Vertex and Edge Residual Mean Distances: New Resilience Measures for Telecommunication Networks

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Abstract

Any telecommunication network is subject to a node or link failure at any given time. Such a failure may impact the quality of the services provided by the network, and therefore the network resilience. In this paper, we define two new measures for evaluating network resilience with respect to a node or link failure: the vertex residual mean distance and the edge residual mean distance, in short RMDs. The RMDs are graph invariants which measure the impact of the removal of an arbitrary vertex/edge on the mean distance of the original graph. Some authors investigated different graph invariants before and after a vertex or an edge removal. However, to our knowledge, none of them incorporated graph invariant with vertex/edge removal. We study the RMDs on diverse graph classes, such as cycle graphs, complete graphs, twin graphs and k-geodetically connected graphs. We establish tight lower bound of the RMDs and graphs reaching them, as well as non-tight upper bound of the RMDs. Moreover, we conjecture tight upper bounds of the RMDs and graphs reaching them. From our results, we point out a family of graphs, i.e. twin graphs, that minimize the vertex residual mean distance and for which the edge residual mean distance is almost as small as that of the complete graph. Notice that each twin graph on n vertices has only $2n-4$ edges, compared to the $n(n-1)/2$ edges of the complete graph. Thus, twin graphs require relatively few edges to provide great levels of resilience. Since, in telecommunications, the number of edges is related to the cost of deploying the network, these results make it interesting to apply twin graphs for the physical topology design of telecommunication networks.

Keywords: Cycle graphs, Complete graphs, Twin graphs, Distance, Wiener index, Transmission, Telecommunication networks, Node failure, Link failure, Resilience.

1 Introduction

When designing a survivable telecommunication network, it is crucial to ensure that each pair of nodes belongs to at least one cycle, so the traffic can be rerouted in the case of a single node or link failure. However, the length of these cycles may impact the quality of the services provided by the network. If we define resilience as the capability of the network to provide a good service quality after link or node failure, the size of the cycle to which belong each pair of vertices is related to network resilienc. The shorter the cycles

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are, the lower the impact, and the higher the resilience of the topology.

It is immediate to see that the cycle graph and the complete graph are extremal topologies with respect to that impact. In a cycle graph, all pairs of nodes belong to the same cycle and they share the protection resources, whereas in a complete graph, each pair of nodes belongs to at least a triangle and has its own protection resources.

The length of the cycle relying each pair of nodes sums up the length of the working path (i.e. the path used during normal network operation) and the length of the backup path (i.e. the path used when a failure occurs). Thus, the impact is proportional to the relative difference between the lengths of the working and the backup paths. Therefore, by averaging the relative distances before and after each possible failure, for all node pairs, it is possible to estimate the lengths of the cycles, so the resilence of the topology.

To measure the resilience of a graph, we propose in this paper the *vertex* and the *edge residual mean distances* as new graph invariants based on the average distances after a single vertex or a edge removal. By means of these invariants, it is possible to measure how far is a given topology from cycles and complete graphs, with respect to resilience. By studying their lower and upper bounds, it is possible to identify families of graphs that minimizes or maximizes resilience. Also, it is possible to use them for solving the physical topology design, given the required level of resilience and the other possible constraints.

In the literature, Najjar and Gaudiot [1] define a probabilistic measure of network fault tolerance whereas Barker, Ramirez-Marquez and Rocco [2] provided two resilience-based component importance measures. Some authors [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] studied diverse invariants, centrality measures or functional robustness after a vertex or an edge removal. However, none of them offer a measure that involves invariants and vertex, or edge, removals. To our knowledge, the vertex and the edge residual mean distances are the only such measures.

This paper is organized as follows. Section 2 introduces the main concepts used through the paper. In Section 3, we present our results. Firstly, we define in Subsection 3.1 the vertex and edge residual mean distances. Secondly, we compute them for some graph classes in Subsection 3.2. Thirdly, we establish in Subsection 3.3 lower and upper bounds on the residual mean distances. Finally, in Section 3.4 we present and analyze vertex and edge residual mean distances results of all 2-connected graphs up to 9 vertices. Section 4 brings our conclusions and future research directions.

2 Preliminaries

A *graph* is a mathematical structure $G = (\mathrm{V}(G), \mathrm{E}(G))$, where $\mathrm{V}(G)$ and $\mathrm{E}(G)$ are referred to as the *vertex set* of G and the *edge set* of G, respectively. Naturally, the elements of $\mathrm{V}(G)$ are called *vertices* and the elements of $\mathrm{E}(G)$ are called *edges*. Every edge is composed of two vertices related to each other. It is well known that when G is a simple graph (i.e. a graph without directed or multiple edges and without loops), then $\mathrm{E}(G) \subseteq \{uv | u, v \in \mathrm{V}(G)\}$. In this paper, the term *order* refers to the number of vertices and is denoted by $n = |V(G)|$, whereas the term *size* refers to the number of edges and is denoted by $m = |E(G)|$. All graphs under study are simple graphs of order $n \geqslant 4$, and we consider $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$.

Let G be a graph with vertex set $\mathrm{V}(G)$. Given two vertices v_1 and v_2 in $\mathrm{V}(G)$, a (v_1, v_2)-*geodesic* in G is a minimum-size path that connects v_1 to v_2. The *distance* between v_1 and v_2 corresponds to the size of a (v_1, v_2)-geodesic. As a matter of notation, we use $\mathrm{d}_G(\cdot, \cdot)$ for expressing mathematically the distance between two vertices in the graph G. If there is no path connecting two vertices v_1 and v_2 in G then $\mathrm{d}_G(v_1, v_2) = \infty$. A graph G is *connected* if, for every pair of vertices v_1, $v_2 \in \mathrm{V}(G)$, there exists a path between v_1 and v_2 in G. Equivalently, $\mathrm{d}(v_1, v_2) < \infty$ for all $v_1, v_2 \in V(G)$. If there is at least one pair of nodes $v_1, v_2 \in V(G)$

such taht $\mathrm{d}(v_1, v_2) = \infty$, we say that G is *disconnected*.

The *vertex connectivity* of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal from G results in a disconnected graph (or in a single vertex). A connected graph G is *k-connected* if $\kappa(G) \geqslant k$. Analogously, the *edge connectivity* of a graph G, denoted by $\kappa'(G)$, is the minimum number of edges whose removal from G results in a disconnected graph. A connected graph G is *k-edge-connected* if $\kappa'(G) \geqslant k$. The k-connected graphs can be characterized as follows. G is a k-connected graph if and only if each pair of vertives in G is connected by at least k vertex-disjoint paths [13].

A k-connected graph G is called *k-geodetically connected* (k-GC) if the removal from G of at least k vertices is required to increase the distance between any pair of non-adjacent vertices in G [14]. The k-GC graphs can be characterized as follows. G is a k-GC graph if and only if each pair of non-adjacent vertices in G is connected by at least k vertex-disjoint geodesics [14]. Notice that the recognition of geodetically connected graphs can be done in polynomial-time [15]. Moreover, as mentioned in [14, 16], an equivalent definition of k-GC graph is *k-geodetically edge-connected* (k-GEC), i.e. the removal from G of at least k edges is required to increase the distance between any pair of non-adjacent vertices in G.

The minimum-size 2-GC graphs are also known as minimum self-repairing graphs, and they were fully characterized by Farley and Proskurowski [17], as follows: apart from K_3, i.e., the complete graph on 3 vertices, and Q_3, i.e., the cube graph on 8 vertices, the *twin graphs* are the only minimum-size 2-GC graphs. The *twin graphs* are recursively defined as follows: the C_4, i.e., the cycle graph on 4 vertices, is a twin graph; if $\amalg$ is a twin graph, the graph obtained by connecting a new vertex to a *twin pair* in $\amalg$ using two new edges is also a twin graph, where a *twin pair* is a pair of vertices with the same open neighbourhood. In general many different twin graphs on $n+1$ vertices can be generated from a given twin graph on n vertices, since twin pairs are not unique. From the definition, it follows that every twin graph is bipartite and planar with every face isomorphic to a cycle C_4. Moreover, the size of any twin graph on n vertices is given by $m = 2(n-2)$ [17].

For every vertices v_i and v_j in $\mathrm{V}(G)$, one can naturally define a matrix $\mathbf{d}_{ij}(G) := d_G(v_i, v_j)$, i.e. a matrix in which every element at the ith row and jth column is the distance between the vertices v_i and v_j in G. We shall call $\mathbf{d}(G)$ the *distance matrix* of a graph G.

Considering a graph G on n vertices, the *transmission* $\mathrm{T}_G(v_i)$ of a vertex v_i in G is given by $\mathrm{T}_G(v_i) := \sum_{j=1}^{n} \mathbf{d}_{ij}(G)$ [18]. Then, the *Wiener index* $\mathrm{W}(G)$ of the graph G is defined as [19]:

$$\mathrm{W}(G) := \frac{1}{2}\sum_{i=1}^{n} \mathrm{T}_G(v_i).$$

Given a graph $G = (\mathrm{V}(G), \mathrm{E}(G))$ on n vertices, the *mean distance* of G can be written as:

$$\bar{\mathrm{d}}(G) = \frac{\sum_{i=1}^{n}\sum_{j>i} \mathbf{d}_{ij}(G)}{\frac{n(n-1)}{2}} = \frac{2\,\mathrm{W}(G)}{n(n-1)} = \frac{\mathrm{W}(G)}{\mathrm{W}(K_n)},$$

where K_n denotes the complete graph on n vertices.

3 Results

In this section, we define our metrics, state and prove some results for particular graph classes and some other results that ensure the coherency of the applications: lower and upper bounds on the residual mean distances are established whereas a computational analysis of these latter is made at the end of this section.

3.1 Measuring the impact of single vertex or edge removal

In order to measure the behavior of graphs against single vertex or edge removal, we define what we call *residual mean distances* (RMDs for short), as follows in Definition 1 and in Definition 2. Given a vertex $v \in \mathrm{V}(G)$, let us denote by $G \setminus \{v\}$ (or simply $G \setminus v$) the graph obtained from G by deleting its vertex v and all edges adjacent to v. Analogously, given an edge $e \in \mathrm{E}(G)$, we denote by $G \setminus \{e\}$ (or simply $G \setminus e$) the graph obtained from G by deleting its edge e.

Definition 1 *Let G be a 2-connected graph. The* vertex residual mean distance *of G, denoted by $\varepsilon_v(G)$, is defined as:*

$$\varepsilon_v(G) := \frac{1}{n} \sum_{i=1}^{n} \left[\bar{\mathrm{d}}\,(G \setminus v_i) - \bar{\mathrm{d}}\,(G)\right].$$

An equivalent definition of the vertex residual mean distance of G is

$$\varepsilon_v(G) := \frac{1}{n} \left[\sum_{i=1}^{n} \bar{\mathrm{d}}\,(G \setminus v_i)\right] - \bar{\mathrm{d}}\,(G).$$

Since $\varepsilon_v(G)$ averages the difference between the mean distance of G and its mean distances by every vertex removal, we propose it as a measure of how vertex failures impact a network topology, in terms of distances. Using this definition, if G has vertex connectivity 1, there is a vertex in G whose removal results in a disconnected graph. In this case, the impact of the removal is considered huge and $\varepsilon_v(G) = \infty$. Otherwise, if G is 2-connected, the mean distance may increase or decrease when a vertex is removed.

As edge failures are also common in many applications, we define the edge residual mean distance of a graph G, which is denoted by $\varepsilon_e(G)$. This invariant is analogue to the ε_v for the case of edge removal. In the same way, if G has edge connectivity 1, there is an edge in G whose removal results in a disconnected graph. In this case, the impact of the removal is considered huge and $\varepsilon_e(G) = \infty$. If G is 2-edge-connected, the mean distance always increases when an edge is removed.

Definition 2 *Let G be a 2-edge-connected graph. The* edge residual mean distance *ε_e of G is defined as:*

$$\varepsilon_e(G) := \frac{1}{m} \sum_{i=1}^{m} \left[\bar{\mathrm{d}}\,(G \setminus e_i) - \bar{\mathrm{d}}\,(G)\right].$$

In words, $\varepsilon_e(G)$ is the average of the differences between the mean distances by every edge removal, and the mean distance of the original graph G.

An equivalent definition of the edge residual mean distance of G is

$$\varepsilon_e(G) := \frac{1}{m} \left[\sum_{i=1}^{m} \bar{\mathrm{d}}\,(G \setminus e_i)\right] - \bar{\mathrm{d}}\,(G).$$

3.2 RMDs of families of graphs

We first take a look on what happens with K_n, the complete graph on n vertices, which is expected to be the most resilient topology against any kind of removal. Next, we focuse on C_n, the cycle graph on n vertices, which is expected to be the less resilient topology against any kind of removal. Finally, the study of k-GC graphs exposes that these graphs, generally less dense than the complete graphs, have good resilience.

3.2.1 RMDs of complete graphs

Theorem 1 *Let K_n be the complete graph of order n. It holds that:*

$$\varepsilon_v(K_n) = 0, \tag{1}$$
$$\varepsilon_e(K_n) = \frac{1}{\mathrm{W}(K_n)} = \frac{2}{n(n-1)}. \tag{2}$$

Proof *Notice that the removal of a vertex in the graph K_n gives us the resulting graph K_{n-1}. So, as $\bar{\mathrm{d}}(K_n) = 1$, $n \geqslant 3$, we have*

$$\begin{aligned}\varepsilon_v(K_n) &= \frac{1}{n}\sum_{i=1}^{n}\left[\bar{\mathrm{d}}(K_n \setminus v_i) - \bar{\mathrm{d}}(K_n)\right]\\ &= \frac{1}{n}\sum_{i=1}^{n}(1-1)\\ &= 0.\end{aligned}$$

Now, notice that the removal of an edge in K_n increases the Wiener index of the graph by 1. Therefore, $\bar{\mathrm{d}}(K_n \setminus e) = 1 + \frac{2}{n(n-1)}$ and

$$\begin{aligned}\varepsilon_e(K_n) &= \frac{1}{m}\sum_{i=1}^{m}\left(\bar{\mathrm{d}}(K_n \setminus e) - \bar{\mathrm{d}}(K_n)\right)\\ &= \frac{2}{n(n-1)}\sum_{i=1}^{m}\left[\left(1 + \frac{2}{n(n-1)}\right) - 1\right]\\ &= \frac{2}{n(n-1)}\\ &= \frac{1}{\mathrm{W}(K_n)}.\end{aligned}$$

□

3.2.2 RMDs of cycle graphs

Theorem 2 *Let C_n be the cycle graph of order n. It holds that:*

$$\varepsilon_v(C_n) = \begin{cases} \frac{n(n-4)}{12(n-1)}, & \text{if } n \text{ even} \\ \frac{n-3}{12}, & \text{if } n \text{ odd} \end{cases} \tag{3}$$

$$\varepsilon_e(C_n) = \begin{cases} \frac{n^2-4}{12(n-1)}, & \text{if } n \text{ even} \\ \frac{n+1}{12}. & \text{if } n \text{ odd} \end{cases} \tag{4}$$

Proof *Notice that when a vertex v is removed from the cycle graph C_n, the resulting graph is the path graph P_{n-1} on $n-1$ vertices, for which the Wiener index is given by:*

$$\mathrm{W}(C_n \setminus v) = \mathrm{W}(P_{n-1}) = \sum_{k=1}^{n-2} k(n-1-k) = \frac{n(n-1)(n-2)}{6}.$$

From the last equality, we can deduce that $\bar{\mathrm{d}}(C_n \setminus v) = \frac{n}{3}$, *for any vertex* v *in* C_n.

Besides, notice that after removing an edge e *from the cycle graph* C_n, *the resulting graph is the path graph* P_n *on* n *vertices and*

$$\mathrm{W}(C_n \setminus e) = \mathrm{W}(P_n) = \frac{(n+1)n(n-1)}{6} = \frac{n(n^2-1)}{6}.$$

From the previous equality, we can deduce that $\bar{\mathrm{d}}(C_n \setminus e) = \frac{n+1}{3}$, *for any edge* e *in* C_n.

Now, to prove the theorem, we will consider two cases, depending on the parity of n*:*

1. *When* n *is even:*

   First, we have, for all $v \in V(C_n)$,

   $$\mathrm{T}_{C_n}(v) = \frac{n}{2} + 2\sum_{k=1}^{\frac{n}{2}-1} k = \frac{n}{2} + (\frac{n}{2}-1)\frac{n}{2} = \frac{n^2}{4},$$

   and

   $$\mathrm{W}(C_n) = \frac{1}{2}\sum_{i=1}^{n} \mathrm{T}_{C_n}(v_i) = \frac{1}{2}\sum_{i=1}^{n} \frac{n^2}{4} = \frac{n^3}{8},$$

   and so

   $$\bar{\mathrm{d}}(C_n) = \frac{n^2}{4(n-1)}.$$

   Therefore, we obtain

   $$\varepsilon_v(C_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{n}{3} - \frac{n^2}{4(n-1)}\right] = \frac{n(n-4)}{12(n-1)}$$

   $$\varepsilon_e(C_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{n+1}{3} - \frac{n^2}{4(n-1)}\right] = \frac{n^2-4}{12(n-1)}.$$

2. *When* n *is odd:*

   For any vertex $v \in V(C_n)$, *the transmission of* v *in* C_n *is:*

   $$\mathrm{T}_{C_n}(v) = 2\sum_{k=1}^{\frac{n-1}{2}} k = \frac{(n-1)(n+1)}{4} = \frac{n^2-1}{4}.$$

   In addition, the Wiener index of C_n *is*

   $$\mathrm{W}(C_n) = \frac{1}{2}\sum_{i=1}^{n} \frac{n^2-1}{4} = \frac{n(n^2-1)}{8},$$

   which implies that the mean distance of C_n *is*

   $$\bar{\mathrm{d}}(C_n) = \frac{n+1}{4}.$$

Consequently, the residual mean distances become:

$$\varepsilon_v(C_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{n}{3} - \frac{n+1}{4}\right] = \frac{n-3}{12},$$

$$\varepsilon_e(C_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{n+1}{3} - \frac{n+1}{4}\right] = \frac{n+1}{12}.$$

□

One can verify that for both vertex and edge residual mean distances, the extension to the real number system $\mathbb{R}_{>1}$ (i.e., the set of all real numbers greater than 1) of the functions that give us the values of ε_v and ε_e for odd cycles is an oblique asymptote of the extension functions for even cycles.

3.2.3 RMDs of k-GC graphs

By definition of k-GC graphs, for a fixed $k \geqslant 2$, the removal of a single vertex in any k-GC graph has a local impact, as it does not change the distances between other pairs of vertices. From this observation, we provide in Lemma 1 an alternative characterization of the k-GC graphs, using the concepts of transmission and Wiener index. Figure 1 illustrates the ideas shown in this lemma for a k-GC graph. Then, using this lemma, we prove Theorem 3, which gives us the value of the ε_v for k-GC graphs.

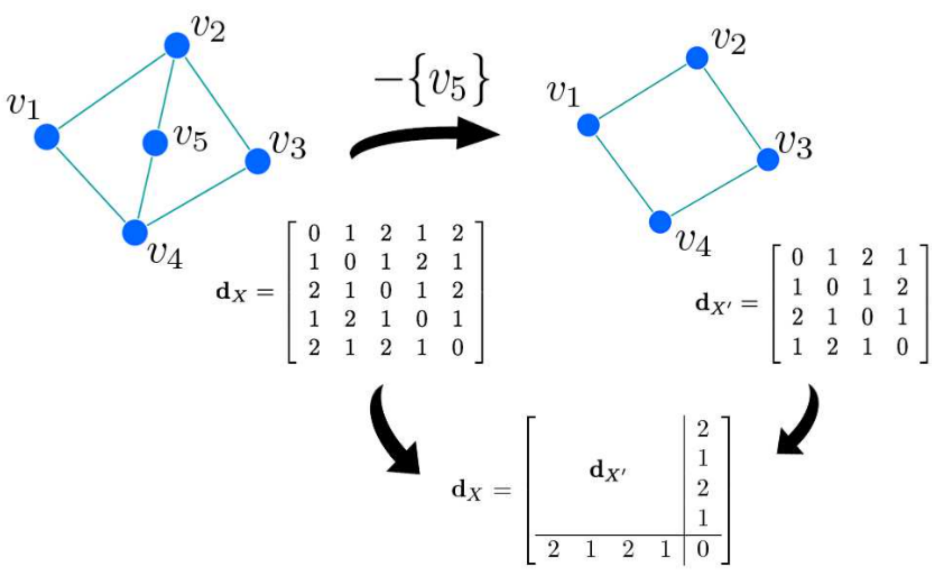


Figure 1: For a k-GC graph X and $v \in \mathrm{V}(X)$ let $X' = X \setminus v$. The matrix $\mathbf{d}_{X'}$ is a submatrix of the matrix $\mathbf{d}_X$.

Lemma 1 *Let G be a k-GC graph of order n, $k \geqslant 2$, and consider $v \in \mathrm{V}(G)$. Then,*

$$\mathrm{W}(G \setminus v) = \mathrm{W}(G) - \mathrm{T}_G(v).$$

Proof *Since G is a k-GC graph, $k \geqslant 2$, there are at least k disjoint geodesics connecting each pair of non-adjacent vertices in G. Thus, the removal of a single vertex v will not change the distance between any pair* *of vertices $u, w \neq v$. Therefore:*

$$\begin{aligned}
\mathrm{W}(G) &= \frac{1}{2}\,\mathrm{T}_G(v) + \frac{1}{2}\sum_{u\in \mathrm{V}(G\setminus v)} \mathrm{T}_G(u) \\
&= \frac{1}{2}\sum_{u\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,v) + \frac{1}{2}\sum_{u\in \mathrm{V}(G\setminus v)} \left[\mathrm{d}_G(u,v) + \sum_{w\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,w)\right] \\
&= \frac{1}{2}\sum_{u\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,v) + \frac{1}{2}\sum_{u\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,v) + \frac{1}{2}\sum_{u,w\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,w) \\
&= \sum_{u\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,v) + \frac{1}{2}\sum_{u,w\in \mathrm{V}(G\setminus v)} \mathrm{d}_{G\setminus v}(u,w) \\
&= \mathrm{T}_G(v) + \mathrm{W}(G\setminus v).
\end{aligned}$$

□

Theorem 3 *If G is a k-GC graph of n vertices, then*

$$\varepsilon_v(G) = 0. \tag{5}$$

Proof *Since G is k-GC, by Lemma 1, the equation $\mathrm{W}\left(G\setminus u\right) = \mathrm{W}\left(G\right) - \mathrm{T}_G\left(u\right)$ clearly holds for every vertex u of G. Also, notice that for every $i = 1,\ldots,n$, we have*

$$\bar{\mathrm{d}}\left(G\setminus v_i\right) = \frac{2}{\left(n-1\right)\left(n-2\right)}\mathrm{W}\left(G\setminus v_i\right) = \frac{2}{\left(n-1\right)\left(n-2\right)}\left(\mathrm{W}\left(G\right) - \mathrm{T}_G\left(v_i\right)\right).$$

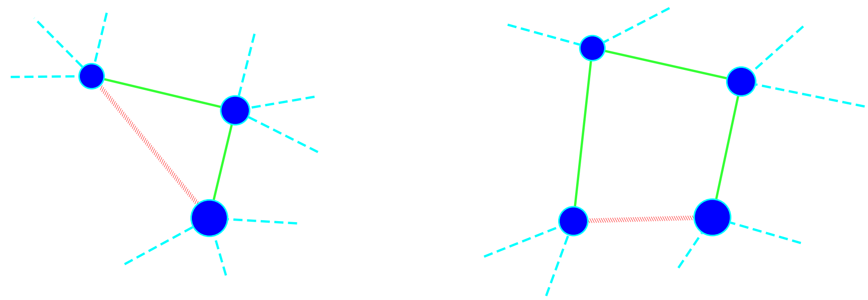
Then,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\bar{\mathrm{d}}\left(G\setminus v_i\right) &= \frac{2}{n\left(n-1\right)\left(n-2\right)}\sum_{i=1}^{n}\left(\mathrm{W}\left(G\right) - \mathrm{T}_G\left(v_i\right)\right) \\
&= \frac{2}{n(n-1)(n-2)}\left(n\mathrm{W}(G) - \sum_{i=1}^{n}\mathrm{T}_G(v_i)\right) \\
&= \frac{2}{n(n-1)(n-2)}\left(n\mathrm{W}(G) - 2\mathrm{W}(G)\right) \\
&= \frac{2\mathrm{W}\left(G\right)}{n\left(n-1\right)\left(n-2\right)}\left(n-2\right) \\
&= \frac{2\mathrm{W}\left(G\right)}{n\left(n-1\right)} \\
&= \bar{\mathrm{d}}\left(G\right).
\end{aligned}$$

So, the vertex residual mean distance in k-GC graphs is always 0. □

In words, if G is a k-GC graph, the mean of the mean distances of G after the removal of each vertex equals the mean distance of G. It does not mean, however, that the mean distance of G remains unchanged after removing any vertex. Indeed, the mean distance of a k-GC graph may increase or decrease after a single vertex removal.

For the edge removal, we see the behavior of k-GC graphs thanks to the existence of a C_4 subgraph between any pair of adjacent vertices. Indeed, in Figure 2a, the existence of a triangle given any 3 vertices has as consequence the minimal impact by edge failure, increasing the Wiener index by 1 when the failure occurs. In Figure 2b, the existence of a cycle C_4 makes the impact being the increment of the Wiener index by 2 instead of 1 when one of the 4 edges in the cycle fails.



(a) Triangle in an arbitrary graph. The existence of a triangle given any 3 vertices has as consequence the minimal impact by edge failure, increasing the Wiener index by 1 when the failure occurs.

(b) In a k-GC graph, given two vertices distance 2 apart, they have at least 2 common neighbors and form a cycle on 4 vertices, which makes the impact being the increment of the Wiener index by 2 instead of 1 when one of the 4 edges in the cycle fails.

Figure 2: Situation after an edge removal: in graphs with a triangle (a), and in k-GC graphs (b).

We bound the edge residual mean distance of k-GC graphs in Theorem 4. First, we prove the following useful lemma.

Lemma 2 *If G is a k-GC graph, then for every edge e in $E(G)$ the following equation holds:*

$$\mathrm{W}(G \setminus e) = \begin{cases} \mathrm{W}(G) + 1 & \text{if } e \text{ belongs to } C_3 \\ \mathrm{W}(G) + 2 & \text{otherwise.} \end{cases}$$

Proof *Let e be an edge of G with endpoints u and v. The removal of e implies that*

$$\mathrm{T}_{G\setminus e}(u) = \begin{cases} \mathrm{T}_G(u) + 1 & \text{if } e \text{ belongs to } C_3 \\ \mathrm{T}_G(u) + 2 & \text{otherwise} \end{cases}$$

and

$$\mathrm{T}_{G\setminus e}(v) = \begin{cases} \mathrm{T}_G(v) + 1 & \text{if } e \text{ belongs to } C_3 \\ \mathrm{T}_G(v) + 2 & \text{otherwise} \end{cases}$$

since when e does not belong to a triangle, e must belong to a subgraph isomorphic to C_4 because G is k-GC. Also, notice that if $w \in \mathrm{V}(G) \setminus \{u, v\}$, then $\mathrm{T}_{G\setminus e}(w) = \mathrm{T}_G(w)$. □

Theorem 4 *If G is a k-GC graph on n vertices, then:*

$$\frac{2}{n(n-1)} \leqslant \varepsilon_e(G) \leqslant \frac{4}{n(n-1)}. \tag{6}$$

Moreover, the upper bound is reached if and only if G is C_3-free, in particular if G is a twin graph, and the lower bound is reached if and only if every edge of G belongs to a triangle.

Proof *Since G is a k-GC graph on m edges and since*

$$\begin{aligned}\varepsilon_e(G) &= \frac{1}{m}\sum_{i=1}^{m} \bar{\mathrm{d}}\,(G \setminus e_i) - \bar{\mathrm{d}}\,(G) \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{2\mathrm{W}(G \setminus e_i)}{n(n-1)} - \bar{\mathrm{d}}(G) \\ &= \frac{2}{mn(n-1)}\sum_{i=1}^{m} \mathrm{W}(G \setminus e_i) - \bar{\mathrm{d}}(G),\end{aligned}$$

we obtain

$$\begin{array}{rcccl} \frac{2}{mn(n-1)}\sum_{i=1}^{m}(\mathrm{W}(G)+1) - \bar{\mathrm{d}}(G) & \leqslant & \varepsilon_e(G) & \leqslant & \frac{2}{mn(n-1)}\sum_{i=1}^{m}(\mathrm{W}(G)+2) - \bar{\mathrm{d}}(G) \\ \frac{2}{n(n-1)}(\mathrm{W}(G)+1) - \bar{\mathrm{d}}(G) & \leqslant & \varepsilon_e(G) & \leqslant & \frac{2}{n(n-1)}(\mathrm{W}(G)+2) - \bar{\mathrm{d}}(G) \\ \bar{\mathrm{d}}(G) + \frac{2}{n(n-1)} - \bar{\mathrm{d}}(G) & \leqslant & \varepsilon_e(G) & \leqslant & \bar{\mathrm{d}}(G) + \frac{4}{n(n-1)} - \bar{\mathrm{d}}(G) \\ \frac{2}{n(n-1)} & \leqslant & \varepsilon_e(G) & \leqslant & \frac{4}{n(n-1)}. \end{array}$$

By Lemma 2, $\varepsilon_e(G)$ reaches the upper bound if and only if G is C_3-free and the lower bound if and only if every edge of G belongs to a triangle. □

Worth noting, the last theorems give us the following results. For every k-GC graph G on n vertices, we have:

$$\varepsilon_v(G) = \varepsilon_v(K_n) = 0,$$
$$\varepsilon_e(K_n) \leqslant \varepsilon_e(G) \leqslant 2\varepsilon_e(K_n).$$

Thus, by the study of the residual mean distances, the k-GC graphs behave exactly such as the complete graphs regarding single vertex removal and very close to complete graphs regarding single edge removal, which makes them a great choice for the topology design of a telecommunication network.

Since edges in general refer to cost, the topology design focuses on minimum-size graphs. Notice that, even though the impact on the edge residual mean distance is greater for twin graphs on n vertices than for K_n, each twin graph has only $2n-4$ edges, a linear number compared to the quadratic number of edges in the complete graph, i.e. $(n^2-n)/2$ edges. Thus, twin graphs are good candidates for telecommunication network design. It is also important to observe that, for each fixed n, there is a family of twin graphs on n vertices reaching these results.

3.3 Lower and upper bounds of the RMDs

As expected, the lower bound on both edge and vertex residual mean distances is attained by K_n, as shown in Theorem 6. Fortunately, much less edges are needed in order to get the same level of resilience provided by the complete graph, for both vertex and edge removal. For this reason, we introduce a new graph: let $K^*_{2,n-2}$ be the complete split graph with vertex partition $(2, n-2)$, i.e. the complete bipartite graph $K_{2,n-2}$ with an edge added between the vertices of degree $n-2$. In this section, we first study the vertex and edge residual mean distances of $K^*_{2,n-2}$. Then Theorem 6 bounds the RMDs of any graph and provides a characterization of the minimum-size graphs reaching the lower bound for the RMDs. Moreover, we state a conjecture on a tight upper bound of the RMDs and on the minimum-size graphs reaching this bound.

Theorem 5 *Let $K^*_{2,n-2}$ be the complete split graph with the vertex partition $(2, n-2)$. It holds that:*

$$\varepsilon_v\left(K^*_{2,n-2}\right) = 0,$$
$$\varepsilon_e\left(K^*_{2,n-2}\right) = \frac{2}{n(n-1)} = \frac{1}{\mathrm{W}(K_n)}.$$

Proof *Let consider the complete split graph as depicted in Figure 3 with the partition (A, B).*

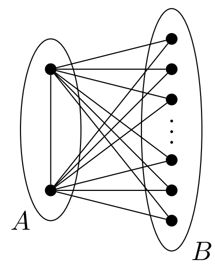


Figure 3: The complete split graph $K^*_{2,n-2}$ with vertex partition (A, B).

First, we compute its vertex residual mean distance. Let $v_A \in A$ and $v_B \in B$. Then

$$\begin{array}{rcl}
\mathrm{W}(K^*_{2,n-2}) & = & (n-1) + (n-2) + 2\left(\frac{n(n-1)}{2} - (2n-3)\right), \\
 & = & n^2 - 3n + 3, \\
\mathrm{W}(K^*_{2,n-2} \setminus v_A) & = & \mathrm{W}(K_{1,n-2}) \\
 & = & (n-2) + 2\frac{(n-2)(n-3)}{2} \\
 & = & (n-2)^2, \\
\mathrm{W}(K^*_{2,n-2} \setminus v_B) & = & \mathrm{W}(K^*_{2,n-3}) \\
 & = & (n-1)^2 - 3(n-1) + 3 \\
 & = & n^2 - 5n + 7.
\end{array}$$

Therefore, we obtain the following mean distances:

$$\begin{array}{rcl}
\bar{\mathrm{d}}(K^*_{2,n-2}) & = & \frac{2n^2-6n+6}{n^2-n}, \\
\bar{\mathrm{d}}(K^*_{2,n-2} \setminus v_A) & = & \frac{2n-4}{n-1}, \\
\bar{\mathrm{d}}(K^*_{2,n-2} \setminus v_B) & = & \frac{2n^2-10n+14}{(n-1)(n-2)}.
\end{array}$$

Accordingly,

$$\varepsilon_v(K^*_{2,n-2}) = \frac{1}{n}\left(2\left(\frac{2n-4}{n-1}\right) + (n-2)\frac{2n^2-10n+14}{(n-1)(n-2)}\right) - \frac{2n^2-6n+6}{n^2-n} = 0.$$

Now, we compute the edge residual mean distance of $K^*_{2,n-2}$. Let e_A be the edge whose both endpoints are in A and let f be any other edge. Since every edge of $K^*_{2,n-2}$ belongs to a triangle, then

$$\mathrm{W}(K^*_{2,n-2} \setminus e_A) = \mathrm{W}(K^*_{2,n-2}) + 1 = \mathrm{W}(K^*_{2,n-2} \setminus f).$$

Accordingly, for any arbitrary edge e, we obtain

$$\mathrm{W}(K^*_{2,n-2} \setminus e) = n^2 - 3n + 4,$$

and

$$\bar{\mathrm{d}}(K^*_{2,n-2} \setminus e) = \frac{2n^2-6n+8}{n^2-n}.$$

Because the graph $K^*_{2,n-2}$ has $2n-3$ edges, its edge residual mean distance is thus

$$\varepsilon_e(K^*_{2,n-2}) = \frac{1}{2n-3}(2n-3)\left(\frac{2n^2-6n+8}{n^2-n}\right) - \frac{2n^2-6n+6}{n^2-n} = \frac{2}{n^2-n} = \frac{1}{\mathrm{W}(K_n)}.$$

□

Theorem 6 *Let G be a 2-connected graph on n vertices, then we have*

$$0 \leqslant \varepsilon_v(G) \leqslant \frac{n-3}{3}.$$

Let G be a 2-edge-connected graph on n vertices and m edges, then we have

$$\frac{2}{n(n-1)} \leqslant \varepsilon_e(G) \leqslant \frac{n-2}{3}.$$

Moreover, the minimum-size graphs reaching the lower bound for ε_v are the minimum-size 2-GC graphs, particularly twin graphs. The minimum-size graphs reaching the lower bound for ε_e are 2-GC graphs on n vertices and $2n-3$ edges such that every edge belongs to a triangle. In particular, the complete split graph with the vertex partition $(2, n-2)$ is such a graph.

Proof *Firstly, we prove the lower bound about the vertex residual mean distance. Let v_i be an arbitrary vertex of G. The distance in $G \setminus v_i$ between any pair of vertices v_s and v_t cannot be smaller than the distance in G between them, i.e. $\mathrm{d}_{G\setminus v_i}(v_s, v_t) \geqslant \mathrm{d}_G(v_s, v_t)$. Accordingly, $\mathrm{W}(G \setminus v_i) \geqslant \mathrm{W}(G) - \mathrm{T}_G(v_i)$. By making the sum on every vertices, we obtain*

$$\begin{aligned}
\sum_{i=1}^{n} \mathrm{W}(G \setminus v_i) &\geqslant \sum_{i=1}^{n} (\mathrm{W}(G) - \mathrm{T}_G(v_i)) \\
&= n\,\mathrm{W}(G) - \sum_{i=1}^{n} \mathrm{T}_G(v_i) \\
&= n\,\mathrm{W}(G) - 2\,\mathrm{W}(G) \\
&= (n-2)\,\mathrm{W}(G).
\end{aligned}$$

Therefore, the following inequalities are equivalent:

$$\begin{aligned}
\sum_{i=1}^{n} \mathrm{W}(G \setminus v_i) &\geqslant (n-2)\,\mathrm{W}(G) \\
\sum_{i=1}^{n} \frac{2\,\mathrm{W}(G \setminus v_i)}{(n-1)(n-2)} &\geqslant n\ \frac{2\,\mathrm{W}(G)}{n(n-1)} \\
\sum_{i=1}^{n} \bar{\mathrm{d}}(G \setminus v_i) &\geqslant n\ \bar{\mathrm{d}}(G).
\end{aligned}$$

And the last inequality is trivially equivalent to $\varepsilon_v(G) \geqslant 0$.

Secondly, we prove the lower bound about the edge residual mean distance. Let e_i be an arbitrary edge of G. The distance in $G \setminus e_i$ between any pair of vertices v_i and v_j cannot be smaller than the distance in G between them, i.e. $\mathrm{d}_G(v_i, v_j) \leqslant \mathrm{d}_{G\setminus e_i}(v_i, v_j)$. Moreover, if e_i is composed by vertices v_s and v_t, then

$\mathrm{d}_G(v_s, v_t) = 1 < \mathrm{d}_{G\setminus e_i}(v_s, v_t)$, *and so* $\mathrm{W}(G \setminus e_i) \geqslant \mathrm{W}(G) + 1$. *Therefore, we obtain*

$$\begin{array}{rcl} \varepsilon_e(G) & = & \frac{1}{m}\sum_{i=1}^{m} \bar{\mathrm{d}}(G \setminus e_i) - \bar{\mathrm{d}}(G) \\ & = & \frac{1}{m}\sum_{i=1}^{m} \frac{\mathrm{W}(G\setminus e_i)}{\mathrm{W}(K_n)} - \frac{\mathrm{W}(G)}{\mathrm{W}(K_n)} \\ & \geqslant & \frac{1}{m}\sum_{i=1}^{m} \frac{\mathrm{W}(G)+1}{\mathrm{W}(K_n)} - \frac{\mathrm{W}(G)}{\mathrm{W}(K_n)} \\ & = & \frac{1}{m}\sum_{i=1}^{m} \frac{1}{\mathrm{W}(K_n)} \\ & = & \frac{1}{\mathrm{W}(K_n)} \\ & = & \frac{2}{n(n-1)}. \end{array}$$

Now, we prove the upper bound on the residual mean distances. We recall that $\mathrm{W}(P_n) = \frac{n(n^2-1)}{6}$ *as proved in Theorem 2. Moreover, as mentioned in [20], for any graph* H *on* n *vertices, we have*

$$\mathrm{W}(K_n) \leqslant \mathrm{W}(H) \leqslant \mathrm{W}(P_n).$$

For the vertex residual mean distance, we obtain thus

$$\begin{array}{rcl} \varepsilon_v(G) & = & \frac{1}{n}\sum_{i=1}^{n} \frac{\mathrm{W}(G\setminus v_i)}{\mathrm{W}(K_{n-1})} - \frac{\mathrm{W}(G)}{\mathrm{W}(K_n)} \\ & \leqslant & \frac{1}{n}\sum_{i=1}^{n} \frac{\mathrm{W}(P_{n-1})}{\mathrm{W}(K_{n-1})} - \frac{\mathrm{W}(K_n)}{\mathrm{W}(K_n)} \\ & = & \frac{\mathrm{W}(P_{n-1})}{\mathrm{W}(K_{n-1})} - 1 \\ & = & \frac{(n-1)((n-1)^2-1)}{6}\frac{2}{(n-1)(n-2)} - 1 \\ & = & \frac{n-3}{3}, \end{array}$$

and for the edge residual mean distance, we have

$$\begin{array}{rcl} \varepsilon_e(G) & = & \frac{1}{m}\sum_{i=1}^{m} \frac{\mathrm{W}(G\setminus e_i)}{\mathrm{W}(K_n)} - \frac{\mathrm{W}(G)}{\mathrm{W}(K_n)} \\ & \leqslant & \frac{1}{m}\sum_{i=1}^{m} \frac{\mathrm{W}(P_n)}{\mathrm{W}(K_n)} - \frac{\mathrm{W}(K_n)}{\mathrm{W}(K_n)} \\ & = & \frac{n(n^2-1)}{6}\frac{2}{n(n-1)} - 1 \\ & = & \frac{n-2}{3}. \end{array}$$

By Lemma 3 and Lemma 4, we know that if the graph is not 2*-GC then both lower bounds are not reached by this graph. Therefore, graphs reaching one of the lower bound must be* 2*-GC. By Theorem 3, the vertex residual mean distance of any* 2*-GC graph is* 0*. Thus the minimum-size graphs reaching the lower bound for* ε_v *are the minimum-size* 2*-GC graphs, i.e.* C_3*, the cube and twin graphs. Notice that twin graphs and the cube are bipartite, in particular* C_3*-free, and have* $2n-4$ *edges. By Theorem 4, among* 2*-GC graphs, the lower bound on the edge residual mean distance is reached by graphs such that every edge belongs to a triangle. Accordingly, such graphs have at least* $2n-3$ *egdes. Moreover,* $K^*_{2,n-2}$ *is a* 2*-GC graph with* n *vertices and* $2n-3$ *edges such that every edge belongs to a triangle. Thus, the minimum-size graph reaching the lower bound for* ε_e *are* 2*-GC graphs on* n *vertices and* $2n-3$ *edges such that every edge belongs to a triangle.* □

Lemma 3 *Let* G *be a graph on* n *vertices. We assume that* G *is not* 2*-GC, i.e. there exist three different vertices* t, s_1 *and* $s_2 \in V(G)$ *such that*

$$\mathrm{d}_{G\setminus t}(s_1, s_2) > \mathrm{d}_G(s_1, s_2).$$

Then

$$0 < \varepsilon_v(G).$$

Proof *From the proof of Lemma 1, we deduce for any $v \in V(G)$,*

$$\begin{aligned}
\mathrm{W}(G) &= \sum_{u\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,v) + \frac{1}{2}\sum_{u,w\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,w) \\
&\leqslant \sum_{u\in \mathrm{V}(G\setminus v)} \mathrm{d}_G(u,v) + \frac{1}{2}\sum_{u,w\in \mathrm{V}(G\setminus v)} \mathrm{d}_{G\setminus v}(u,w) \\
&= \mathrm{T}_G(v) + \mathrm{W}(G\setminus v).
\end{aligned}$$

Thus, $\mathrm{W}(G\setminus v) \geqslant \mathrm{W}(G) - \mathrm{T}_G(v)$ and the inequality is strict if $v = t$. Moreover, by adapting the proof of Theorem 3, we obtain

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \bar{\mathrm{d}}\left(G\setminus v_i\right) &= \frac{2}{n\left(n-1\right)\left(n-2\right)}\sum_{i=1}^{n}(\mathrm{W}(G\setminus v_i)) \\
&> \frac{2}{n\left(n-1\right)\left(n-2\right)}\sum_{i=1}^{n}\left(\mathrm{W}\left(G\right) - \mathrm{T}_G\left(v_i\right)\right) \\
&= \bar{\mathrm{d}}\left(G\right).
\end{aligned}$$

Accordingly, $\varepsilon_v(G) > 0$. □

Lemma 4 *Let G be a graph on n vertices. We assume that G is not 2-GEC, i.e. there exist an edge $f \in E(G)$ and two different vertices s_1 and $s_2 \in V(G)$ such that s_1 and s_2 are not the endpoints of f and*

$$\mathrm{d}_{G\setminus f}(s_1,s_2) > \mathrm{d}_G(s_1,s_2).$$

Then

$$\frac{2}{n(n-1)} < \varepsilon_e(G).$$

Proof *By a similar argumentation from Lemma 2, $\mathrm{W}(G\setminus e) \geqslant \mathrm{W}(G)+1$ and the inequality is strict if $e = f$. Moreover, by Theorem 4, we deduce that*

$$\begin{aligned}
\varepsilon_e(G) &= \tfrac{2}{mn(n-1)}\textstyle\sum_{e\in \mathrm{E}(G)} \mathrm{W}(G\setminus e) - \bar{\mathrm{d}}(G), \\
&> \tfrac{2}{mn(n-1)}\textstyle\sum_{e\in \mathrm{E}(G)}(\mathrm{W}(G)+1) - \bar{\mathrm{d}}(G) \\
&= \tfrac{2}{n(n-1)}.
\end{aligned}$$

□

Lemma 5 *Let G be a 2-GC graph with n vertices and $2n-3$ edges. We assume that every edge in G belongs to a triangle and the maximum degree of G is $n-1$. Then G is isomorphic to $K^*_{2,n-2}$.*

Proof *Let v be a vertex with degree $n-1$ in G. Since G is 2-GC, then $G\setminus v$ is still connected. Moreover, the number of edges in $G\setminus v$ is the number of edges in G minus all edges adjacent to v, i.e. $(2n-3)-(n-1) = n-2$. So $G\setminus v$ is a tree. Besides, the diameter of G is at most 2. Since G is 2-GC, it is also the case of $G\setminus v$. The only tree with diameter at most 2 is a star. Thus $G \cong K^*_{2,n-2}$.* □

From a mathematical point of view, the cycle is the minimum-size graph among all 2-connected graphs. Intuitively, this graph is the less resilient. Therefore, we conjecture that C_n is the only minimum-size graph reaching the upper bound on both vertex and edge residual mean distances. This conjecture is verified for graphs from 4 up to 9 vertices, as observed in Subsection 3.4.

Conjecture 1 *Let G be a 2-connected graph on n vertices, then we have*

$$\varepsilon_v(G) \leqslant \varepsilon_v(C_n)$$

Let G be a 2-edge-connected graph on n vertices and m edges, then we have

$$\varepsilon_e(G) \leqslant \varepsilon_e(C_n).$$

Moreover, the upper bound for minimum-size graphs is reached only by C_n, for both ε_v and ε_e.

We are particularly interested in minimum-size graphs minimizing the impacts of vertex and edge removal, because the number of edges is related to the cost of deploying the network. Notice however that there are many other graphs reaching the lower and the upper bounds of ε_v and ε_e stated in Theorem 6 and in Conjecture 1, if the minimum-size constraint is not required.

3.4 Computational analysis of the RMDs

In this section, we graphically illustrate the results obtained in the previous subsections, highlighting the values of the RMDs for C_n, K_n and twin graphs from the ones for all the 2-connected graphs on a certain number of vertices.

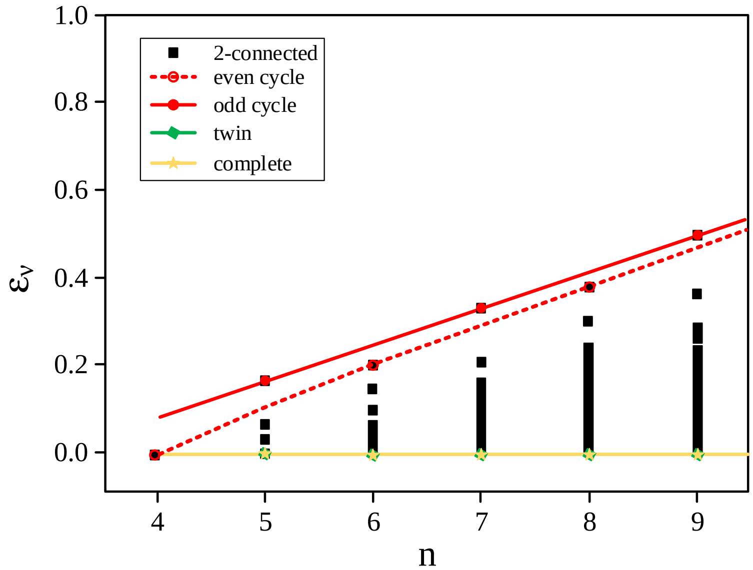
For that purpose, we have computed both vertex and edge residual mean distances for all 2-connected graphs from 4 up to 9 vertices, which totalizes $201,726$ graphs. These graphs were generated by using the Nauty library [21, 22], and the twin graphs were generated by the recursive process found in [17] and stated in Section 2.

Figure 4 shows the values of the RMDs for all 2-connected graphs from 4 up to 9 vertices. As shown in Figure 4a, the values of ε_v for every graph are located between the complete graphs and the cycles. The twin graphs (actually, any k-GC graph) behave such as the complete graphs with respect to vertex failure resilience. For the values of ε_e, illustrated in Figure 4b, the twins $\mathbb{II}$ are close to the complete graphs, as predicted, as $\varepsilon_e(\mathbb{II}) = 2\varepsilon_e(K_n)$ and $\varepsilon_e(K_n)$ decreases as the number of vertices grows.

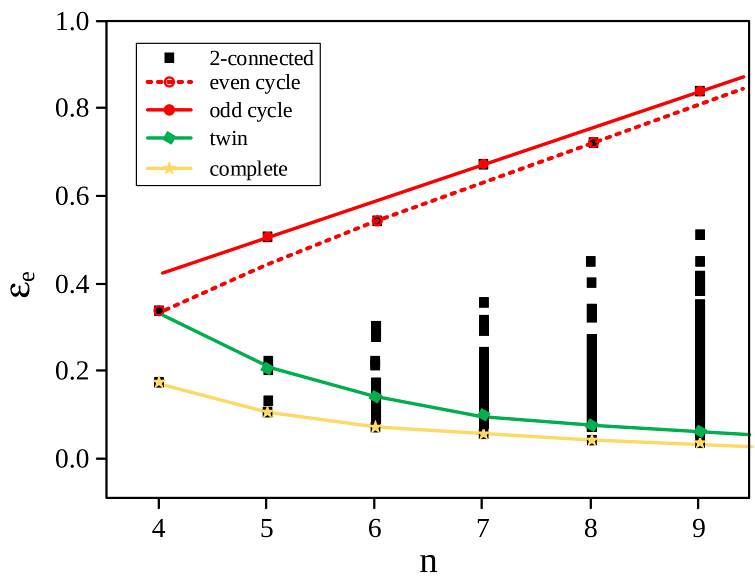
As we can see, the behavior of the proposed measures is coherent with the intuitive prediction. For cycle graphs, greater values of n mean greater difference between the mean distance before and after the removal of a vertex or an edge. But in some graph classes, the removal of a vertex or an edge represents a kind of "local phenomena": as n increases, the impact of the removal becomes more and more negligible to the whole network, making ε_v and ε_e decreasing functions.

In order to see in more details the behavior of the RMDs, we plot in Figure 5 the values of vertex and edge residual mean distances, with respect to the number of edges, for all 2-connected graphs on 9 vertices. Figure 5a shows the values of ε_v, whereas the values of ε_e are shown in Figure 5b. In the first figure, the cycle and the complete graph are alone in their positions, but all the twins are in the same point, since they have the same size and the same value of the ε_v for any fixed number of vertices. An analogous sentence holds for ε_e.

As proved, the twins are the minimum-size graphs that reach $\varepsilon_v = 0$. The twins can not reach the same ε_e of the complete graph, but for their size, i.e. $m = 2n - 4$, they minimize the ε_e. Moreover, as we know, just one more edge, totalizing $m = 2n - 3$, is needed to reach the value of ε_e of the complete graphs. We observe that, although both ε_v and ε_e tend to decrease as the number of edges increases, for a fixed number of vertices, it is not needed to have lots of edges in order to reach their minimum values.

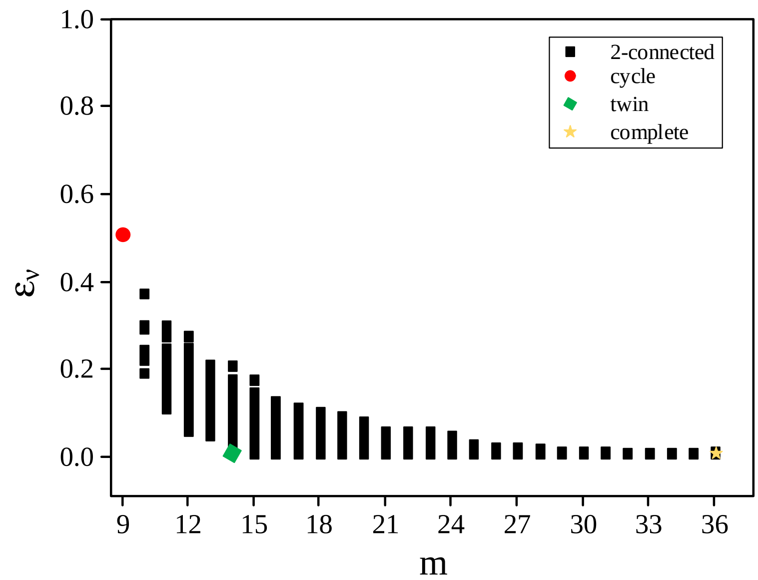


(a) The vertex residual mean distance, ε_v, versus the number of vertices for all 2-connected graphs from 4 up to 9 vertices. Highlighted in red are the graphs maximizing ε_v, i.e. the cycle graphs (Equation 3), and in yellow-green the graphs minimizing ε_v, i.e. the complete graphs (Equation 1), and the twin graphs (Equation 5).

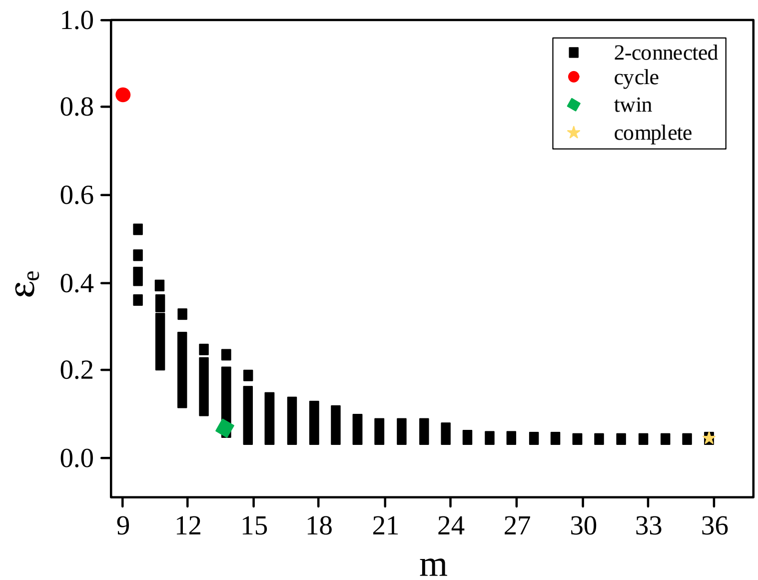


(b) The edge residual mean distance, ε_e, versus the number of vertices for all 2-connected graphs from 4 up to 9 vertices. Highlighted in red are the graphs maximizing ε_e, i.e. the cycle graphs (Equation 4), and in yellow the graphs minimizing ε_e, i.e. the complete graphs (Equation 2). The results for the twin graphs (Equation 6) are highlighted in green.

Figure 4: The vertex residual mean distance (a), and the edge residual mean distance (b), versus the number of vertices for all 2-connected graphs from 4 up to 9 vertices.



(a) The vertex residual mean distance, ε_v, versus the number of edges for all 2-connected graphs on 9 vertices. Highlighted are the cycle, the complete graph, and the twin graphs.



(b) The edge residual mean distance, ε_e, versus the number of edges for all 2-connected graphs on 9 vertices. Highlighted are the cycle, the complete graph, and the twin graphs.

Figure 5: The vertex residual mean distance (a), and the edge residual mean distance (b), versus the number of edges for all 2-connected graphs on 9 vertices.

4 Concluding remarks and future work

As we have seen, among the 2-connected graphs, the ones that minimize, respectively maximize, the number of edges are the cycles and the complete graphs, respectively. For a fixed number of vertices, a twin graph is much closer to a cycle than a complete graph, regarding the number of edges. However, the ε_e of a twin graph is close to the value of the ε_e of a complete graph and its ε_v equals the ε_v of a complete graph, revealing a suitable aspect of the twin graphs: they require relatively few edges and provide great resilience with respect to vertex or edge failure. Figure 6 illustrates the relative position of twin graphs, compared to cycle and complete graph. Notice that these comparisons are out of scale and their main purpose is to illustrate the behavior of the proposed invariants.

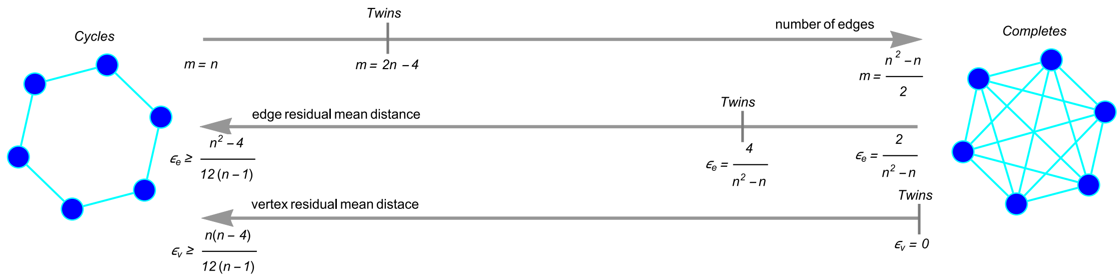


Figure 6: Comparisons between cycles, complete graphs and twin graphs, with respect to the number of edges, the vertex and the edge residual mean distances.

After what was said in the previous section, one question might remain: how do we obtain a graph that has a value of ε_e between the twins and the complete graphs? We know that the graphs with intermediate values of ε_e are obtained depending on belonging of edges to triangles. In this way, one can reduce ε_e by adding one or more edges into a twin graph so as to increase the number of triangles. From this procedure emerges a new family of graphs, which could behave closely to the complete graph regarding single edge and vertex removals by the study of the residual mean distances, but require few edges more than the twin graphs. It leads to the suggestion of investigating this new family of graphs, and comparing their properties in the context of the topology design of a telecommunication network.

Moreover, another open question is to characterize the minimum-size 2-GC graphs such that every edge belongs to a triangle. We know that $K^*_{2,n-2}$ is such a graph. However, are there any others? For graphs with maximum degree $n-1$, the answer is no, by Lemma 5. With the help of the software computer system AutoGraphiX [23, 24], we obtained that $K^*_{2,n-2}$ is the only graph answering to our query for graphs up to 12 vertices. Besides, we observe that in this graph, all edges, except one, are in only one triangle. This property could be a characterization of minimum-size 2-GC graphs with every edge in a triangle. Accordingly, we state the following conjecture.

Conjecture 2 *The only minimum-size* 2*-GC graph such that every edge belongs to a triangle is isomorphic to* $K^*_{2,n-2}$.

Notice however that the graph $K^*_{2,n-2}$ can not be obtained by the above-mentioned procedure from twin graphs. Another question arises then: can we characterize (not necessarily minimum-size) 2-GC graphs such that every edge belongs to a triangle?

As future research directions, we consider to generalize the definition of the RMDs, for taking into account an arbitrary number of vertex/edge removals. We also consider, instead of using the mean distance in the definition of the RMDs, to investigate other graph invariants such as domination number, diameter, vertex cover number, depending on the needs of particular applications.

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