From Cartesian product graph to a robust communication network

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Abstract

We present some results on the building of networks with a small average distance and a bounded maximum degree, in order to design efficient data centers. Moreover, due to real technical constraints, we need that the network possesses some robustness, especially after an edge or a vertex removal.

From a starting point, we are interested by the *Cartesian product graphs.* From two graphs G and H, we build the Cartesian product graph $G\Box H$ in the following way :

$$V(G\Box H) = \{uv|u \in V(G) \text{ and } v \in V(H)\}$$

and

$$E(G\Box H) = \{uv - u'v'|(u = u' \text{ and } v - v' \in E(H)) \text{ or } (u - u' \in E(G) \text{ and } v = v')\}.$$

The main advantage is that the number of vertices of the result is the product of those from G and H, whereas regarding the most other invariants (for instance average distance or maximum degree), one of the result is the sum of those from G and H. Following properties illustrate this concept.

Property 1 *The degree $\delta_{uv}^{G\Box H}$ of the vertex uv in $G\Box H$ is equal to*

$$\delta_{uv}^{G\Box H} = \delta_u^G + \delta_v^H.$$

Property 2 *The geodesic distance between vertices uv and $u'v'$ in $G\Box H$ is given by:*

$$d^{G\Box H}(uv, u'v') = d^G(u, u') + d^H(v, v').$$

Moreover, we study the robustness of the Cartesian product after an edge/vertex removal :

- the diameter does not change,
- the distance between pair of vertices cannot increase by more than 2,
- the number of vertex-disjoint, resp. edge-disjoint, shortest paths between pair of vertices mainly cannot decrease by more than 1.

Due to these properties, Cartesian product of graphs is interesting for the purpose of data center conception for which we need graphs with a large number of vertices, moderate average distance, bounded maximum degree and robustness.

Some deeper analysis of Cartesian product from the robustness point of view shows that one of its most interesting feature is the presence of a large number of C_4 (from the product of each edge of G with each edge of H). A special case is when each induced P_3 belongs to a C_4. In this special case, removing any edge/vertex does not change the geodesic distance between any pair of other vertices. Notice that if the presence of C_4 is related to robustness, the minimization of the average distance with bounded maximum degree implies to avoid C_4 as soon as the graph has a minimum number of vertices.

Given these two properties, the search of robust graphs with bounded maximum degree and minimum average distance implies some compromise. The search of efficient networks thus becomes a multi objective problem in which robustness is maximized while the average distance is minimized upon graphs with bounded maximum degree.

Starting from Cartesian product of graphs, this talk explores ways to construct graphs with bounded maximum degree that have a small average distance and for which the impact of any edge/vertex removal upon the geodesic distances is limited.

Out of curiosity, by using the Variable Neighborhood Search with the computer system AutoGraphiX [1, 2], we find a candidate on 64 vertices of maximum degree 6 and average distance 2.35, which is quite good since the theoretical bound is 2.33. However, its structure is not simple and this graphs has no regularity (except the degree). Therefore, it is not usable in practice and it does not respect the robustness constraint. Based on the needed properties, other candidates were constructed with slightly larger average distance but a better structure and robustness.

Bibliography

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