The price of connectivity for dominating set: upper bounds and complexity

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Abstract

In the first part of this paper, we investigate the interdependence of the connected domination number $\gamma_c(G)$ and the domination number $\gamma(G)$ in some hereditary graph classes. We prove the following results:

- A connected graph G is (P_6, C_6)-free if and only if $\gamma_c(H) \leqslant \gamma(H)+1$ for every connected induced subgraph H of G. Moreover, there are (P_6, C_6)-free graphs with arbitrarily large domination number attaining this bound.
- For every connected (P_8, C_8)-free graph G, it holds that $\gamma_c(G)/\gamma(G) \leqslant 2$, and this bound is attained by connected (P_7, C_7)-free graphs with arbitrarily large domination number. In particular, the bound $\gamma_c(G) \leqslant 2\gamma(G)$ is best possible even in the class of connected (P_7, C_7)-free graphs.
- The general upper bound of $\gamma_c(G)/\gamma(G) < 3$ is asymptotically sharp on connected (P_9, C_9)-free graphs.

In the second part, we prove that the following decision problem is Θ_2^p-complete, for every fixed rational $1 < r < 3$: Given a graph G, is $\gamma_c(G)/\gamma(G) \leqslant r$? Loosely speaking, this means that deciding whether the ratio of $\gamma_c(G)$ and $\gamma(G)$ is bounded by some rational number r with $1 < r < 3$ is as hard as computing both $\gamma_c(G)$ and $\gamma(G)$ explicitly.

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1 Introduction

A *dominating set* of a graph G is a vertex subset X such that every vertex not in X has a neighbor in X. The minimum size of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is called a *minimum dominating set.*

Dominating sets have been intensively studied in the literature. The main interest in dominating sets is that they are relevant both theoretically and practically. Moreover, there are interesting variants of domination and many of them are well-studied. A good introduction into the topic is given by Haynes, Hedetniemi and Slater [8].

Let G be a connected graph. A *connected dominating set* of G is a dominating set X of G that induces a connected subgraph. A connected dominating set of G is called *minimal* if none of its proper subsets is a connected dominating set of G, and *minimum* if G has no connected dominating set of smaller size. The size of a minimum connected dominating set of G is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. Among the applications of connected dominating sets is the routing of messages in mobile ad-hoc networks. Blum, Ding, Thaeler and Cheng [2] explain the usefulness of connected dominating sets in this context.

Duchet and Meyniel [5] observed that for every connected graph G it holds that $\gamma_c(G) \leqslant 3\gamma(G) - 2$. As an immediate consequence, every connected graph G satisfies

$$\gamma_c(G)/\gamma(G) < 3. \tag{1}$$

Loosely speaking, the price of connectivity for dominating set of a graph G, $\gamma_c(G)/\gamma(G)$, is strictly bounded by 3.

Let P_k be the induced path on k vertices and let C_k be the induced cycle on k vertices.

Observation 1. *It holds that*

$$\lim_{k\to\infty} \gamma_c(P_k)/\gamma(P_k) = 3 = \lim_{k\to\infty} \gamma_c(C_k)/\gamma(C_k). \tag{2}$$

In particular, the upper bound (1) is asymptotically sharp in the class of paths and in the class of cycles.

The price of connectivity has been introduced by Cardinal and Levy [4, 10] for the vertex cover problem. They showed that it is bounded by $2/(1 + \varepsilon)$ in graphs with average degree εn, where n is the number of vertices. In a companion paper to the present paper, the price of connectivity for vertex cover is studied by Camby, Cardinal, Fiorini and Schaudt [3]. Belmonte, van 't Hof, Kamiński and Paulusma [1] study the price of connectivity for feedback vertex set in hereditary graph classes. In a similar spirit, Schaudt [12] studied the ratio between the connected domination number and the total domination number. Fulman [6] and Zverovich [16] investigated the ratio between the independence number and the upper domination number, the latter being the maximum size of a minimal dominating set. Many results in this area concern graph classes defined by forbidden induced subgraphs. This line of research stems from the classical theory of perfect graphs, for which the clique number and the chromatic number are equal in every induced subgraph [7].

To present our results, we use the following standard notation. If G and H are two graphs, we say that G is *H-free* if H does not appear as an induced subgraph of G. Furthermore, if G is H_1-free and H_2-free for some graphs H_1 and H_2, we say that G is *(H_1, H_2)-free.* Besides, if X is a vertex set of the graph G, we say that $G[X]$ is the subgraph induced by X in G. Our starting point is the following result by Zverovich [15].

Theorem 1 (Zverovich [15]). *The following assertions are equivalent for every graph G.*

(i) For every connected induced subgraph H of G it holds that $\gamma_c(H) = \gamma(H)$.

(ii) G is (P_5, C_5)-free.

In the first part of this paper, we aim for similar bounds in the class of connected (P_k, C_k)-free graphs for $k \geqslant 6$. The properties of connected dominating sets in P_6-free graphs have been studied before, e.g. by Liu, Peng and Zhao [11] and later van 't Hof and Paulusma [9]. However, since $\gamma_c(C_k) = \gamma_c(P_k)$ and $\gamma(C_k) = \gamma(P_k)$ for every $k \geqslant 3$, it seems reasonable to forbid C_k and P_k.

We prove the following.

- A connected graph G is (P_6, C_6)-free if and only if $\gamma_c(H) \leqslant \gamma(H) + 1$ for every connected induced subgraph H of G. Moreover, there is an infinite family of (P_6, C_6)-free graphs with arbitrarily large values of $\gamma(G)$ attaining this bound.
- For every connected (P_8, C_8)-free graph G, it holds that $\gamma_c(G)/\gamma(G) \leqslant 2$, and this bound is attained by connected (P_7, C_7)-free graphs with arbitrarily large values of γ. In particular, the bound $\gamma_c(G) \leqslant 2\gamma(G)$ is best possible even in the class of connected (P_7, C_7)-free graphs.
- The general upper bound of $\gamma_c(G)/\gamma(G) < 3$ is asymptotically sharp on connected (P_9, C_9)-free graphs.

In the second part of this paper, we study the computational complexity of the price of connectivity. The class $\Theta_2^P = P^{\mathrm{NP}[\log]}$ is defined as the class of decision problems solvable in polynomial time by a deterministic Turing machine that is allowed use $\mathcal{O}(\log n)$ many queries to an NP-oracle, where n is the size of the input. We prove that the following decision problem is Θ_2^p-complete, for every fixed rational $1 < r < 3$: Given a graph G, is $\gamma_c(G)/\gamma(G) \leqslant r$? Loosely speaking, this means that deciding whether the ratio of $\gamma_c(G)$ and $\gamma(G)$ is bounded by some rational number r with $1 < r < 3$ is as hard as computing both $\gamma_c(G)$ and $\gamma(G)$ explicitly. And this remains true even if r is not part of the input.

The proof of this complexity result is similar to the analogous result for vertex cover which the authors of the present paper, together with Jean Cardinal and Samuel Fiorini, developed in [3].

2 Upper bounds

We will use the following lemma several times. Note that this lemma is concerned with minimal connected dominating sets.

Lemma 1. *Let G be a connected graph that is (P_k, C_k)-free for some $k \geqslant 4$ and let X be a minimal connected dominating set of G. Then $G[X]$ is P_{k-2}-free.*

Proof. Suppose that there is an induced path $(v_1, v_2, \ldots, v_{k-2})$ on $k-2$ vertices in $G[X]$. As X is minimal, $X \setminus \{v_1\}$ is not a connected dominating set. Hence, $X \setminus \{v_1\}$ is not a dominating set or $G[X \setminus \{v_1\}]$ is disconnected. In the first case, there is a vertex $v'_1 \in V \setminus X$ whose only neighbor in X is v_1. In the second case, the vertices $v_2, \ldots, v_{k-2}$ are contained in a single connected component of $G[X \setminus \{v_1\}]$. Thus, there is a neighbor of v_1 in X, say v'_1, that is not adjacent to any member of $\{v_2, \ldots, v_{k-2}\}$. In both cases, there is a vertex $v'_1 \notin \{v_1, v_2, \ldots, v_{k-2}\}$ whose only neighbor among $\{v_1, v_2, \ldots, v_{k-2}\}$ is v_1. Similarly, there is a vertex $v'_{k-2} \notin \{v_1, v_2, \ldots, v_{k-2}\}$ whose only neighbor among $\{v_1, v_2, \ldots, v_{k-2}\}$ is v_{k-2}. But then $G[\{v'_1, v_1, v_2, \ldots, v_{k-2}, v'_{k-2}\}]$ is isomorphic to P_k or to C_k, depending on the adjacency of v'_1 and v'_{k-2}. This is a contradiction to the choice of G. □

For a graph G and $v \in V(G)$ we denote by $N_G[v]$ the closed neighborhood of v in G. Our first result establishes the upper bound $\gamma_c(G) \leqslant \gamma(G) + 1$ for a connected (P_6, C_6)-free graph G.

Theorem 2. *For every connected graph G, the following assertions are equivalent:*

(a) For every connected induced subgraph H of G it holds that $\gamma_c(H) \leqslant \gamma(H)+1$.

(b) G is (P_6, C_6)-free.

Proof. It is straightforward to see that $\gamma_c(P_6) = \gamma_c(C_6) = 4$ and $\gamma(P_6) = \gamma(C_6) = 2$. Thus, (a) implies (b).

Now let $G = (V, E)$ be a connected (P_6, C_6)-free graph and let H be a connected induced subgraph of G. Observe that H is (P_6, C_6)-free. To see that (b) implies (a), it suffices to prove that $\gamma_c(H) \leqslant \gamma(H) + 1$. For this, let D be a minimum dominating set of H. Let $D_1, D_2, \ldots, D_k$ be the vertex sets of the connected components of $H[D]$. Let $C \subseteq V$ be an inclusionwise minimal set such that $H[D \cup C]$ is connected, and let $X \subseteq D \cup C$ be a minimal connected dominating set of H. By Lemma 1, $H[X]$ is P_4-free.

Let $I \subseteq \{1, 2, \ldots, k\}$ be such that $i \in I$ if and only if $D_i \cap X = \emptyset$. For every $i \in I$, pick $x_i \in X$ such that x_i has a neighbor in D_i (this is always possible, since X is a dominating set). Note that every x_i belongs to C, and that the x_i do not have to be distinct. Let $S = \bigcup_{i \notin I}(D_i \cap X) \cup \{x_j : j \in I\}$.

Assume first that $H[S]$ is connected. Then $H[D \cup \{x_i : i \in I\}]$ is connected, and so $C = \{x_i : i \in I\}$, since C was chosen minimal such that $H[D \cup C]$ is connected. Thus, $X = S$, which gives

$$\gamma_c(H) \leqslant |X| = |S| \leqslant \sum_{i \notin I} |D_i \cap X| + |I| \leqslant |D| = \gamma(H).$$

Now assume that $H[S]$ is not connected. Among other authors, Seinsche [13] proved that every P_4-free graph with at least two vertices is either disconnected, or its complement is disconnected. In particular, this applies to $H[X]$. Since the complement of $H[S]$ is connected, but the complement of $H[X]$, $\overline{H[X]}$, is disconnected, the graph $\overline{H[X]}$ has a connected component containing S and at least one other connected component Y. In H, every vertex in $V(Y)$ is adjacent to every vertex in S. Let $y \in V(Y)$. Then $H[D \cup \{x_i : i \in I\} \cup \{y\}]$ is connected, and so $C = \{x_i : i \in I\} \cup \{y\}$, since C was chosen minimal such that $H[D \cup C]$ is connected. Thus, $X = S \cup \{y\}$, which gives

$$\gamma_c(H) \leqslant |X| = |S| + 1 \leqslant \sum_{i \notin I} |D_i \cap X| + |I| + 1 \leqslant |D| + 1 = \gamma(H) + 1.$$

This completes the proof. □

To see that the bound given by Theorem 2 is best possible, consider the infinite family $\{F_k : k \geqslant 2\}$ of (P_6, C_6)-free graphs where F_k is the graph obtained from $K_{1,k}$ by subdividing each edge exactly once. Clearly $\gamma_c(F_k) = k + 1 = \gamma(F_k) + 1$.

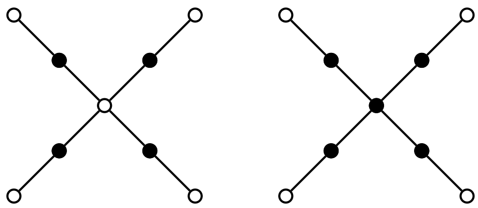

Figure 1: The black vertices indicate a minimum dominating set (resp. a minimum connected dominating set) of F_4.

Theorem 3. *For every connected (P_8, C_8)-free graph G, it holds that $\gamma_c(G)/\gamma(G) \leqslant 2$, and this bound is attained by connected (P_7, C_7)-free graphs with arbitrarily large values of γ. In particular, the bound $\gamma_c(G) \leqslant 2\gamma(G)$ is best possible even in the class of connected (P_7, C_7)-free graphs.*

Proof. Let $G = (V, E)$ be a connected (P_8, C_8)-free graph and let D be a minimum dominating set of G. Let $D_1, D_2, \ldots, D_k$ be the vertex sets of the connected components of $G[D]$.

It is clear that if $k = 1$, then $\gamma_c(G) = \gamma(G)$. So assume that $G[D]$ has at least two components, that is, $k \geqslant 2$. Let $C \subseteq V$ be an inclusionwise minimal set such that $G[D \cup C]$ is connected, and let $X \subseteq D \cup C$ be a minimal connected dominating set of G. By Lemma 1, $G[X]$ is P_6-free.

Let us first assume that $\gamma(G) = |D| \leqslant 3$. If $k = 2$, then $\gamma_c(G) \leqslant \gamma(G) + 2$, and since $\gamma_c(G) \geqslant \gamma(G) \geqslant 2$, this implies that $\gamma_c(G)/\gamma(G) \leqslant 2$. So we may assume that $k = 3$. Then D is an independent set of size 3. Let $D = \{x, y, z\}$.

Since D is a dominating set of G, every vertex of C has a neighbour among D. Hence, the distance in $G[D \cup C]$ between x and y or z, y say, is at most 3. Similarly, the distance between z and x or y, y say, is at most 3. So, at most four vertices of C suffice to connect x, y and z. By the minimality of C, $|C| \leqslant 4$. Thus, $|X| \leqslant |D| + |C| \leqslant 7$.

Suppose that $|X| = 7$, i.e., $|C| = 4$ and $X = D \cup C$. Because $|C| = 4$, by the previous argumentation, we can suppose that there are two vertices u and v in C such that (x, u, v, y) is a shortest path in $G[X]$ between x and y, and there are two vertices u' and v' in C such that (y, u', v', z) is a shortest path in $G[X]$ between y and z. Because $|C| = 4$, all of u, v, u', v' are distinct by minimality of C. Then $\{u, v, u', v'\} \cap D = \emptyset$ and $C = \{u, v, u', v'\}$. If x, resp. z, is adjacent to at least one of u' or v', resp. u or v, then by minimality of C, $C = \{u', v'\}$, resp. $C = \{u, v\}$, a contradiction. Otherwise, if u, resp. u', is adjacent to at least one of u' or v', resp. u or v, then by minimality of C, $C = \{u, u', v'\}$, resp. $C = \{u', u, v\}$, a contradiction. Otherwise $G[X]$ is isomorphic to P_7, or contains an induced P_6, depending on the adjacency between v and v', a contradiction.

This means $|X| \leqslant 6$ and thus $\gamma_c(G) \leqslant |X| \leqslant 6 = 2\gamma(G)$.

Now let us assume that $\gamma(G) = |D| \geqslant 4$. Let $I \subseteq \{1, 2, \ldots, k\}$ be such that $i \in I$ if and only if $D_i \cap X = \emptyset$. For every $i \in I$, pick $x_i \in X$ such that x_i has a neighbor in D_i; this is always possible, since X is a dominating set of G. Note that every x_i belongs to C, and that the x_i do not have to be distinct. Let $S = \bigcup_{i \notin I}(D_i \cap X) \cup \{x_j : j \in I\}$.

It is shown by van 't Hof and Paulusma [9] that every connected P_6-free graph has a connected dominating set Z for which the following holds: $G[Z]$ is either isomorphic to C_6 or contains a complete bipartite graph as spanning subgraph. Let Y be such a connected dominating set of $G[X]$.

Assume first that $G[Y]$ is isomorphic to C_6. Let $u_1, u_2, \ldots, u_6$ be a consecutive ordering of the vertices of the C_6. Suppose that $Y' = \{u_1, u_2, u_3, u_4\}$ is not a dominating set of $G[X]$. Then there is a vertex $z \in X$ with $N_G[z] \cap Y' = \emptyset$. Without loss of generality, z is adjacent to u_5. But then $G[Y' \cup \{u_5, z\}]$ is isomorphic to P_6, a contradiction to the fact that $G[X]$ is P_6-free. Thus Y' is a connected dominating set of $G[X]$.

Since $\{x_j : j \in I\} \subseteq X$ and Y' is a connected dominating set of $G[X]$, $G[Y' \cup \{x_j : j \in I\}]$ is connected. Thus, $G[D \cup Y' \cup \{x_j : j \in I\}]$ is connected: every D_i with $D_i \cap X \neq \emptyset$ has a neighbour in Y', and every D_i with $D_i \cap X = \emptyset$ has a neighbour in $\{x_j : j \in I\}$, namely x_i.

As $(Y' \cup \{x_j : j \in I\}) \subseteq X$, $(Y' \cup \{x_j : j \in I\}) \setminus D \subseteq C$. By the minimality of C, $C = (Y' \cup \{x_j : j \in I\}) \setminus D$. Thus, $X \subseteq S \cup Y'$, which gives

$$\gamma_c(G) \leqslant |X| \leqslant |S| + |Y'| \leqslant \sum_{i \notin I} |D_i \cap X| + |I| + 4 \leqslant |D| + 4 \leqslant 2\gamma(G).$$

Now assume that $G[Y]$ contains a complete bipartite graph as a spanning subgraph. Let (A, B) be a bipartition of this complete bipartite graph. For each $1 \leqslant i \leqslant k$, pick $y_i \in Y$ with the following property. If $D_i \cap X \neq \emptyset$, $N_{G[X]}[y_i] \cap D_i \neq \emptyset$, and if $D_i \cap X = \emptyset$, $y_i \in N_{G[X]}[x_i]$. These y_i exist since Y is a dominating set of $G[X]$. We can assume that $A \cap \{y_i : 1 \leqslant i \leqslant k\} \neq \emptyset$.

If $B \cap \{y_i : 1 \leqslant i \leqslant k\} \neq \emptyset$, the graph $G[D \cup \{x_i : i \in I\} \cup \{y_j : 1 \leqslant j \leqslant k\}]$ is connected. As $\{x_i : i \in I\} \cup \{y_j : 1 \leqslant j \leqslant k\} \subseteq X$, $(\{x_i : i \in I\} \cup \{y_j : 1 \leqslant j \leqslant k\}) \setminus D \subseteq C$. By the minimality of C, $C = (\{x_i : i \in I\} \cup \{y_j : 1 \leqslant j \leqslant k\}) \setminus D$. Thus $X \subseteq S \cup \{y_j : 1 \leqslant j \leqslant k\}$, which gives

$$\gamma_c(G) \leqslant |X| \leqslant |S| + k \leqslant \sum_{i \notin I} |D_i \cap X| + |I| + k \leqslant |D| + k \leqslant 2\gamma(G). \quad (3)$$

So we may assume that $B \cap \{y_i : 1 \leqslant i \leqslant k\} = \emptyset$. Pick any $z \in B$. Since D is a dominating set of G, there is an index $1 \leqslant l \leqslant k$ such that $N_G[z] \cap D_l \neq \emptyset$. Hence, $G[D \cup \{x_i : i \in I\} \cup \{y_j : 1 \leqslant j \leqslant k, j \neq l\} \cup \{z\}]$ is connected. So, $C = (\{x_i : i \in I\} \cup \{y_j : 1 \leqslant j \leqslant k, j \neq l\} \cup \{z\}) \setminus D$ and thus $X \subseteq S \cup \{y_j : 1 \leqslant j \leqslant k, j \neq l\} \cup \{z\}$, which gives (3). This completes the proof of the bound $\gamma_c(G) \leqslant 2\gamma(G)$ for (P_8, C_8)-free graphs.

The bound $\gamma_c(G) \leqslant 2\gamma(G)$ is attained by an infinite number of connected (P_7, C_7)-free graphs G, given by the following construction. For every $k \in \mathbb{N}$, let H_k be the graph obtained from K_k by attaching to each vertex a pendant path on two vertices in the way illustrated in Figure 2. It is easily seen that, for all $k \in \mathbb{N}$, $\gamma_c(H_k)/\gamma(H_k) = 2$. □

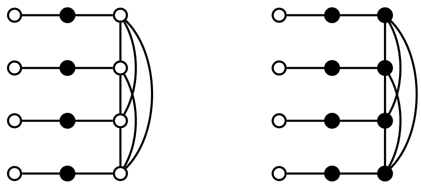

Figure 2: The black vertices indicate a minimum dominating set (respectively a minimum connected dominating set) of H_4.

A similar construction shows that (1) is asymptotically sharp in the class of connected (P_9, C_9)-free graphs, in the sense that there is an infinite family $\{G_k : k \in \mathbb{N}\}$ of (P_9, C_9)-free graphs such that $\lim_{k\to\infty} \gamma_c(G_k)/\gamma(G_k) = 3$. For every $k \in \mathbb{N}$, let G_k be the graph obtained from K_k by attaching to each vertex a pendant path on three vertices in the way illustrated in Figure 3. It is easy to check that for every $k \geqslant 2$, $\gamma(G_k) = k + 1$ and $\gamma_c(G_k) = 3k$. Furthermore, G_k is (P_9, C_9)-free.

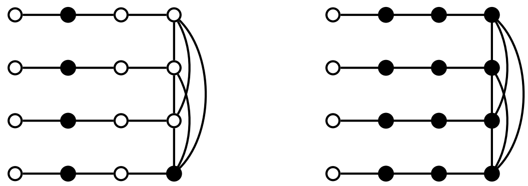

Figure 3: The black vertices indicate a minimum dominating set (respectively a minimum connected dominating set) of G_4.

2.1 A Conjecture

We end this section with a conjecture that came up during our research. As Theorem 3 shows, it holds that $\gamma_c(G) \leqslant 2\gamma(G)$ for every connected (P_8, C_8)-free graph G. However, $\gamma_c(P_8)/\gamma(P_8) = 2 = \gamma_c(C_8)/\gamma(C_8)$, i.e., both P_8 and C_8 do not violate the bound given by Theorem 3. While intensively searching for minimal connected graphs G with $\gamma_c(G) > 2\gamma(G)$, we got the strong impression that P_9, C_9 and H, the graph depicted in Figure 4, might be the only minimal graphs. If this is true, the following conjecture holds.

Conjecture 1. *For every connected (P_9, C_9, H)-free graph G, where H is the graph depicted in Figure 4, it holds that $\gamma_c(G) \leqslant 2\gamma(G)$.*

Note that for any $G \in \{P_9, C_9, H\}$, $\gamma_c(G) > 2\gamma(G)$. Hence, if true, Conjecture 1 would give a characterization of the largest class of connected graphs that is closed under connected induced subgraphs where $\gamma_c(G) \leqslant 2\gamma(G)$ holds.

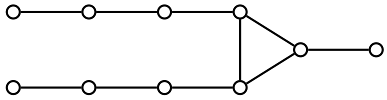

Figure 4: The graph H from Conjecture 1.

3 Complexity result

The class Θ_2^P, sometimes denoted $P^{\mathrm{NP}[\log]}$, is defined as the class of decision problems solvable in polynomial time by a deterministic Turing machine that is allowed use $\mathcal{O}(\log n)$ many queries to an NP-oracle, where n is the size of the input.

Theorem 4. *Let $1 < r < 3$ be a fixed rational number. Given a connected graph G, the problem of deciding whether $\gamma_c(G)/\gamma(G) \leqslant r$ is Θ_2^p-complete.*

It is easy to see that the above decision problem belongs to Θ_2^p, since both $\gamma(G)$ and $\gamma_c(G)$ can be computed using logarithmically many queries to an NP-oracle by binary search. Thus, Theorem 4 is a negative result: loosely speaking, it tells us that deciding whether the ratio of $\gamma_c(G)$ and $\gamma(G)$ is bounded by some constant is as hard as computing both $\gamma_c(G)$ and $\gamma(G)$ explicitly, and this remains true even if the constant is not part of the input.

A *vertex cover* of a graph G is a set $X \subseteq V(G)$ such that every edge of G is incident to at least one member of X. The minimum size of a vertex cover of G is denoted by $\tau(G)$. Our reduction is from the decision problem whether for two given graphs G and H it holds that $\tau(G) \geqslant \tau(H)$, which is known to be Θ_2^p-complete due to Spakowski and Vogel [14].

Before proving Theorem 4, we need the following two lemmas.

Lemma 2. *Given a connected graph G with $n \geqslant 2$ vertices, one can construct in linear time a graph G' such that $\gamma(G') = n + \tau(G)$ and $\gamma_c(G') = 3n - 1$.*

Proof. With each vertex $v \in V(G)$, associate three vertices v, v', v'' in $V(G')$. Moreover, for each edge $e \in E(G)$, define a vertex e of $V(G')$. We may treat $V(G)$ and $E(G)$ as subsets of $V(G')$. Let

$$E(G') := \bigcup_{e=uv\in E(G)} \{ue, ve\} \cup \bigcup_{v\in V(G)} \{vv', v'v''\}.$$

Let D be a minimum dominating set of G'. Since every vertex of the form v'' is of degree 1, we can assume that $v' \in D$ for every $v \in V(G)$.

Assume that $e \in D$ for some $e = uv \in E(G)$. Since $u', v' \in D$, the set $D \setminus \{e\}$ dominates all vertices of G' except e. Thus $(D \setminus \{e\}) \cup \{u\}$ is a minimum dominating set of G'. So we may assume $D \cap E(G) = \emptyset$. Hence, $D \cap V(G)$ is a vertex cover of G, and so $\gamma(G') = |D| \geqslant n + \tau(G)$.

Similarly, if T is a vertex cover of G, the set $\{v' : v \in V(G)\} \cup T$ is a dominating set of G. Hence, $\gamma(G') \leqslant n + \tau(G)$. This gives $\gamma(G') = n + \tau(G)$.

To see that $\gamma_c(G') = 3n - 1$, let C be a minimum connected dominating set of G'. Recall that $n \geqslant 2$ and G contains at least one edge. Thus, the vertices v and v', for every $v \in V(G)$, are cut-vertices of G' and therefore contained in C. Moreover, C contains no vertex of the form v''.

Observe that the set $C \cap E(G)$ defines a minimum spanning tree of G, and so $|C \cap E(G)| = n - 1$. Summarizing, $\gamma_c(G') = |C| = 2n + n - 1 = 3n - 1$. □

Lemma 3. *Given a graph G with n vertices and m edges, one can construct in linear time a graph G' such that $\gamma(G') = n+m+1$ and $\gamma_c(G') = n+m+1+\tau(G)$.*

Proof. We start from the construction given in the proof of Lemma 2. We add two vertices w, w' and, for every $e \in E(G)$, a vertex e'. Then we put edges joining w to every vertex of the form v', where $v \in V(G)$, and to w'. We also put edges joining e and e', for every $e \in E(G)$.

For each edge $e \in E(G)$, the corresponding vertex $e \in V(G')$ is adjacent to the degree-one vertex e'. Thus it can be considered, without loss of generality, to be part of any minimum dominating set of G'. The same remark holds for every vertex v', where $v \in V(G)$, and for w. Now the union $D \subseteq V(G')$ of those vertices is a dominating set of G', hence we have $\gamma(G') = n + m + 1$.

It remains to compute $\gamma_c(G')$. The previous dominating set D is not connected, as $G'[D]$ has exactly $m + 1$ connected components: one for each edge of G, and one induced by w and the vertices of the form v'. To make it connected, we need to add the fewest possible additional vertices $v \in V(G)$. Every such vertex v will link the component containing v to every vertex $e \in E(G)$ of G' such that $v \in e$. Hence the minimum number of additional vertices to add to C is exactly the size $\tau(G)$ of a minimum vertex cover of G. Hence $\gamma_c(G') = n + m + 1 + \tau(G)$. □

Proof of Theorem 4. Let $r = r_1/r_2$ be a fixed rational number with $1 < r < 3$. We have already argued why the decision problem is in Θ_2^p, so we proceed to proving the Θ_2^p-hardness. Let G and H be two graphs. We reduce from the Θ_2^p-complete decision problem of deciding whether $\tau(G) \geqslant \tau(H)$ [14]. We may w.l.o.g. assume that G and H are both connected: otherwise we may add an isolated vertex to G and then a universal vertex, and proceed similar with H. Denoting the resulting graphs by G' and H', we see that $\tau(G') = \tau(G) + 1$ and $\tau(H') = \tau(H) + 1$. The reduction consists of the following five steps.

Step 1. We choose an arbitrary vertex $v \in V(G)$. Starting with r_2 disjoint copies of G, we connect all r_2 copies of v to a new vertex w. We then attach a pendant vertex w' to w. The graph obtained we denote by G_{r_2}. Let $n_G = |V(G)|$. Clearly, $\tau(G_{r_2}) = r_2\tau(G) + 1$ and $|V(G_{r_2})| = r_2 n_G + 2$.

Similarly we construct H_{r_1} from H. Let $n_H = |V(H)|$ and $m_H = |E(H)|$. Clearly, $\tau(H_{r_1}) = r_1\tau(H) + 1$, $|V(H_{r_1})| = r_1 n_H + 2$, and $|E(H_{r_1})| = r_1 m_H + r_1 + 1$.

Step 2. We apply Lemma 2 to G_{r_2} to get G'_{r_2}. We obtain

$$\begin{aligned}\gamma(G'_{r_2}) &= |V(G_{r_2})| + \tau(G_{r_2}) \\ &= r_2\tau(G) + r_2 n_G + 3, \\ \gamma_c(G'_{r_2}) &= 3|V(G_{r_2})| - 1 \\ &= 3r_2 n_G + 5.\end{aligned}$$

We apply Lemma 3 to H_{r_1} to get H'_{r_1}, and obtain

$$\begin{aligned}\gamma(H'_{r_1}) &= |V(H_{r_1})| + |E(H_{r_1})| + 1\\ &= r_1(n_H + m_H + 1) + 4,\\ \gamma_c(H'_{r_1}) &= \tau(H_{r_1}) + |V(H_{r_1})| + |E(H_{r_1})| + 1\\ &= r_1\tau(H) + r_1(n_H + m_H + 1) + 5.\end{aligned}$$

Step 3. We construct a new graph U by taking the disjoint union of two copies of G'_{r_2} and two copies of H'_{r_1}, picking a vertex from each of these four graphs that is adjacent to a degree-one vertex, and then adding any possible edge between these four vertices. Observe that there exists a minimum connected dominating set in each of the four copies containing the picked vertex.

By the construction of U,

$$\begin{aligned}\gamma_c(U) &= 2\gamma_c(G'_{r_2}) + 2\gamma_c(H'_{r_1})\\ &= 2r_1\tau(H) + 2(r_1(n_H + m_H + 1) + 3r_2n_G + 10),\\ \gamma(U) &= 2\gamma(G'_{r_2}) + 2\gamma(H'_{r_1})\\ &= 2r_2\tau(G) + 2(r_1(n_H + m_H + 1) + r_2n_G + 7).\end{aligned}$$

Step 4. Let

$$\begin{aligned}\varphi_1 &= r_1(n_H + m_H + 1) + 3r_2n_G + 10,\\ \varphi_2 &= r_1(n_H + m_H + 1) + r_2n_G + 7.\end{aligned}$$

Let $p = \max\{|\varphi_1 - 3\varphi_2|, |\varphi_2 - \varphi_1|\}$, and

$$\begin{aligned}a &= p(3r_2 - r_1) + (\varphi_1 - 3\varphi_2),\\ b &= p(r_1 - r_2) + (\varphi_2 - \varphi_1).\end{aligned}$$

By definition of p, $a \geqslant |\varphi_1 - 3\varphi_2|(3r_2 - r_1) + (\varphi_1 - 3\varphi_2) \geqslant |\varphi_1 - 3\varphi_2|(3r_2 - r_1 - 1)$ and $b \geqslant |\varphi_2 - \varphi_1|(r_1 - r_2) + (\varphi_2 - \varphi_1) \geqslant |\varphi_2 - \varphi_1|(r_1 - r_2 - 1)$. Since $r_1 > r_2$ and $3r_2 > r_1$, then a and b are non-negative integers. Moreover $a, b \in \mathcal{O}(\varphi_1 + \varphi_2)$. An easy calculation shows that

$$a + 3b + 2\varphi_1 = 2pr_1 \text{ and } a + b + 2\varphi_2 = 2pr_2. \tag{4}$$

We now construct a graph U' from U as follows. Let v be a vertex in U of degree 1 (such a vertex is always present). Let P^a be the graph obtained from the chordless path with vertex set $\{u_1, u_2, \ldots, u_a\}$ by attaching a pendant vertex to every member of $\{u_1, u_2, \ldots, u_a\}$. Let P^b be the graph obtained from the chordless path with vertex set $\{v_1, v_2, \ldots, v_{3b}\}$ by attaching a pendant vertex to every member of $\{v_3, v_6, \ldots, v_{3b}\}$. Let U' be the graph obtained from the disjoint union of U, P^a and P^b by putting an edge from v to u_1 and to v_1. Since $a, b \in \mathcal{O}(\varphi_1 + \varphi_2)$, the above procedure can be done in linear time in the size of the graph U and thus in the size of the input.

By the construction of U', we obtain

$$\begin{aligned}\gamma_c(U') &= \gamma_c(U) + a + 3b\\ &= 2r_1\tau(H) + a + 3b + 2\varphi_1\\ &\overset{(4)}{=} 2r_1\tau(H) + 2pr_1,\end{aligned}$$

and

$$\begin{aligned}\gamma(U') &= \gamma(U) + a + b \\ &= 2r_2\tau(G) + a + b + 2\varphi_2 \\ &\overset{(4)}{=} 2r_2\tau(G) + 2pr_2.\end{aligned}$$

Recalling $r = r_1/r_2$, we have

$$\frac{\gamma_c(U')}{\gamma(U')} = \frac{2r_1\tau(H) + a + 3b + 2\varphi_1}{2r_2\tau(G) + a + b + 2\varphi_2} = \frac{2r_1\tau(H) + 2pr_1}{2r_2\tau(G) + 2pr_2} = r\frac{\tau(H) + p}{\tau(G) + p}.$$

Thus, $\gamma_c(U')/\gamma(U') \leqslant r$ if and only if $\tau(H) \leqslant \tau(G)$. This completes the proof. □

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