Price of Connectivity for the vertex cover problem and the dominating set problem: conjectures and investigation of critical graphs

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Abstract

The vertex cover problem and the dominating set problem are two well-known problems in graph theory. Their goal is to find the minimum size of a vertex subset satisfying some properties. Both hold a connected version, which imposes that the vertex subset must induce a connected component. To study the interdependence between the connected version and the original version of a problem, the Price of Connectivity (PoC) was introduced by Cardinal and Levy [8, 14] as the ratio between invariants from the connected version and the original version of the problem.

Camby, Cardinal, Fiorini and Schaudt [5] for the vertex cover problem, Camby and Schaudt [7] for the dominating set problem characterized some classes of PoC-Near-Perfect graphs, hereditary classes of graphs in which the Price of Connectivity is bounded by a fixed constant. Moreover, only for the vertex cover problem, Camby & al. [5] introduced the notion of critical graphs, graphs that can appear in the list of forbidden induced subgraphs characterization. By definition, the Price of Connectivity of a critical graph is strictly greater than that of any proper induced subgraph.

In this paper, we prove that for the vertex cover problem, every critical graph is either isomorphic to a cycle on 5 vertices or bipartite. To go further in the previous studies, we also present conjectures on PoC-Near-Perfect graphs and critical graphs with the help of the computer software GraphsInGraphs [4]. Moreover, for the dominating set problem, we investigate critical trees and we show that every minimum dominating set of a critical graph is independent.

keywords: vertex cover, connected vertex cover, dominating set, connected dominating set, forbidden induced subgraph, extremal graph.

1 Introduction

This section contains basic background and a brief state-of-the-art.

1.1 Background

All fundamental background is explained by Diestel [9]. We recall here some notions that we need.

Let G and H be two graphs. We say that G *contains* H if G contains an induced subgraph isomorphic to H. When G does not contain H, G is H*-free*. Moreover, G is $(H_1, \dots, H_k)$*-free* if G does not contain H_i for any $i \in \{1, \dots, k\}$.

A *vertex cover* is a vertex subset X such that every edge of G has at least one endpoint in X. The minimum size of a vertex cover, denoted by $\tau(G)$, is called the *vertex cover number* of G. A vertex cover of size $\tau(G)$ is called *minimum*.

A *connected vertex cover* of G is a vertex cover X such that the induced subgraph $G[X]$ is connected. When G is not connected, we require that $G[X]$ has the same number of connected components as G. The *connected vertex cover number*, denoted by $\tau_c(G)$, of a graph G is the minimum size of a connected vertex cover of G. Naturally, a connected vertex cover of size $\tau_c(G)$ is called *minimum*.

The *Price of Connectivity* of a graph G for the vertex cover problem is defined as

$$\frac{\tau_c(G)}{\tau(G)}.$$

A *dominating set* of a graph G is a vertex subset X such that every vertex either is in X or has a neighbor in X. The *domination number* of G is the minimum size of a dominating set of G and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is called *minimum*.

A *connected dominating set* of G is a dominating set X of G that induces a connected subgraph. If G is disconnected, we ask that $G[X]$ has the same number of connected components as G. The minimum size of a connected dominating set is the *connected domination number* and is denoted $\gamma_c(G)$.

Similarly to the vertex cover problem, the *Price of Connectivity* of a graph G for the dominating set problem is defined as

$$\frac{\gamma_c(G)}{\gamma(G)}.$$

A set X of vertices in a graph G is called *independent* if the induced subgraph $G[X]$ contains no edge, i.e.

$$E(G[X]) = \emptyset.$$

We denote by P_k the path on k vertices and by C_k the cycle on k vertices.

1.2 State-of-the-art

In 1972, Karp identified 21 NP-hard problems, among which finding a minimum vertex cover of a graph. Cardinal and Levy [8, 14] introduced the Price of Connectivity for the vertex cover problem, as defined in the previous subsection. Lately, Camby, Cardinal, Fiorini and Schaudt [5] studied rigorously this new graph invariant. We will examine in detail their results in Section 2. Besides, several researchers studied the interdependence between other graphs invariants.

Zverovich [18] characterized, in terms of list of forbidden induced subgraphs, perfect connected-dominant graphs, here called *PoC*-Perfect graphs, graphs for which the connected domination number and the domination number are equal for all induced subgraphs. Duchet and Meyniel [10], Tuza [17] established some results on these graph invariants. Some years ago, Camby and Schaudt [7] translated the Price of Connectivity from the vertex cover problem to the dominating set problem. Likewise for the vertex cover problem, we will explore in detail their results in Section 3.

In the same spirit, Summer and Moore [16] introduced the class of domination perfect graphs, related to the independent domination number and to the domination number, and Zverovich and Zverovich [19] characterized it. However, Fulman [11] found a counter-example. Then, Zverovich and Zverovich [20] corrected the characterization. Unfortunately, the last one was still inexact. Camby and Klein [6] finally revised the theorem and proposed a polynomial-time algorithm that transforms from any minimum dominating set into an independent one, for every graph in the so-characterized class.

Zverovich and Zverovich [21] considered the ratio between the independence number and the upper domination number whereas Schaudt [15] studied the ratio between the connected domination number and the total domination number.

Recently, Belmonte, van 't Hof, Kamiński and Paulusma [1, 2, 3] investigated the Price of Connectivity for the feedback vertex set while Hartinger, Johnson, Milanič and Paulusma [12, 13] studied the Price of Connectivity for cycle transversals.

This paper is divided into two parts: one dedicated to the vertex cover problem and another one to the dominating set problem. In each section, we state a whole previous work on the Price of Connectivity for the concerning problem. Then we present new results: conjectures (if any) and investigation on critical graphs.

2 Vertex cover problem

2.1 Previous work

First, the Price of Connectivity of any graph lies in the interval $[1, 2)$, since a vertex cover X of a connected graph G such that $G[X]$ has k connected components can be connected by adding at most $k-1$ vertices.

For the vertex cover problem, the class of *PoC*-Perfect graphs, i.e. the hereditary class of graphs G for which $\tau_c(G) = \tau(G)$, was characterized by Camby, Cardinal, Fiorini and Schaudt [5].

Theorem 1 (Camby & al. [5]). *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\tau_c(H) = \tau(H)$.

(ii) G is (P_5, C_5, C_4)-free.

(iii) G is chordal and P_5-free.

Camby & al. [5] introduced the notion of *PoC*-Near-Perfect graphs with threshold t, for a fixed $t \in [1, 2)$, and they characterized them for $t \leqslant \frac{3}{2}$. A graph G is said *PoC*-Near-Perfect graphs with threshold t if every induced subgraph H of G satisfies $\tau_c(H) \leqslant t\ \tau(H)$. The following theorem states for $t = \frac{4}{3}$.

Theorem 2 (Camby & al. [5]). *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\tau_c(H) \leqslant \frac{4}{3}\ \tau(H)$.

(ii) G is (P_5, C_4)-free.

For any fixed $t \in \left[1, \frac{4}{3}\right)$, the characterization of *PoC*-Near-Perfect graphs with threshold t is the same as in Theorem 1. The next interesting threshold is $\frac{3}{2}$, i.e. the Price of Connectivity of C_4 and of P_5. So, for any fixed $t \in \left[\frac{4}{3}, \frac{3}{2}\right)$, the characterization of these graphs with threshold t is like in Theorem 2.

Theorem 3 (Camby & al. [5]). *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\tau_c(H) \leqslant \frac{3}{2}\ \tau(H)$.

(ii) G is $(P_7, C_6, \Delta_1, \Delta_2)$-free, where Δ_1 and Δ_2 are depicted in Figure 1.

After investigating *PoC*-Near-Perfect graphs, they turned their attention to *critical graphs*, i.e. graphs G for which the Price of Connectivity of any proper induced subgraph H of G is strictly smaller than the Price of Connectivity of G, since these graphs appear in a forbidden induced subgraph characterization of the *PoC*-Near-Perfect graphs for some threshold $t \in [1, 2)$. They also defined a smaller class of graphs: a *strongly critical graph* is a graph G for which every proper (not necessarily induced) subgraph H of G has a Price of Connectivity that is strictly smaller than the Price of Connectivity of G.

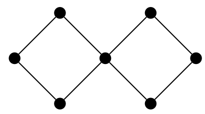

Figure 1: An illustration of graphs Δ_1 (on the left) and Δ_2 (on the right).

Camby & al. [5] obtained a characterization of critical graphs in the class of chordal graphs. Beforehand, they defined 1-*special trees*. A tree T is 1-*special* if it is obtained from another tree by subdividing each edge exactly once and then attaching a pendant vertex to every leaf of the resulting graph (See Figure 2 for an example).

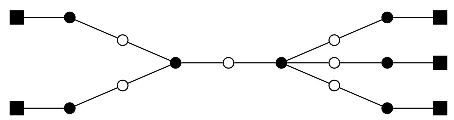

Figure 2: A 1-special tree constructed from another tree H (vertices indicated by filled circles) where each edge of H is exactly subdivided once.

Theorem 4 (Camby & al. [5])**.** *For a chordal graph G, the following assertions are equivalent:*

(i) G is a 1-*special tree.*

(ii) G is strongly critical.

(iii) G is critical.

Moreover, they found the following theorem about strongly critical graphs.

Theorem 5 (Camby & al. [5])**.** *Let G be a strongly critical graph.*

(i) Every minimum vertex cover of G is independent. In particular, G is bipartite.

(ii) If G has a cutvertex, then G is a 1-*special tree.*

2.2 New results

2.2.1 Conjectures

With the help of GraphsInGraphs [4], the computer aided Graph Theory software that relates graphs and their induced subgraphs, we establish two new conjectures on PoC-Near-Perfect graphs and critical graphs. Indeed, we obtain the list of critical graphs up to 10 vertices, in which appears C_4, C_5, C_6, P_5, P_7, Δ_1 and Δ_2, as expected, and the graphs depicted in Figure 3.

Conjecture 1. *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\tau_c(H) \leqslant \frac{5}{3}\ \tau(H)$.

(ii) G is $(H_i)_{i=1}^{10}$-free, where graphs $H_1, \ldots, H_{10}$ are depicted in Figure 3.

Conjecture 2. *Every critical graph is a cactus, i.e. a connected graph in which any two cycles have at most one vertex in common.*

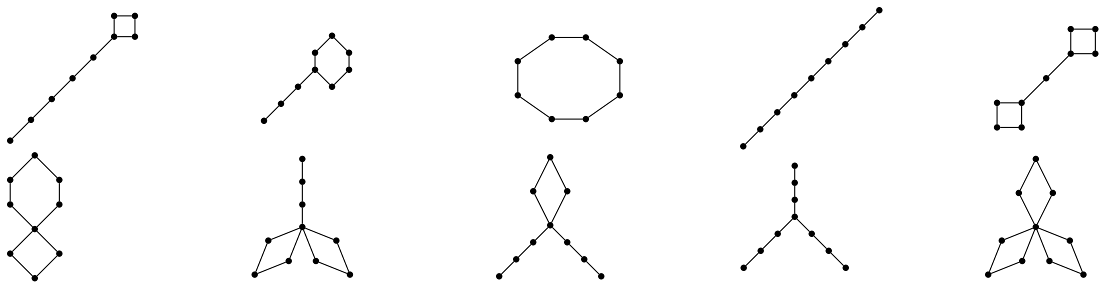

Figure 3: Graphs $H_1, \ldots, H_{10}$ from Conjecture 1.

2.2.2 Critical graphs

In Theorem 5, Camby & al. proved that every strongly critical graph is bipartite. Here, we extend the result to the class of critical graphs, except the cycle C_5 on 5 vertices.

Theorem 6. *A critical graph G is either isomorphic to C_5, or bipartite. Moreover, when G is bipartite, every minimum vertex cover of G is independent.*

Proof. The proof arises directly from Lemma 1 and Lemma 2. □

Lemma 1. *Let G be a critical graph with $\tau_c(G) \leqslant \frac{3}{2}\tau(G)$. Then G is isomorphic to C_4, to C_5 or to P_5.*

Proof. First, we assume that $G \ncong P_5$ and $G \ncong C_4$. We prove that $\tau_c(G) \leqslant \frac{4}{3}\tau(G)$. Otherwise by Theorem 2, the graph G contains either P_5 or C_4. However,

$$\frac{\tau_c(P_5)}{\tau(P_5)} = \frac{\tau_c(C_4)}{\tau(C_4)} = \frac{3}{2} \geqslant \frac{\tau_c(G)}{\tau(G)},$$

which is a contradiction with G critical graph.

Now, we prove that G is isomorphic to C_5. We assume that it is not the case. Since $1 \leqslant \frac{\tau_c(H)}{\tau(H)} < \frac{\tau_c(G)}{\tau(G)} \leqslant \frac{4}{3}$ for all induced subgraph H of G, we deduce that by Theorem 1, G is not (C_4, C_5, P_5)-free and by Theorem 2, G is (C_4, P_5)-free. Accordingly, G contains C_5 or $G \cong C_5$. But in the first case,

$$\frac{\tau_c(C_5)}{\tau(C_5)} = \frac{4}{3} \geqslant \frac{\tau_c(G)}{\tau(G)},$$

which is again in contradiction with G critical graph. Thus $G \cong C_5$. □

Lemma 2. *Let G be a critical graph with $\frac{\tau_c(G)}{\tau(G)} > \frac{3}{2}$. Every minimum vertex cover of G is independent. In particular, G is bipartite.*

Proof. Let X be a minimum vertex cover of G. We assume that there are two adjacent vertices u and v in X. Consider $H = G \setminus \{u, v\}$ the resulting graph after deleting vertices u and v. Since $X \setminus \{u, v\}$ is a vertex cover of H, and any vertex cover of H with vertices u and v is a vertex cover of G,

$$\tau(G) = \tau(H) + 2.$$

Moreover, $\tau_c(G) \leqslant \tau_c(H) + 3$ because any connected vertex cover of H with u, v and an arbitrary vertex from $N_G(u) \cup N_G(v)$ gives a connected vertex cover of G. Accordingly,

$$\frac{3}{2} < \frac{\tau_c(G)}{\tau(G)} \leqslant \frac{\tau_c(H) + 3}{\tau(H) + 2}.$$

Because

$$\frac{\tau_c(H) + 3}{\tau(H) + 2} \in \left[\frac{3}{2}, \frac{\tau_c(H)}{\tau(H)}\right],$$

we obtain that $\frac{\tau_c(H)}{\tau(H)} = \max\left(\frac{3}{2}, \frac{\tau_c(H)}{\tau(H)}\right)$ and so $\frac{\tau_c(G)}{\tau(G)} \leqslant \frac{\tau_c(H)}{\tau(H)}$, which is a contradiction with G critical. □

3 Dominating set problem

3.1 Previous work

First, the Price of Connectivity of any graph G for the dominating set problem lies in the interval $[1, 3)$ because a dominating set X of a connected graph G can be turned into a connected dominating set by adding at most $2k - 2$ vertices, where k is the number of connected components of $G[X]$.

For the dominating set problem, Zverovich [18] established a characterization of PoC-Perfect graphs in the following theorem.

Theorem 7 (Zverovich [18])**.** *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\gamma_c(H) = \gamma(H)$.

(ii) G is (P_5, C_5)-free.

Camby and Schaudt [7] investigated the class of PoC-Near-Perfect graphs, similarly defined as in Section 2. Actually, they proved that every (P_6, C_6)-free graph G satisfies $\gamma_c(G) \leqslant \gamma(G) + 1$, which directly yields the next characterization of PoC-Near-Perfect graph with threshold $\frac{3}{2}$.

Theorem 8 (Camby and Schaudt [7])**.** *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\gamma_c(H) \leqslant \frac{3}{2}\gamma(H)$.

(ii) G is (P_6, C_6)-free.

They also attempted to characterize PoC-Near-Perfect graphs with threshold 2: they found a subclass of this class.

Theorem 9 (Camby and Schaudt [7])**.** *For every (P_8, C_8)-free graph G, it holds that*

$$\gamma_c(G) \leqslant 2\,\gamma(G).$$

Conjecture on PoC-Near-Perfect graphs

Camby and Schaudt [7] have already established the following conjecture, confirmed by the computer software GraphsInGraphs [4].

Conjecture 3 (Camby and Schaudt [7])**.** *The following assertions are equivalent for every graph G:*

(i) For every induced subgraph H of G it holds that $\gamma_c(H) \leqslant 2\,\gamma(H)$.

(ii) G is (P_9, C_9, H)-free, where the graph H is depicted in Figure 4.

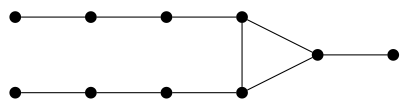

Figure 4: The graph H from Conjecture 3.

3.2 New results

We now turn our attention to *critical graphs*, which is, graphs G for which the Price of Connectivity of every proper induced subgraph H of G is strictly smaller than the Price of Connectivity of G, similarly to what Camby & al. [5] did for the vertex cover problem. These are exactly the graphs that can appear in a minimal forbidden induced subgraph characterization of the PoC-Near-Perfect graphs for some threshold $t \in [1, 3)$. A perhaps more tractable class of graphs is the class of *strongly critical* graphs, defined as the graphs G for which every proper (not necessarily induced) subgraph H of G has a Price of Connectivity that is strictly smaller than the Price of Connectivity of G. It is clear that every strongly critical graph is critical, but the converse is not true. For instance, C_5 is critical, but not strongly critical. Notice that every (strongly) critical graph is connected.

3.2.1 Critical trees

Let T be a tree. We call T 2-*special* if T is obtained from another tree (with at least one edge) by subdividing each edge either once or twice and then attaching a pendent vertex to every leaf of the resulting graph (see Figure 5 for an example).

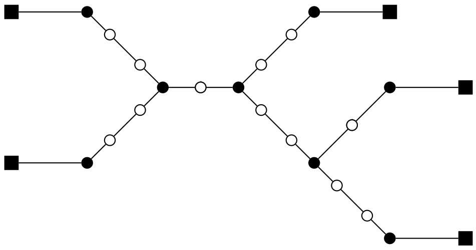

Figure 5: A 2-special tree constructed from another tree (vertices indicated by filled circles) by subdividing each edge either once or twice (subdivision vertices are indicated by hollow circles) and then attaching a pendent vertex (indicated by squares) to every leaf.

The next result gives a partial characterization of the class of critical trees. However, the class of 2-special trees turns out to be too restricted. We need a new definition.

A tree T is called *peculiar* if

- the neighbor of every leaf has degree 2,
- every minimum dominating set D of T is independent and
- every vertex $v \in V(T) \setminus D$ with degree at least 3 has only one neighbor in D, i.e. $|N_T(v) \cap D| = 1$.

See Figure 6 for an example.

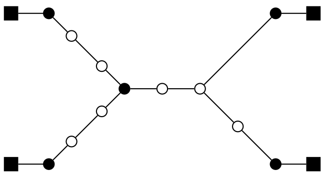

Figure 6: A peculiar tree where a minimum dominating set contains vertices indicated by filled circles and leaves are indicated by squares.

In spite of our initial expectation, Theorem 4 cannot be straightforwardly adapted to the domination case because 2-special trees are too restricted.

Theorem 10. *For a tree G, the following assertions are equivalent:*

(i) G is a peculiar critical tree.

(ii) G is strongly critical.

(iii) G is critical.

Moreover, if G is critical and if the degree of any $v \in V(G) \setminus D$, where D is an arbitrary minimum dominating set of G, is at most 2, then G is a 2-special tree built on an initial tree H, where $V(H)$ is a minimum dominating set.

Before proving the theorem, we show the following useful lemma.

Lemma 3. *Let G be a critical graph. For every minimum dominating set D of G, there does not exist a bridge of G with endpoints in D.*

Proof. Suppose there exists a bridge x_1x_2 with $x_1, x_2 \in D$. The removal of the edge x_1x_2 results in two connected subgraphs of G, which we denote by G_1 and G_2 respectively. We can assume that $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. Let G_1' be the graph obtained from G_1 by attaching a pendent vertex to x_1. Similarly, let G_2' be the graph obtained from G_2 by attaching a pendent vertex to x_2.

We observe that $D \cap V(G_1)$ is a dominating set of G_1' and $D \cap V(G_2)$ is a dominating set of G_2'. Thus

$$\gamma(G) \geqslant \gamma(G_1') + \gamma(G_2'). \tag{1}$$

On the other hand, let $D_{c,1}$ be a minimum connected dominating set of G_1' and $D_{c,2}$ be a minimum connected dominating set of G_2'. We can assume that $D_{c,1} \subseteq V(G_1)$ and $D_{c,2} \subseteq V(G_2)$, since for $i = 1, 2$, G_i' is the graph G_i with a pendent vertex. It is clear that $x_1 \in D_{c,1}$ and $x_2 \in D_{c,2}$. Thus $D_{c,1} \cup D_{c,2}$ is a connected dominating set of G. Since $D_{c,1} \cap D_{c,2} = \emptyset$,

$$\gamma_c(G) \leqslant \gamma_c(G_1') + \gamma_c(G_2'). \tag{2}$$

But (1) and (2) say that

$$\gamma_c(G)/\gamma(G) \leqslant \max\{\gamma_c(G_1')/\gamma(G_1'), \gamma_c(G_2')/\gamma(G_2')\}. \tag{3}$$

Since both G_1' and G_2' are isomorphic to induced subgraphs of G, (3) is a contradiction to the fact that G is critical. □

Proof of Theorem 10. It is obvious that (ii) implies (iii). Firstly, we show that (i) implies (ii): every peculiar critical tree is strongly critical. Let G be a peculiar critical tree. Let G' be a proper (not necessarily induced) subgraph of G and $C_1, \ldots, C_k$ be the vertex sets of its connected components. Because G is a tree, G' is the disjoint union of $G[C_i]$ for $i = 1, \ldots, k$. Moreover a minimum (resp. connected) dominating set of G is the union of a minimum (resp. connected) dominating set of each connected component of G', so

$$\frac{\gamma_c(G')}{\gamma(G')} = \frac{\sum_{1\leqslant i\leqslant k} \gamma_c(G[C_i])}{\sum_{1\leqslant i\leqslant k} \gamma(G[C_i])} \leqslant \max\left\{\frac{\gamma_c(G[C_i])}{\gamma(G[C_i])}\middle| i = 1, \ldots, k\right\} < \frac{\gamma_c(G)}{\gamma(G)},$$

since G is critical.

Secondly, we prove that (iii) implies (i): every critical tree is peculiar. For this, let $G = (V, E)$ be a critical tree. Let D be a minimum dominating set of G.

First we show that D is an independent set. Suppose there are $x, y \in D$ such that $xy \in E$. Since G is a tree, xy is a bridge, a contradiction to Lemma 3.

Now we show that every member of $V \setminus D$ with degree at least 3 has only one neighbor in D. For this, let $x \in V \setminus D$ with $|N_G(x)| \geqslant 3$. Suppose that $|N_G(x) \cap D| \geqslant 2$, hence let $d_1, d_2 \in N_G(x) \cap D$. Let $X_1, X_2, \ldots, X_k$ be the vertex sets of the connected components of $G - x$. By assumption, $k \geqslant 3$. Suppose that $d_1 \in X_1$ and $d_2 \in X_2$. Let

$$H_1 = G - \bigcup_{i \neq 2,3} X_i$$

and

$$H_2 = G - (X_2 \cup X_3).$$

We observe that

$$\gamma(G) \geqslant \gamma(H_1) + \gamma(H_2). \tag{4}$$

Since x is a cutvertex of H_1, x is contained in every connected dominating set of H_1. Therefore

$$\gamma_c(G) \leqslant \gamma_c(H_1) + \gamma_c(H_2). \tag{5}$$

By the same argumentation from Lemma 3, (4) and (5) yield a contradiction to the fact that G is critical. This proves that every vertex of $V \setminus D$ with degree at least 3 has only one neighbor in D. From the discussion above, D is an independent set.

Moreover, two degree-1 vertices, say x and y, cannot have the same neighbor since $\gamma_c(G-x) = \gamma_c(G)$, $\gamma(G-x) = \gamma(G)$ and G is critical. The neighbor y of any degree-1 vertex x must have its degree equal to 2. Otherwise, clearly $\gamma(G-x) \leqslant \gamma(G)$. The vertex y is a cutvertex of the tree $G-x$, hence $\gamma_c(G-x) = \gamma_c(G)$. We obtain a contradiction since G is critical. All in all, G is peculiar.

Now, we prove that the critical tree G is 2-special if every member of $V \setminus D$ has a degree at most 2, for a minimum dominating set D. By the previous argument, we know that any minimum dominating set is independent. We can suppose that D does not contain a leaf. Otherwise, we could replace in D the degree-1 vertex by its neighbor such that D is an independent dominating set and every vertex $v \notin D$ has degree at most 2. Consider the initial tree H defined as followed: $V(H) = D$ and $E(H) = \{uv|$ there exists a path P_{uv} in $(V \setminus D) \cup \{u, v\}$ from u to $v\}$. Because D is a dominating set of G, if uv is an edge in H, then the length of the path P_{uv} in G is either 2 or 3. All in all, G is a 2-special tree. This completes the proof. □

Notice that Theorem 10 is quite similar to Theorem 4 for the vertex cover problem but the hypothesis in the domination version is stronger. Indeed, for the dominating set problem, in the class of chordal graphs, the three assumptions from Theorem 10 are not equivalent: the graph from Conjecture 3 is a critical chordal graph without being a tree.

Now, we investigate the relations between graph classes: 2-special trees, peculiar trees and critical trees. Figure 7 illustrates the situation.

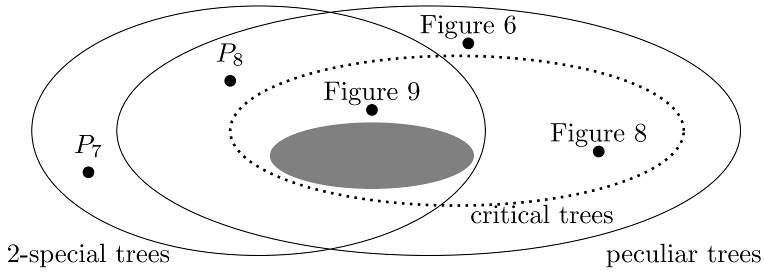

Figure 7: The situation around critical trees, where the gray area represents special trees with a double subdivision.

Not all peculiar trees are critical. For instance, the graph depicted in Figure 6 (whose Price of Connectivity is 12/5) is not critical because it contains an induced subgraph H with a higher Price of Connectivity. Indeed, for instance, H could be the graph obtained from a claw, $K_{1,3}$, by subdividing each edge exactly thrice. Furthermore, the graph illustrated by Figure 8 is a peculiar critical tree which is not 2-special. Also, we point out that not all 2-special trees are critical, for instance P_8 contains an induced P_6 with the same Price of Connectivity.

Moreover, by Proposition 1, every 2-special tree built on the initial tree H, where all edges of H are subdivided exactly twice in G, is critical. These graphs are represented by the gray area in Figure 7. However, the converse in the class of 2-special trees is not true because the graph illustrated by Figure 9 is critical.

Proposition 1. *Let G be a 2-special tree built on the initial tree H. If all edges of H are subdivided twice in G, then G is critical.*

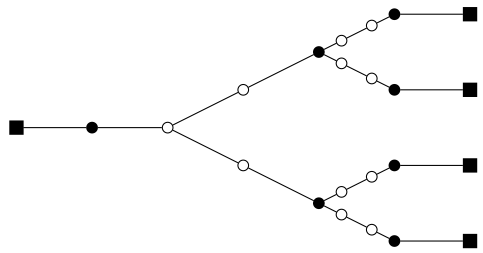

Figure 8: A peculiar critical tree which is not special, where a minimum dominating set contains vertices indicated by filled circles and leaves are indicated by squares.

Proof. Notice that $\gamma_c(G) = 3\gamma(G) - 2$. Let G' be a proper induced subgraph of G. We can assume that G' is connected, because otherwise

$$\frac{\gamma_c(G')}{\gamma(G')} = \frac{\sum_{i=1}^{k} \gamma_c(G[C_i])}{\sum_{i=1}^{k} \gamma(G[C_i])} \leqslant \max\left\{ \frac{\gamma_c(G[C_i])}{\gamma(G[C_i])} \,\middle|\, i = 1, \ldots, k \right\},$$

where C_i, for $i = 1, \ldots, k$, is the vertex set of connected components in G'. First, we suppose that $\gamma(G') = \gamma(G)$. We know that a minimum connected dominating set of a tree (with one vertex of degree > 1) is the set of vertices with degree at least 2, i.e. all vertices which are not leaves, or in other words, the set of internal vertices. Observe that the number of internal vertices of G' is strictly smaller than this of G because every neighbor of a leaf in G has a degree 2. Thus $\gamma_c(G') < \gamma_c(G)$ and

$$\frac{\gamma_c(G')}{\gamma(G')} < \frac{\gamma_c(G)}{\gamma(G)}.$$

Now, assume that $\gamma(G') > \gamma(G)$. Because G is a tree and G' is connected, a minimum connected dominating set of G' is a subset of the internal vertex set of G, hence $\gamma_c(G') \leqslant \gamma_c(G)$. Trivially, we obtain

$$\frac{\gamma_c(G)}{\gamma(G)} > \frac{\gamma_c(G')}{\gamma(G')}.$$

It remains the case where $\gamma(G') < \gamma(G)$. Since $\gamma_c(G') \leqslant 3\gamma(G') - 2$,

$$\frac{\gamma_c(G')}{\gamma(G')} \leqslant \frac{3\gamma(G') - 2}{\gamma(G')} = 3 - \frac{2}{\gamma(G')} < 3 - \frac{2}{\gamma(G)} = \frac{3\gamma(G) - 2}{\gamma(G)} = \frac{\gamma_c(G)}{\gamma(G)}.$$

□

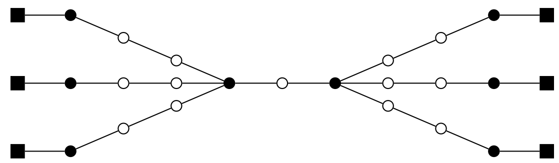

Figure 9: A 2-special critical tree constructed from another tree H (vertices indicated by filled circles) where each edge of H is not necessarily subdivided twice.

Besides, the following proposition gives a necessary condition for a 2-special tree to be critical. First, we prove a short lemma.

Lemma 4. *Every 2-special tree G built on an initial tree H with $|V(H)| \geqslant 3$ contains a proper induced path on 6 vertices.*

Proof. Let P be a longest path in H. Then P has $k \geqslant 3$ vertices, since $|V(H)| \geqslant 3$. Thus, the corresponding path P' in G contains all vertices from P and at least one for each edge. Moreover, P' can be extended in its endpoints by two vertices, by construction of G. Accordingly, G contains a path on $|V(P')|+2 \geqslant (2k-1)+2 \geqslant 7$ vertices. Besides, being a tree implies that any path is induced. Thus, G contains a proper induced path on 6 vertices. □

Proposition 2. *Let G be a 2-special critical tree built on the initial tree H with $V(H)$ a minimum dominating set. Suppose that $|V(H)| \geqslant 3$.*

(i) Let x be a leaf of H and y be its neighbor in H. Then the edge xy is subdivided exactly twice in G.

(ii) Let v be a vertex in H and $N_H(v)$ be its neighborhood. Then there exists at least one $u \in N_H(v)$ such that the edge uv is subdivided exactly twice in G.

Proof. By Lemma 4, observe that $\gamma_c(P_6)/\gamma(P_6) = 2 < \gamma_c(G)/\gamma(G)$ since G is critical. First, we prove (i). Suppose that there exists a leaf x in H and its neighbor y where xy is subdivided exactly once in G. Let x' be the pendent vertex of x in G. Consider $G' = G - \{x, x'\}$ the induced subtree built on $H - \{x\}$. It is easy to see that $\gamma(G) \geqslant \gamma(G') + 1$ and $\gamma_c(G) \leqslant \gamma_c(G') + 2$. Thus,

$$\frac{\gamma_c(G)}{\gamma(G)} \leqslant \frac{\gamma_c(G')+2}{\gamma(G')+1} \leqslant \max\left(\frac{\gamma_c(G')}{\gamma(G')}, 2\right) = \frac{\gamma_c(G')}{\gamma(G')}. \tag{6}$$

The last equality is true because $\gamma_c(G)/\gamma(G) > 2$. Thus, we obtain a contradiction since G is critical.

It remains to show (ii). Let v be a vertex in H. Suppose that for every $u \in N_H(v)$, the edge uv is subdivided only once in G. We can suppose that $d_H(v) \geqslant 2$ because the case of leaves was studied previously. Let $u_1, \ldots, u_k$ be the neighbors of v in H with $k \geqslant 2$ and v_i be the midpoint of the edge $u_i v$ in G, for all i. Consider $C_1, \ldots, C_j$ the vertex sets of the connected components of $G - v$. Since G is a tree, we have that $j = k$ and we can assume that $v_i, u_i \in C_i$ for all i.

Let G_1 be the subgraph of G induced by C_1 and G_2 be the subgraph of G induced by $\cup_{i=2}^k C_i \cup \{v, v_1, u_1\}$. We observe that $V(H) \cap C_1$ is a dominating set of G_1 and $((V(H) \cap V(G_2)) \setminus \{v\}) \cup \{v_1\}$ is a dominating set of G_2. Thus

$$\gamma(G) \geqslant \gamma(G_1) + \gamma(G_2). \tag{7}$$

On the other hand, let $D_{c,1}$ be a minimum connected dominating set of G_1 and $D_{c,2}$ be a minimum connected dominating set of G_2. We can assume that $D_{c,1} \subseteq V(G_1) \setminus \{v_1\}$ and $D_{c,2} \subseteq V(G_2) \setminus \{u_1\}$. It is clear that $u_1 \in D_{c,1}$ and $v_1 \in D_{c,2}$. Thus $D_{c,1} \cup D_{c,2}$ is a connected dominating set of G. Since $D_{c,1} \cap D_{c,2} = \emptyset$,

$$\gamma_c(G) \leqslant \gamma_c(G_1) + \gamma_c(G_2). \tag{8}$$

But (7) and (8) say that

$$\gamma_c(G)/\gamma(G) \leqslant \max\{\gamma_c(G_1)/\gamma(G_1), \gamma_c(G_2)/\gamma(G_2)\}. \tag{9}$$

Since both G_1 and G_2 are induced subgraphs of G, (9) is a contradiction to the fact that G is critical. □

3.2.2 Critical graphs

As proved for the vertex cover problem in Theorem 6, we state a similar result for the dominating set problem.

Theorem 11. *Every minimum dominating set of a critical graph is independent.*

Proof. The proof arises directly from Lemma 5 and Lemma 6 and because every minimum dominating set of C_5, P_5, C_6 or P_6 is independent. □

Lemma 5. *Let G be a critical graph with $\frac{\gamma_c(G)}{\gamma(G)} \leqslant 2$. Then G is isomorphic to one of the following graphs: C_5, P_5, C_6 or P_6.*

Proof. First, we suppose that $\frac{\gamma_c(G)}{\gamma(G)} > \frac{3}{2}$. By Theorem 8, the graph G strictly contains either P_6 or C_6, or is isomorphic to P_6 or to C_6. But in the first case,

$$\frac{\gamma_c(P_6)}{\gamma(P_6)} = \frac{\gamma_c(C_6)}{\gamma(C_6)} = 2 \geqslant \frac{\gamma_c(G)}{\gamma(G)},$$

which is a contradiction with G critical graph. So, $G \cong P_6$ or $G \cong C_6$.

Now, we prove that if $\frac{\gamma_c(G)}{\gamma(G)} \leqslant \frac{3}{2}$ then G is isomorphic either to C_5, or to P_5. Since

$$1 \leqslant \frac{\gamma_c(H)}{\gamma(H)} < \frac{\gamma_c(G)}{\gamma(G)}$$

for all induced subgraph H of G, we deduce that by Theorem 7, G is not (C_5, P_5)-free. Accordingly, G strictly contains either C_5 or P_5, or G is isomorphic to C_5 or to P_5. In the first case,

$$\frac{\gamma_c(C_5)}{\gamma(C_5)} = \frac{\gamma_c(P_5)}{\gamma(P_5)} = \frac{3}{2} \geqslant \frac{\gamma_c(G)}{\gamma(G)},$$

which is again in contradiction with G critical graph. Thus $G \cong C_5$ or $G \cong P_5$. □

Lemma 6. *Let G be a critical graph with $\frac{\gamma_c(G)}{\gamma(G)} > 2$. Every minimum dominating set of G is independent.*

Proof. Let X be a minimum dominating set of G. We suppose that two adjacent vertices u and v are in X. Consider $H = G \setminus (N_G[u] \cup N_G[v])$ the resulting graph after deleting the closed neighborhood of vertices u and v. Since $X \setminus \{u, v\}$ is a dominating set of H,

$$\gamma(G) \geqslant \gamma(H) + 2.$$

Besides, $\gamma_c(G) \leqslant \gamma_c(H) + 4$ because any connected dominating set of H with u, v and two additional vertices to obtain a connected component gives a connected dominating set of G. Accordingly,

$$2 < \frac{\gamma_c(G)}{\gamma(G)} \leqslant \frac{\gamma_c(H) + 4}{\gamma(H) + 2}.$$

Because

$$\frac{\gamma_c(H) + 4}{\gamma(H) + 2} \in \left[2, \frac{\gamma_c(H)}{\gamma(H)}\right],$$

we obtain that $\frac{\gamma_c(H)}{\gamma(H)} = \max\left(2, \frac{\gamma_c(H)}{\gamma(H)}\right)$ and so $\frac{\gamma_c(G)}{\gamma(G)} \leqslant \frac{\gamma_c(H)}{\gamma(H)}$, which is a contradiction with G critical. □

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