems, among which finding a minimum vertex cover of a graph. In 2008, Cardinal and Levy [7, 10] introduced the Price of Connectivity for the vertex cover problem. Lately, Camby, Cardinal, Fiorini and Schaudt [5] studied more in depth this new graph invariant. Besides, several researchers studied the interdependence between other graphs invariants.

Original version	VERTEX COVER PROBLEM	DOMINATING SET PROBLEM
value of the minimum size name of this value	$\tau(G)$ vertex cover number	$\gamma(G)$ domination number

Connected version	VERTEX COVER PROBLEM	DOMINATING SET PROBLEM
value of the minimum size name of this value	$\tau_c(G)$ connected vertex cover number	$\gamma_c(G)$ connected domination number

Price of Connectivity	VERTEX COVER PROBLEM	DOMINATING SET PROBLEM
value	$\frac{\tau_c(G)}{\tau(G)}$	$\frac{\gamma_c(G)}{\gamma(G)}$

Table 1: Notations for the vertex cover problem and the dominating set problem.

Zverovich [11] characterized, in terms of list of forbidden induced subgraphs, *PoC-Perfect graphs*, graphs for which the connected domination number and the domination number are equal for all induced subgraphs. Some years ago, Camby and Schaudt [6] translated the Price of Connectivity from the vertex cover problem to the dominating set problem and investigated it.

Recently, Belmonte, vant Hof, Kamiński and Paulusma [1, 2, 3] studied the Price of Connectivity for the feedback vertex set while Hartinger, Johnson, Milanič and Paulusma [8, 9] investigated the Price of Connectivity for cycle transversals.

2 Our results

2.1 The vertex cover problem

2.1.1 Conjectures on PoC-Near-Perfect graphs and critical graphs

With the help of the computer software GraphsInGraphs [4], we establish two new conjectures on *PoC*-Near-Perfect graphs and critical graphs. As a generalization of *PoC*-Perfect graphs, *PoC-Near-Perfect graphs* are graphs such that the Price of Connectivity of all induced subgraphs is bounded by a fixed constant, whereas, by definition, a *critical graph* has a Price of Connectivity strictly greater than that of all proper induced subgraphs.

Conjecture 1. *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\tau_c(H) \leqslant \frac{5}{3}\ \tau(H)$.

(ii) G is $(H_i)_{i=1}^{10}$-free, where graphs $H_1, \ldots, H_{10}$ are depiced in Figure 1.

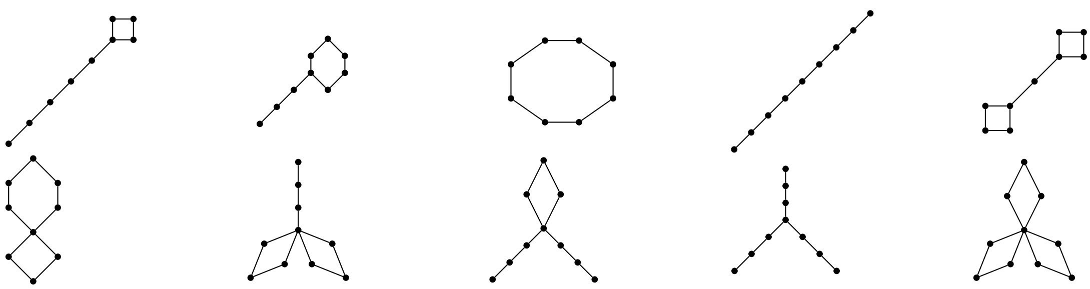

Figure 1: Graphs $H_1, \ldots, H_{10}$ from Conjecture 1.

Conjecture 2. *Every critical graph is a cactus.*

2.1.2 Critical graphs

Camby *& al.* [5] proved that every strongly critical graph, graph whose its price of connectivity is strictly greater than that of all proper (not necessarily induced) subgraphs, is bipartite. Here, we extend the result to the class of critical graphs, expect the cycle C_5 on 5 vertices.

Theorem 1. *A critical graph G is either isomorphic to C_5, or bipartite. Moreover, when G is bipartite, every minimum vertex cover of G is independent.*

2.2 Dominating set problem

2.2.1 Critical trees

Let T be a tree. We call T *special* if T is obtained from another tree (filled circle vertices in the example of Figure 2) by subdividing each edge either once or twice (hollow circle vertics) and then attaching a pendent vertex to every leaf of the resulting graph (square vertices).

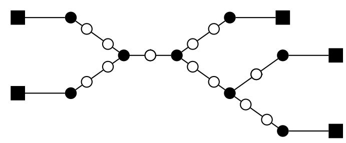

Figure 2: A special tree

The next result gives a partial characterization of the class of critical trees. However, the class of special trees turns out to be too restricted. We need a new definition.

We call a tree T *peculiar* if the neighbor of every leaf has degree 2, every minimum dominating set D of T is independent and every vertex $v \in V(T) \setminus D$ with degree at least 3 has only one neighbor in D, i.e. $|N_T(v) \cap D| = 1$. See Figure 3 for an example of peculiar tree where a minimum dominating set contains vertices indicated by filled circles and leaves are indicated by squares.

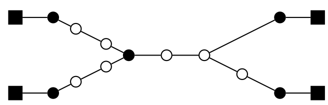

Figure 3: A peculiar tree.

Theorem 2. *For a tree G, the following assertions are equivalent:*

(i) G is a peculiar critical tree.

(ii) G is strongly critical.

(iii) G is critical.

Moreover, if G is critical and if the degree of any $v \in V(G) \setminus D$, where D is an arbitrary minimum dominating set of G, is at most 2, then G is a special tree built on an initial tree H, where $V(H)$ is a minimum dominating set.

Figure 4 illustrates the relations between graph classes: special trees, peculiar trees and critical trees. Not all peculiar trees are critical. For instance, the graph depicted in Figure 3 (whose Price of Connectivity is 12/5) is not critical because it contains as an induced subgraph the graph, with a higher Price of Connectivity, obtained from $K_{1,3}$ by subdividing each edge exactly thrice. Furthermore, the graph illustrated by Figure 5 is a peculiar critical tree which is not special. Also, we point out that not all special trees are critical, for instance P_8 contains an induced

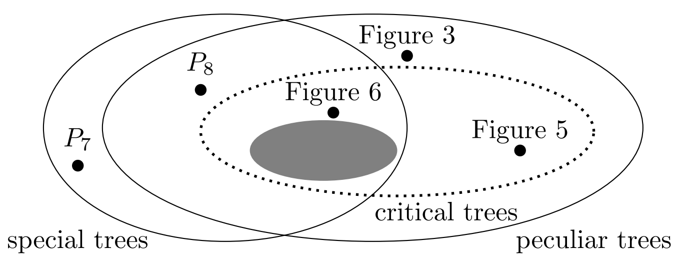

Figure 4: The situation around critical trees.

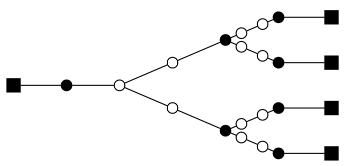

Figure 5: A peculiar critical tree, not special.

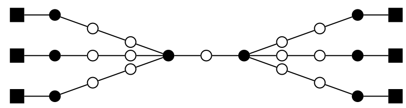

Figure 6: A special critical tree.

P_6 with the same Price of Connectivity.

Moreover, by Proposition 3, every special tree built on the initial tree H, where all edges of H are subdivided exactly twice in G, is critical. These graphs are represented by the gray area in Figure 4. However, the converse in the class of special trees is not true because the graph illustrated by Figure 6 is critical.

Proposition 3. *Let G be a special tree built on the initial tree H. If all edges of H are subdivided twice in G, then G is critical.*

2.2.2 Critical graphs

Theorem 4. *Every minimum dominating set of a critical graph is independent.*

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