A new characterization of P_k-free graphs

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Abstract. The class of graphs that do not contain an induced path on k vertices, P_k-free graphs, plays a prominent role in algorithmic graph theory. This motivates the search for special structural properties of P_k-free graphs, including alternative characterizations.
Let G be a connected P_k-free graph, $k \geq 4$. We show that G admits a connected dominating set whose induced subgraph is either P_{k-2}-free, or isomorphic to P_{k-2}. Surprisingly, it turns out that every minimum connected dominating set of G has this property.
This yields a new characterization for P_k-free graphs: a graph G is P_k-free if and only if each connected induced subgraph of G has a connected dominating set whose induced subgraph is either P_{k-2}-free, or isomorphic to C_k. This improves and generalizes several previous results; the particular case of $k = 7$ solves a problem posed by van 't Hof and Paulusma [A new characterization of P_6-free graphs, COCOON 2008].
In the second part of the paper, we present an efficient algorithm that, given a connected graph G, computes a connected dominating set X of G with the following property: for the minimum k such that G is P_k-free, the subgraph induced by X is P_{k-2}-free or isomorphic to P_{k-2}.
As an application our results, we prove that Hypergraph 2-Colorability, an NP-complete problem in general, can be solved in polynomial time for hypergraphs whose vertex-hyperedge incidence graph is P_7-free.
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1 Introduction

A *dominating set* of a graph G is a vertex subset X such that every vertex not in X has a neighbor in X. Dominating sets have been intensively studied in the literature. The main interest in dominating sets is due to their relevance on both theoretical and practical side. Moreover, there are interesting variants of domination and many of them are well-studied.

A *connected dominating set* of a graph G is a dominating set X whose induced subgraph, henceforth denoted $G[X]$, is connected. As usual, a connected dominating set such that every proper subset is not a connected dominating set is called a *minimal connected dominating set.* A connected dominating set of minimum size is called a *minimum connected dominating set.*

We use the following standard notation. Let P_k be the induced path on k vertices and let C_k be the induced cycle on k vertices. If G and H are two graphs, we say that G is *H-free* if H does not appear as an induced subgraph of G. Furthermore, if G is H_1-free and H_2-free for some graphs H_1 and H_2, we say that G is *(H_1, H_2)-free.* If two graphs G and H are isomorphic, we write $G \cong H$.

The class of P_k-free graphs has received a fair amount of attention in the theory of graph algorithms. Given an NP-hard optimization problem, it is often fruitful to study its complexity when the instances are restricted to P_k-free graphs.

Let us mention two recent results in this direction: the polynomial time algorithm to compute a stable set of maximum weight, given by Lokshtanov *et al.* [10], and the result of Hoang *et al.* [6] showing that k-COLORABILITY is efficiently solvable on P_5-free graphs. The proof of the latter result relies on the fact that a connected P_5-free graph has a dominating clique or a dominating P_3.

Theorem 1 (Bácso and Tuza [1]). *Let G be a connected P_5-free graph. Then G has a dominating clique or a dominating induced P_3.*

An immediate implication of this result is the following.

Theorem 2 (Bácso and Tuza [1], Cozzens and Kelleher [4]). *Let G be a graph. The following assertions are equivalent.*

(i) G is P_5-free.
(ii) Every induced subgraph H of G admits a connected dominating set X such that $H[X]$ is a clique or $H[X] \cong C_5$.

Later, van 't Hof and Paulusma [13] obtained a characterization for the class of P_6-free graphs in the flavour of Theorem 2. An earlier, slightly weaker result was given by Liu *et al.* [8], and the particular case of triangle free graphs was discussed before by Liu and Zhou [9].

Theorem 3 (van 't Hof and Paulusma [13]). *Let G be a graph. The following assertions are equivalent.*

(i) G is P_6-free.
(ii) Every connected induced subgraph H of G admits a connected dominating set X such that $H[X]$ has a complete bipartite spanning subgraph or $H[X] \cong C_6$.

Complementing Theorem 3, van 't Hof and Paulusma give a polynomial time algorithm that, given a connected P_6-free graph, computes a connected dominating set X such that $G[X]$ has a complete bipartite spanning subgraph or $G[X] \cong C_6$.

In view of Theorems 2 and 3, two questions arise. The first one is whether condition (ii) of Theorem 3 can be tightened, such that $H[X]$ is a P_4-free graph or $G[X] \cong C_6$. Note that if $H[X]$ is P_4-free, it is a connected cograph, and in particular has a complete bipartite spanning subgraph. This condition is the direct analogue of condition (ii) of Theorem 2 for P_6-free graphs. The advantage of the strenghtened version is of course that the structure of cographs is well understood and more restricted compared to the class of graphs having a spanning complete bipartite graph.

The second question is whether similar characterizations can be given for the class of P_k-free graphs, for $k > 6$. In their paper, van 't Hof and Paulusma [13] explicitly ask for such a characterization in the case of $k = 7$.

1.1 Our contribution

In this paper, we give an affirmative answer to these two questions. We show that every connected P_k-free graph, $k \geq 4$, admits a connected dominating set whose induced subgraph is either P_{k-2}-free, or isomorphic to P_{k-2}. Surprisingly, it turns out that every minimum connected dominating set has this property.

Theorem 4. *Let G be a connected P_k-free graph, $k \geq 4$, and let X be any minimum connected dominating set of G. Then $G[X]$ is P_{k-2}-free, or $G[X] \cong P_{k-2}$.*

From this result we derive the following characterization of P_k-free graphs.

Theorem 5. *Let G be a graph and $k \geq 4$. The following assertions are equivalent.*

(i) G is P_k-free.
(ii) Every connected induced subgraph H of G admits a connected dominating set X such that $H[X]$ is P_{k-2}-free or $H[X] \cong C_k$.

We now come to the algorithmic dimension of the problem. The proof of Theorem 4 is constructive in the sense that it yields an algorithm to compute, given a P_k-free graph, a connected dominating set whose induced subgraph is either P_{k-2}-free, or isomorphic to P_{k-2}. However, recall that the computation of a longest induced path in a graph is an NP-hard problem, as shown in Garey and Johnson [5, p. 196]. In other words, there is little hope of computing in polynomial time the minimum k for which the input graph is P_k-free. To overcome this obstacle, our algorithm can only make implicite use of the absent induced P_k, which is the main difficulty here.

Theorem 6. *Given a connected graph G on n vertices and m edges, one can compute in time $\mathcal{O}(n^5(n+m))$ a connected dominating set X with the following property: for the minimum $k \geq 3$ such that G is P_k-free, $G[X]$ is P_{k-2}-free or $G[X] \cong P_{k-2}$.*

Our last result is an application of the previous theorems. A *2-coloring* of a hypergraph assigns to each vertex one of two colors, such that each hyperedge contains vertices of both colors. The problem Hypergraph 2-Colorability is to decide whether a given hypergraph admits a 2-coloring. Garey and Johnson [5, p. 221] explain that it is NP-complete in general. One successful approach to deal with this hardness is to put restrictions on the bipartite vertex-hyperedge incidence graph[3] of the input hypergraph.

As an application of Theorem 3, van 't Hof and Paulusma [13] show that Hypergraph 2-Colorability is solvable in polynomial time for hypergraphs with P_6-free incidence graph. Using our results, we settle the case of hypergraphs with P_7-free incidence graph.

Theorem 7. Hypergraph 2-Colorability *can be solved in polynomial time for hypergraphs with P_7-free incidence graph. If it exists, a 2-coloring can be computed in polynomial time.*

The proof of Theorems 4, 5 and 6 we give in the next section. Due to space limitations, the proof of Theorem 7 is omitted. We close the paper with a short discussion of our contribution.

2 Proofs

2.1 Proof of Theorems 4 and 5

We need the following lemma from an earlier paper of ours [3].

Lemma 1 (Camby and Schaudt [3]). *Let G be a connected graph that is (P_k, C_k)-free, for some $k \geq 4$, and let X be a minimal connected dominating set of G. Then $G[X]$ is P_{k-2}-free.*

When applied to P_k-free graphs, which are in particular (P_{k+1}, C_{k+1})-free, the above lemma implies that any minimal connected dominating set induces a P_{k-1}-free graph, for $k \geq 3$. We next prove a simple but useful lemma, which plays a key role also in the proof of Theorem 6. Let X be a connected dominating set of a graph G, and $x \in X$. Assuming that X is a minimal connected dominating set and $|X| \geq 2$, x is a cut-vertex of $G[X]$ or x has a *private neighbor*: a vertex $y \in V(G) \setminus X$ with $N_G(y) \cap X = \{x\}$.

Lemma 2. *Let G be a P_k-free graph, for some $k \geq 4$, and let X be a minimal connected dominating set of G. Assume that there is an induced P_{k-2} in $G[X]$, say on the vertices $x_1, x_2, \ldots, x_{k-2}$. Then any private neighbor y of x_1 is such that $(X \cup \{y\}) \setminus \{x_{k-2}\}$ is a connected dominating set of G.*

Proof. Note that G is in particular (P_{k+1}, C_{k+1})-free and thus, by Lemma 1, $G[X]$ is P_{k-1}-free.

[3] Recall that for a hypergraph $H = (V, E)$ we define the bipartite vertex-hyperedge incidence graph as the bipartite graph on the set of vertices $V \cup E$ with the edges vY such that $v \in V$, $Y \in E$ and $v \in Y$. In the following, we just say the *incidence graph*.

Let $X' := \{x_1, x_2, \ldots, x_{k-2}\}$. Moreover, let y be any private neighbor of x_1, and let $Y := (X \cup \{y\}) \setminus \{x_{k-2}\}$. We have to prove that Y is a connected dominating set of G.

Suppose for a contradiction that $G[Y]$ is not connected. Hence, x_{k-2} is a cut-vertex of $G[X]$. In particular, there is some vertex $y' \in X$ such that $N_G(y') \cap X' = \{x_{k-2}\}$. But then $G[X' \cup \{y'\}] \cong P_{k-1}$, a contradiction.

It remains to show that Y is a dominating set. Suppose the contrary, that is, there is some vertex x' with $N_G[x'] \cap Y = \emptyset$. As X is a dominating set, $N_G[x'] \cap X = \{x_{k-2}\}$. Because x_{k-2} is adjacent to Y and x' is not adjacent to Y, $x' \neq x_{k-2}$. But this means that $G[X' \cup \{y, x'\}] \cong P_k$, a contradiction. □

Now we can state the proof of Theorem 4.

Proof (Proof of Theorem 4.). Let X be a minimum connected dominating set of G. As G is in particular (P_{k+1}, C_{k+1})-free, $G[X]$ is P_{k-1}-free, by Lemma 1. We have to show that $G[X]$ is P_{k-2}-free or isomorphic to P_{k-2}.

To see this, assume there is an induced P_{k-2} in $G[X]$, say on the vertices $x_1, x_2, \ldots, x_{k-2}$. Let $X' := \{x_1, x_2, \ldots, x_{k-2}\}$. Note that x_1 is not a cut-vertex of $G[X]$: otherwise there is some vertex $y' \in X$ such that $N_G(y') \cap X' = \{x_1\}$, and hence $G[X' \cup \{y'\}] \cong P_{k-1}$. This is a contradiction. Thus, x_1 is not a cut-vertex of $G[X]$ and therefore has a private neighbor w.r.t. X, say y_1. By Lemma 2, $Y_1 := (X \cup \{y_1\}) \setminus \{x_{k-2}\}$ is a connected dominating set of G. As X is a minimum connected dominating set, Y_1 is a minimum connected dominating set, too. Moreover, y_1 has no neighbor in $X \setminus \{x_1\}$, in particular in $X \setminus X'$.

By reapplying the argumentation to Y_1 and the induced P_{k-2} on $y_1, x_1, x_2, \ldots,$ x_{k-3}, We obtain a vertex $y_2 \in V(G) \setminus Y_1$ such that $Y_2 := (Y_1 \cup \{y_2\}) \setminus \{x_{k-3}\}$ is a minimum connected dominating set of G and $G[Y_2]$ contains an induced P_{k-2} on the vertices $y_2, y_1, x_1, x_2, \ldots, x_{k-4}$. Moreover, y_2 has no neighbor in $Y_1 \setminus \{y_1\}$, in particular in $X \setminus X'$.

Iteratively, we end up with a minimum connected dominating set Y_{k-2}, which is exactly $(X \setminus X') \cup \{y_1, \ldots, y_{k-2}\}$. Since, for $i = 1, 2, \ldots, k-2$, y_i is not adjacent to $X \setminus X'$ and $G[Y_{k-2}]$ is connected, $X \setminus X'$ must be empty, hence $X = X'$. Thus, $G[X] = G[X'] \cong P_{k-2}$. This completes the proof. □

Proof (Proof of Theorem 5.). Clearly P_k does not have a connected dominating set satisfying (ii). Hence, (ii) implies (i).

Conversely, let H be any connected induced subgraph of G, and let X be a minimum connected dominating set of H. By Theorem 4, $H[X]$ is P_{k-2}-free or $H[X] \cong P_{k-2}$. If $H[X]$ is P_{k-2}-free, the assertion of (ii) is satisfied. Otherwise, let $x_1, x_2, \ldots, x_{k-2}$ be a consecutive ordering of the induced path $H[X]$. In particular, x_1 and x_{k-2} are not cut-vertices of $H[X]$. As X is minimum, there exists a private neighbor y_i of x_i, for $i \in \{1, k-2\}$. It must be that $y_1 y_{k-2} \in E(H)$, since otherwise $H[X \cup \{y_1, y_{k-2}\}] \cong P_k$. Hence, $H[X \cup \{y_1, y_{k-2}\}] \cong C_k$, as desired. So, (i) implies (ii). □

2.2 Proof of Theorem 6

Before we state our algorithm, we need to introduce some notation and definitions. For this, let us assume we are given a connected input graph G on n vertices and m edges. Let X be an arbitrary connected dominating set of G.

By $NC(X)$ we denote the set of vertices in X that are non-cutting in $G[X]$, i.e. for every $x \in NC(X)$, $G[X \setminus \{x\}]$ is connected. Let x be a degree-1 vertex of $G[X]$. We define the *half-path* starting in x to be the maximal path $(x, x_1, x_2, \ldots, x_s)$ in X such that $|N_{G[X]}(x_i)| = 2$ for each $i \in \{1, 2, \ldots, s-1\}$. For example, if the neighbor $y \in X$ of x has degree at least 3, the half-path is simply (x, y). The *length* of the half-path is then s. To each $x \in X$ we assign a weight $w_X(x)$ as follows:

1. if $|N_{G[X]}(x)| \geq 2$, put $w_X(x) = 0$, and
2. if $|N_{G[X]}(x)| = 1$, put $w_X(x) = s$, where s is the length of the half-path starting in x.

Finally, the weight $w(X)$ of the set X given by

$$w(X) = \sum_{x \in X} (w_X(x))^2.$$

See Fig. 1 for an illustration of these definitions.

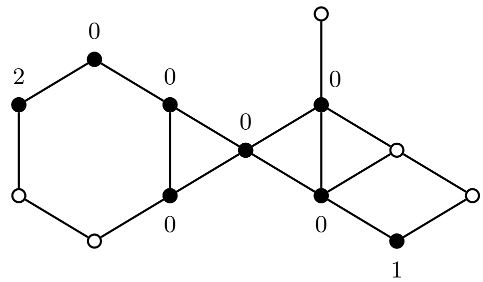

Fig. 1. A graph G. The black vertices form a connected dominating set X of G, with weights w_X as shown. We have $w(X) = 5$.

Let $\mathcal{X}$ be the family of all connected dominating sets of G. We next define a strict partial order $\prec$ on $\mathcal{X}$ as follows. For any two sets $X, Y \in \mathcal{X}$, we put $X \prec Y$ if

1. $|X| > |Y|$, or
2. $|X| = |Y|$ and $w(X) < w(Y)$.

The *height* of the strict poset $(\mathcal{X}, \prec)$ is the maximum set of mutually comparable elements of $\mathcal{X}$.

Lemma 3. *For a connected n-vertex graph G, the height of $(\mathcal{X}, \prec)$ is in $\mathcal{O}(n^3)$.*

Proof. If $G[X]$ is not an induced path, every vertex in X of degree at most 2 in $G[X]$ is contained in at most one half-path. Hence, $\sum_{x\in X} w_X(x) \leq |X|$. If $G[X]$ is an induced path, every vertex appears in at most two half-paths, implying $\sum_{x\in X} w_X(x) \leq 2|X|$. Thus

$$w(X) = \sum_{x\in X} (w_X(x))^2 \leq (\sum_{x\in X} w_X(x))^2 \leq 4|X|^2,$$

and so the weight of a connected dominating set is in $\mathcal{O}(n^2)$. Since there are at most n different possible sizes of connected dominating sets of G, the height of $(\mathcal{X}, \prec)$ is in $\mathcal{O}(n^3)$. $\square$

Proof (Proof of Theorem 6). Assume we are given a connected graph G on n vertices and m edges as input. Our algorithm works as follows, starting with the connected dominating set $Y := V(G)$. Its output is a connected dominating set X with the properties stated in Theorem 6.

1. Compute a minimal connected dominating set $X \subseteq Y$.
2. If $G[X]$ is an induced path, return X and terminate the algorithm.
3. Compute the set $NC(X)$ and the weight $w_X(x)$ for every $x \in NC(X)$.
4. Order the vertices of $NC(X)$ with non-increasing weight w_X, breaking ties arbitrarily. Let that order be $v_1, v_2, \ldots, v_{|NC(X)|}$.
5. For i from 1 to $|NC(X)|$ do the following:
   (a) Compute a private neighbor y_i of v_i w.r.t. X.
   (b) For j from $i+1$ to $|NC(X)|$ do the following:
      i. Check whether $Y_{ij} := (X \cup \{y_i\}) \setminus \{v_j\}$ is a connected dominating set.
      ii. If yes, put $X \leftarrow Y_{ij}$ and go to Step 1.
6. Return X and terminate the algorithm.

We remark that the computation of y_i in Step 5a is always possible, since x_i is non-cutting in $G[X]$ and X is a minimal connected dominating set. The proof is completed by the following sequence of claims.

Claim 1. *When the algorithm terminates, the output X is a connected dominating set and $G[X]$ is P_{k-2}-free or $G[X] \cong P_{k-2}$.*

Since Step 1 is applied before the return is called, X is a minimal connected dominating set. If the algorithm terminates with Step 2, $G[X]$ is P_{k-1}-free by Lemma 1. Hence, either $G[X] \cong P_{k-2}$ or $G[X]$ is P_{k-2}-free.

Now assume that the algorithm terminates in Step 6. In particular, $G[X]$ is not an induced path. Suppose for a contradiction that $G[X]$ contains an induced P_{k-2}, say on the vertices $x_1, x_2, \ldots, x_{k-2}$. Like in the proof of Lemma 2, both x_1 and x_{k-2} cannot be cut-vertices of $G[X]$. Thus, $x_1, x_{k-2} \in NC(X)$.

After Step 4, the vertices of $NC(X)$ are ordered $v_1, v_2, \ldots, v_{|NC(X)|}$ with non-increasing weight. W.l.o.g. $x_1 = v_i$, $x_{k-2} = v_j$, and $i < j$. As X is returned, the set $Y_{ij} := (X \cup \{y_i\}) \setminus \{v_j\}$ is not a connected dominating set, in contradiction to Lemma 2. This proves our claim.

Claim 2. *Let X be a minimal connected dominating set considered in some iteration of the algorithm. Assume that the 'go to' is called in Step 5(b)ii because $Y_{ij} := (X \cup \{y_i\}) \setminus \{v_j\}$ is a connected dominating set. Let X' be the minimal connected dominating set computed in the subsequent Step 1. Then $X \prec X'$.*

Clearly $|X'| \leq |X|$. If $|X'| < |X|$, $X \prec X'$ by definition. So we may assume that $|X'| = |X|$, and hence $X' = Y_{ij}$. It remains to show that $w(X) < w(X')$.

Let $z \in X \setminus \{v_i, v_j\}$ be a degree-1 vertex of $G[X]$, and let $(z, x_1, x_2, \ldots, x_s)$ be a half-path starting in z. As $G[X]$ is not a path, x_s is a cut-vertex of $G[X]$. In particular, $x_s \neq v_j$. Hence, in $G[Y_{ij}]$, $(z, x_1, x_2, \ldots, x_s)$ is the initial segment of a half-path starting in z. In particular, $w_{X'}(z) \geq w_X(z)$.

If v_i is not a degree-1 vertex of $G[X]$, $w_{X'}(v_i) = w_X(v_i) = 0$, and (y_i, v_i) is the initial segment of a half-path starting in y_i. Hence, $w_{X'}(y_i) \geq 1$, and thus

$$w_{X'}(v_i) = 0 \text{ and } w_{X'}(y_i) \geq w_X(v_i) + 1. \tag{1}$$

If the degree of v_i in $G[X]$ is 1, let $(v_i, x_1, x_2, \ldots, x_s)$ be a half-path starting in v_i. Again, x_s is a cut-vertex of $G[X]$, and so $x_s \neq v_j$. Hence, in $G[X']$, $(y_i, v_i, x_1, x_2, \ldots, x_s)$ is the initial segment of a half-path starting in y_i. Again (1) holds.

Summing up, we see that (1) holds, and

$$w_{X'}(z) \geq w_X(z) \text{ for every vertex } z \in X' \setminus \{y_i, v_i\}. \tag{2}$$

We now turn to the vertex v_j. First assume that the degree of v_j in $G[X]$ is at least 2, and thus $w_X(v_j) = 0$. Then, by (2),

$$w(X') - w(X) \geq w_{X'}(y)^2 - w_X(v_j)^2 > 0,$$

and so $w(X') - w(X) > 0$.

Now assume that v_j is a vertex of degree 1 in $G[X]$, and so $w_X(v_j) \geq 1$. Let $N_{G[X]}(v_j) = \{x\}$. As $G[X]$ is not a path, $|N_{G[X]}(x)| \geq 2$, and so $w_X(x) = 0$. Thus $w_{X'}(x) = w_X(v_j) - 1$. Recall that (2) holds, and $w_{X'}(z) \geq w_X(z)$ for every vertex $z \in X' \setminus \{y_i, v_i\}$. We obtain the following inequality.

$$\begin{aligned} w(X') - w(X) &\geq w_{X'}(y_i)^2 + w_{X'}(x)^2 - w_X(v_i)^2 - w_X(v_j)^2 \\ &= (w_{X'}(y_i)^2 - w_X(v_i)^2) - (w_X(v_j)^2 - w_{X'}(x)^2) \\ &\geq [(w_X(v_i) + 1)^2 - w_X(v_i)^2] - [w_X(v_j)^2 - (w_X(v_j) - 1)^2] \end{aligned}$$

But $w_X(v_i) \geq w_X(v_j)$ implies

$$(w_X(v_i) + 1)^2 - w_X(v_i)^2 > w_X(v_j)^2 - (w_X(v_j) - 1)^2,$$

and thus $w(X') - w(X) > 0$ holds as in the previous case.

Hence, $X \prec X'$, proving our claim.

See Fig. 2 for an illustration of Step 5(b)ii.

Claim 3. *The algorithm terminates in $\mathcal{O}(n^5(n + m))$ time.*

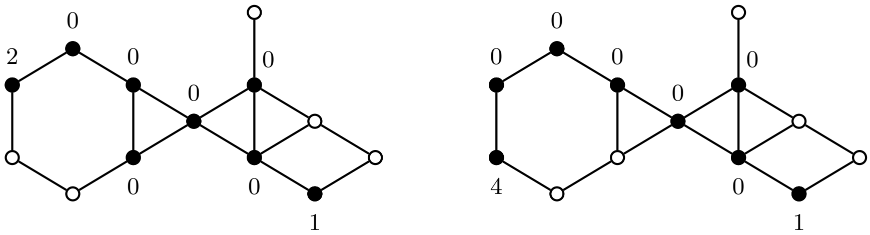

Fig. 2. Before (left) and after (right) an application of Step 5(b)ii. In the next iteration, the algorithm terminates with the right connected dominating set as output.

By Claim 2, each call of the 'go to'-step and the subsequent application of Step 1 result in a connected dominating set that is properly larger in the order $\prec$. By Lemma 3, the height of the poset $(\mathcal{X}, \prec)$, and hence the number of iterations the whole algorithm performs, is in $\mathcal{O}(n^3)$.

It remains to discuss the complexity of the particular steps. For this, recall that it can be checked in time $\mathcal{O}(n+m)$ whether a given vertex subset is a connected dominating set. Consequently, Step 1 can be performed in time $\mathcal{O}(n(n+m))$ by the immediate greedy procedure.

Step 2 and the computation of the weights in Step 3 can both be performed in linear time using the degree sequence of $G[X]$. The computation of the set $NC(X)$ in Step 3 can be done straightforwardly in time $\mathcal{O}(n(n+m))$.

It remains to discuss the complexity of the loop of Step 5. The computation of a private neighbor in Step 5a is clearly done in $\mathcal{O}(n+m)$ time. The inner loop of Step 5b consumes $\mathcal{O}(n)$ checks whether some vertex set is a connected dominating set, requiring $\mathcal{O}(n+m)$ time each. Hence, Step 5 can be done in $\mathcal{O}(n^2(n+m))$ time.

The overall running time computes to $\mathcal{O}(n^5(n+m))$, which completes the proof of both our claim and Theorem 6. $\square$

3 Conclusion

In this paper we gave a description of the structure of connected dominating sets in P_k-free graphs. We have shown that any connected P_k-free graph admits a connected dominating set whose induced subgraph is P_{k-2}-free or isomorphic to P_{k-2}. In fact, any minimum connected dominating set has this property. Loosely speaking, this means that the restricted structure of connected P_k-free graphs results in an even more restricted structure of the induced subgraph of their minimum connected dominating sets.

Although we think that our results are of their own interest, our hope is that they might be useful in other contexts, too. One example we gave is the polynomial time solvability of Hypergraph 2-Colorability for hypergraphs with P_7-free incidence graph. It seems possible that, with more work, one could push this result to hypergraphs with P_8-free incidence graph. However, more interesting would be to know whether there is any k for which Hypergraph

2-Colorability for hypergraphs with P_k-free incidence graph is *not* solvable in polynomial time. So far, we do not have an opinion or an intelligent guess on this question.

Other possible future applications of our results include the coloring of P_k-free graphs. As mentioned earlier, Hoang *et al.* [6] showed that k-Colorability is efficiently solvable on P_5-free graphs, using the fact that a connected P_5-free graph has a dominating clique or a dominating induced P_3. To our knowledge, an open problem, conjectured by Huang [7], in this context is whether 4-colorability can be decided in polynomial time for P_6-free graphs. From Theorem 6 it follows that, given a P_6-free graph, we can efficiently compute a connected dominating set that induces a P_4-free graph (that is a cograph) or a P_4. Of course cographs are less trivial than cliques, especially when it comes to coloring – but that does not rule out an approach similar to that of Hoáng *et al.* [6]. The fact that each vertex of the graph has some neighbor in this cograph leaves a 3-coloring problem for the rest of the graph, once the coloring of the cograph is fixed. Here, one might use the fact that 3-coloring is polynomial time solvable for P_6-free graphs, shown by Randerath and Schiermeyer [11], even in the pre-coloring extension version, proven by Broersma *et al.* [2].

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