# On the impact of one vertex or edge removal on distance-based invariants for Cartesian product graphs

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**Abstract**

In this paper, we summarize some properties of the Cartesian product graphs related to degree and distance-based invariants. Then, we investigate how much a single edge or vertex removal in the Cartesian product of two connected graphs impacts: the distance between any pair of nodes, the average distance, the diameter and the maximum number of (simple, vertex-disjoint or edge-disjoint) shortest paths in the remaining graph.

*Keywords:* Cartesian product graphs; distance-based invariants; edge removal impact; vertex removal impact

## 1. Introduction

The Cartesian product of graphs was introduced by G. Sabidussi [11, 12] in 1957, and it has been studied since 1972 in the context of communication networks [9, 2]. Cartesian product graphs are well suited for network design and analysis, regarding scalability, performance, and fault-tolerance [10], due to the following properties. The Cartesian product $G \square H$ of two connected graphs $G$ and $H$ provides a way of building a graph much larger than the first ones, while keeping relatively small diameter and maximum degree. That is, whereas the order of $G \square H$ is given by the product of the

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orders of $G$ and $H$, its diameter corresponds to the sum of the diameters of $G$ and $H$, and its maximum degree corresponds to the sum of the maximum degrees of $G$ and $H$. Besides this, the (edge) connectivity of $G\Box H$ is never less than the sum of the (edge) connectivities of $G$ and $H$ [11]. Thus, whatever the (edge) connectivities of two connected graphs $G$ and $H$, $G\Box H$ will remain connected after the removal of any single edge or vertex. Such a removal can however impact distance-based invariants of the graph. How much a single edge or vertex removal in $G\Box H$ impacts: the distance between any pair of nodes, the average distance, the diameter and the maximum number of (simple, vertex-disjoint or edge-disjoint) shortest paths in the remaining graph? This paper answers these questions in Section 3, after providing the needed background on the properties of the Cartesian product of graphs in Section 2. Our results are summarized in Section 4. For the uniformity of this paper, Section 2 presents proofs of some known results. For an overview of Cartesian product graphs, we refer the reader to [6, 7, 8].

## 2. Background

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two connected graphs and denote $G\Box H$ the Cartesian product graph of $G$ and $H$. We denote $n_G = |V(G)|$ and $n_H = |V(H)|$ as the number of vertices of $G$ and $H$, respectively, whereas $m_G = |E(G)|$ and $m_H = |E(H)|$ are their number of edges. An edge between vertices $v$ and $v'$ of $G$ are denoted $v - v'$, similarly for the adjacency in $H$ and in $G\Box H$. In this paper, we will assume that $n_G \geq 3$ and $n_H \geq 3$. A definition of the Cartesian product $G\Box H$ of two connected graphs $G$ and $H$ is the following:

$$V(G\Box H) = \{uv | u \in V(G) \text{ and } v \in V(H)\}$$

and

$$E(G\Box H) = \{uv - u'v' | (u = u' \text{ and } v - v' \in E(H)) \text{ or } (u - u' \in E(G) \text{ and } v = v')\}.$$

By abuse of notation, we write $u \in G, v \in H$ or $uv \in G\Box H$ when the situation is without ambiguity.

A first property that directly follows the definition of the Cartesian product graph is:

**Property 1.** *The number of vertices of $G\Box H$ is $n_{G\Box H} = n_G n_H$, and its number of edges is $m_{G\Box H} = m_G n_H + m_H n_G$.*

To simplify the notation, we will denote by $G^v$ the induced subgraph of $G\Box H$ on the vertices $\{uv \mid u \in H\}$ and say that $G^v$ is the copy of $G$ associated to the vertex $v \in H$. Conversely, $H^u$ denotes the induced subgraph on the vertices $\{uv \mid v \in G\}$ and is called the copy of $H$ associated to the vertex $u \in G$.

Obviously, $G^v$ is isomorphic to $G$ and $H^u$ is isomorphic to $H$, and the different copies of $G$ are connected only by edges in copies of $H$ and vice-versa.

A direct consequence of the above definition is that the degree $\delta_{uv}^{G\Box H}$ of the vertex $uv$ in $G\Box H$ is equal to:

$$\delta_{uv}^{G\Box H} = \delta_u^G + \delta_v^H,$$

where $\delta_u^G$ denotes the degree of vertex $u$ in $G$ and $\delta_v^H$ denotes the degree of $v$ in $H$.

**Property 2.** *Let $\Delta(G)$ denote the maximum degree of $G$, then we have:*

$$\Delta(G\Box H) = \Delta(G) + \Delta(H).$$

*Proof.*

$$\begin{aligned}\Delta(G\Box H) &= \max_{uv\in G\Box H} \delta_{uv}^{G\Box H} = \max_{u\in G}\max_{v\in H}(\delta_u^G + \delta_v^H) \\ &= \max_{u\in G}\delta_u^G + \max_{v\in H}\delta_v^H = \Delta(G) + \Delta(H).\end{aligned}$$

□

Let $d^G(u, u')$ denote the geodesic distance between vertices $u$ and $u'$ in $G$, and $d^H(v, v')$ the one between $v$ and $v'$ in $H$.

**Property 3.** *The geodesic distance between vertices $uv$ and $u'v'$ in $G\Box H$ is given by:*

$$d^{G\Box H}(uv, u'v') = d^G(u, u') + d^H(v, v'),$$

*and there exists at least two vertex-disjoint shortest paths between $uv$ and $u'v'$ if $u \neq u'$ and $v \neq v'$.*

*Proof.* Each induced subgraph $G^v$ is associated to a vertex $v \in H$, and each induced subgraph $H^u$ is associated to a vertex $u \in G$. Furthermore, two subgraphs $G^v$ and $G^{v'}$ are adjacent if and only if $v - v' \in E(H)$, which means that the corresponding vertices in the copies of $G$ associated to $v$ and $v'$ are adjacent. A path joining a vertex of $G^v$ to a vertex of $G^{v'}$ will therefore involve copies of vertices in a path from $v$ to $v'$. One shortest path from a vertex of $G^v$ to a vertex of $G^{v'}$ will therefore involve

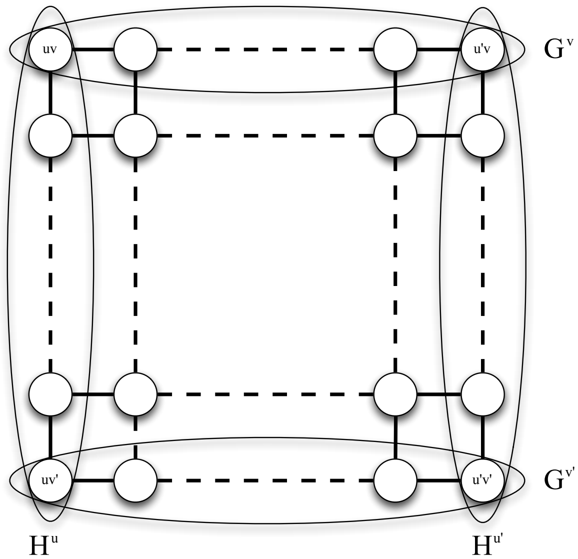


Figure 1: Vertices that may be involved in a shortest path between $uv$ and $u'v'$.

copies of edges in a shortest path from $v$ to $v'$ in $H$. Similarly, one shortest path from a vertex of $H^u$ to a vertex of $H^{u'}$ will involve copies of edges in a shortest path from $u$ to $u'$ in $G$. Since $uv \in H^u$, $uv \in G^v$, $u'v' \in H^{u'}$, $u'v' \in G^{v'}$, we have:

$$d^{G\square H}(uv, u'v') \;=\; d^G(u, u') + d^H(v, v').$$

Furthermore,

$$\begin{aligned} d^{G\square H}(uv, u'v') &= d^{G\square H}(uv, uv') + d^{G\square H}(uv', u'v') && (1)\\ &= d^{G\square H}(uv, u'v) + d^{G\square H}(u'v, u'v'). && (2) \end{aligned}$$

Equation (1) involves vertices both from $H^u$ and $G^{v'}$, whereas Equation (2) involves vertices of $G^v$ and vertices of $H^{u'}$. These paths are vertex-disjoint if $u \neq u'$ and $v \neq v'$ (they will only share the vertices $uv$ and $u'v'$). □

Given a shortest path $u - u_2 - \ldots - u'$ between $u$ and $u'$, and a shortest path $v - v_2 - \ldots - v'$ between $v$ and $v'$, the induced subgraph on the vertices that may belong to a shortest path between $uv$ and $u'v'$ is illustrated in Figure 1.

We say that a graph is 2-*geodetically connected* (2-GC) [4] if for every pair of non-adjacent vertices, there exist at least two edge-disjoint paths connecting these vertices. Accordingly, from the previous property, we can deduce easily the following corollary since all vertex-disjoint paths are edge-disjoint.

**Corollary 1.** *Let $G$ and $H$ be two 2-geodetically connected graphs. Then $G \Box H$ is also 2-geodetically connected.*

Now, let study the diameter of the Cartesian product graph.

**Property 4.** *Let $D(G)$ denote the diameter of $G$, then we have:*

$$D(G \Box H) = D(G) + D(H).$$

*Proof.*

$$\begin{aligned} D(G \Box H) &= \max_{uv, u'v' \in G \Box H} d^{G \Box H}(uv, u'v') \\ &= \max_{u,u' \in G} \max_{v,v' \in H} \left[ d^G(v, v') + d^H(u, u') \right] \\ &= \max_{v,v'} d^G(v, v') + \max_{u,u'} d^H(u, u') \\ &= D(G) + D(H). \end{aligned}$$

□

**Property 5.** *Let $t_u^G = \sum_{u' \in G} d^G(u, u')$ denote the transmission [3, 13] of the vertex $u$ in $G$ and $t_v^H$ the transmission of the vertex $v$ in $H$, then the transmission $t_{uv}^{G \Box H}$ of the vertex $uv$ in $G \Box H$ can be computed by:*

$$t_{uv}^{G \Box H} = n_H t_u^G + n_G t_v^H.$$

*Proof.*

$$
\begin{aligned}
t_{uv}^{G\Box H} &= \sum_{u'v' \in G\Box H} d^{G\Box H}(uv, u'v') \\
&= \sum_{u'v' \in G\Box H} \left[d^{G\Box H}(uv, u'v) + d^{G\Box H}(u'v, u'v')\right] \\
&= \sum_{u' \in G} \sum_{v' \in H} \left[d^{G}(u, u') + d^{H}(v, v')\right] \\
&= \sum_{u' \in G} \left[\sum_{v' \in H} d^{G}(u, u') + \sum_{v' \in H} d^{H}(v, v')\right] \\
&= \sum_{u' \in G} \left[n_H d^{G}(u, u') + \sum_{v' \in H} d^{H}(v, v')\right] \\
&= \sum_{u' \in G} \left[n_H d^{G}(u, u') + t_v^H\right] \\
&= \sum_{u' \in G} \left[n_H d^{G}(u, u')\right] + n_G t_v^H \\
&= n_H t_u^G + n_G t_v^H.
\end{aligned}
$$

□

Let $W(G) = \sum_{u,u' \in G} d^G(u, u') = \frac{1}{2} \sum_{u \in G} t_u^G$ denote the Wiener index [14] of the graph $G$, or the sum of the distances between pairs of vertices of $G$, or the half of the sum of all transmissions of $G$.

**Theorem 1.** *(Theorem 5.9 in [5, 15]) The Wiener index of $G\Box H$ is:*

$$W(G\Box H) = n_H^2 W(G) + n_G^2 W(H).$$

*Proof.*

$$
\begin{aligned}
W(G\square H) &= \frac{1}{2}\sum_{uv\in G\square H} t_{uv} \\
&= \frac{1}{2}\sum_{uv\in G\square H} \left[n_H t_u^G + n_G t_v^H\right] \\
&= \frac{1}{2}\sum_{u\in G}\sum_{v\in H} \left[n_H t_u^G + n_G t_v^H\right] \\
&= \frac{1}{2}\sum_{u\in G} \left[n_H^2 t_u^G + \sum_{v\in H} n_G t_v^H\right] \\
&= \frac{1}{2}\sum_{u\in G} \left[n_H^2 t_u^G + n_G \sum_{v\in H} t_v^H\right] \\
&= \frac{1}{2}\sum_{u\in G} \left[n_H^2 t_u^G + n_G 2W(H)\right] \\
&= \frac{1}{2}\sum_{u\in G} n_H^2 t_u^G + \frac{1}{2}\sum_{u\in G} n_G 2W(H) \\
&= \frac{n_H^2}{2}\sum_{u\in G} t_u^G + n_G W(H)\sum_{u\in G} 1 \\
&= n_H^2 W(G) + n_G^2 W(H).
\end{aligned}
$$

□

**Corollary 2.** *Let*

$$\bar{d}(G) = \frac{\sum_{u,u'\in G} d^G(u,u')}{\binom{n_G}{2}} = \frac{2\sum_{u,u'\in G} d^G(u,u')}{n_G(n_G-1)} = \frac{2W(G)}{n_G(n_G-1)}$$

*denote the average distance of $G$, then we have*

$$\bar{d}(G\square H) = \frac{n_H(n_G-1)\bar{d}(G) + n_G(n_H-1)\bar{d}(H)}{n_G n_H - 1}.$$

*Proof.*

$$\begin{aligned}
\bar{d}(G\square H) &= \frac{2W(G\square H)}{n_{G\square H}(n_{G\square H}-1)} \\
&= \frac{2(n_H^2 W(G)+n_G^2 W(H))}{(n_G n_H)(n_G n_H-1)} \\
&= \frac{2n_H W(G)}{n_G(n_G n_H-1)}+\frac{2n_G W(H)}{n_H(n_G n_H-1)} \\
&= \frac{n_H(n_G-1)\bar{d}(G)}{n_G n_H-1}+\frac{n_G(n_H-1)\bar{d}(H)}{n_G n_H-1} \\
&= \frac{n_H(n_G-1)\bar{d}(G)+n_G(n_H-1)\bar{d}(H)}{n_G n_H-1}.
\end{aligned}$$

□

## 3. Main results

In this section, we present main results after an edge removal or a vertex removal on some graph invariants: distance metrics presented in Section 2 and also the number of three kinds of shortest paths (simple paths, vertex-disjoint paths and edge-disjoint). We recall that in the Cartesian product graph $G\square H$, when we consider an arbitrary edge $uv - u'v'$ in $G\square H$, it means that either $v - v' \in E(H)$ and $u = u'$, or $u - u' \in E(G)$ and $v = v'$.

### *3.1. Distance*

**Theorem 2.** *After the removal of an arbitrary edge, the distance between any pair of vertices of $G\square H$ cannot increase by more than 2.*

*Proof.* Suppose without loss of generality that the edge $uv - uv'$ is removed from $G\square H$. Let $u''v''$ and $u'''v'''$ be any pair of vertices in $G\square H$. First, due to the Property 3, there exist at least two vertex-disjoint shortest paths between $u''v''$ and $u'''v'''$ if $u'' \neq u'''$ and $v'' \neq v'''$, therefore the distance between $u''v''$ and $u'''v'''$ does not change in this case. The case $v'' = v'''$ is obvious as the shortest path between $u''v''$ and $u'''v''$, which is in $G^{v''}$, does not contain the edge $uv - uv'$, which is in $H^u$. Let us consider the case $u'' = u'''$. If $u'' \neq u$, then similarly to the previous case, the edge $uv - uv'$ is in the copy $H^u$, whereas vertices $u''v''$ and $u''v'''$, and any shortest path between them are in the copy $H^{u''}$. Otherwise $u'' = u''' = u$. Suppose that $uv - uv'$ lies in the shortest path between $uv''$ and $uv'''$ (otherwise the distance $d^{G\square H}(uv'', uv''')$ would not be affected by the removal of the edge $uv - uv'$). In this case, to find a path between $uv''$ and $uv'''$, we will avoid the edge $uv - uv'$ in the copy $H^u$ by passing by another copy of $H$: let $u^\star$ be a neighbor of $u$ in $G$. Then the edge $uv - u^\star v$, respectively $uv' - u^\star v'$, is in the copy $G^v$, respectively $G^{v'}$, and thus in $G \Box H$, as well as the edge $u^\star v - u^\star v'$ is in $H^{u^\star}$. Therefore, the edge $uv - uv'$ may be replaced by the three following edges $uv - u^\star v, u^\star v - u^\star v'$ and $u^\star v' - uv'$. There exists then a path from $uv''$ to $uv'''$ whose length is $d^{G \Box H}(uv'', uv''') + 2$. □

**Theorem 3.** *After the removal of a vertex, the distance between any pair of vertices of $G \Box H$ cannot increase by more than 2.*

*Proof.* Suppose the removal of the vertex $uv$ that is on a shortest path between vertices $u'v' \neq uv$ and $u''v'' \neq uv$. Then, two edges adjacent to $uv$ are on that shortest path. Two cases are possible:

1. One edge $uv^i - uv \in E(H^u)$, and the other $uv - u^j v \in E(G^v)$. In this case, replacing these edges by $uv^i - u^j v^i$ and $u^j v^i - u^j v$ will provide a path with exactly the same length as the previous one; and
2. Both edges belong to the same subgraph, either $uv^i - uv \in E(H^u)$ and $uv - uv^j \in E(H^u)$, or $u^i v - uv \in E(G^v)$ and $uv - u^j v \in E(G^v)$. Without loss of generality, suppose the path contains $uv^i - uv \in E(H^u)$ and $uv - uv^j \in E(H^u)$. Let $u^k$ be a neighbor of $u$ in $G$. Then it is possible to replace the edges $uv^i - uv$ and $uv - uv^j$ by the following edges $uv^i - u^k v^i$, $u^k v^i - u^k v$, $u^k v - u^k v^j$ and $u^k v^j - uv^j$, which will result in an increase of the length of the path from $u'v'$ to $u''v''$ by 2.

As a result, the distance between $u'v'$ to $u''v''$ will not increase by more than 2. □

### *3.2. Average distance*

**Theorem 4.** *After the removal of an edge $e = uv - uv' \in E(H^u)$, the average distance of $G \Box H$ does not increase by more than:*

$$\frac{n_H}{n_G(n_G n_H - 1)}.$$

*Proof.* As shown in the proof of Theorem 2, the distance between pairs of vertices may be affected only if they both belong to $H^u$, and the distance increase is bounded by 2. Furthermore, the number of shortest paths affected by the removal of $e$ is less than or equal to $(n_H/2)^2$ [1]. Therefore, the total distance cannot increase by more than $2(n_H/2)^2 = n_H^2/2$. Hence,

$$\begin{aligned} \bar{d}(G \Box H \setminus e) - \bar{d}(G \Box H) &\leq \frac{n_H^2}{n_G n_H (n_G n_H - 1)} \\ &= \frac{n_H}{n_G(n_G n_H - 1)}. \end{aligned}$$

$\square$

**Theorem 5.** *After the removal of a vertex $uv$, the total distance of $G\square H \setminus uv$ does not increase by more than:*

$$(n_H - 1)(n_H - 2) + (n_G - 1)(n_G - 2) - t_{uv}^{G\square H},$$

*and the average distance $\bar{d}(G\square H \setminus uv)$ of $G\square H \setminus uv$ becomes at most:*

$$\frac{1}{\binom{n_G n_H - 1}{2}} \left( (n_H - 1)(n_H - 2) + (n_G - 1)(n_G - 2) + \bar{d}(G\square H)\binom{n_G n_H}{2} - t_{uv}^{G\square H} \right).$$

*Proof.* The argument is similar to that of Theorem 4, except that paths between vertices of $G^v$ and between vertices of $H^u$ may both be affected, and the removal of the vertex $uv$ must be considered as well. The total distance will not increase by more than:

$$2\frac{(n_H - 1)(n_H - 2)}{2} + 2\frac{(n_G - 1)(n_G - 2)}{2} - t_{uv}^{G\square H},$$

since the distance between at most $\frac{(n_H-1)(n_H-2)}{2}$ pairs in $H^u$ and at most $\frac{(n_G-1)(n_G-2)}{2}$ pairs in $G^v$ can be increased by 2. The upper bound of $\bar{d}(G\square H \setminus uv)$ in terms of $\bar{d}(G\square H)$ is directly deduced from the basic definition of average distance. $\square$

*3.3. Diameter*

**Theorem 6.** *The diameter of $G\square H$ remains unchanged after an edge removal.*

*Proof.* Consider $uv$ and $u'v'$ two vertices such that $d^{G\square H}(uv, u'v') = D(G\square H)$. As $n_G \geq 3$ and $n_H \geq 3$, we can suppose that $u \neq u'$ and $v \neq v'$. According to Property 3, there are at least two vertex-disjoint shortest paths between $uv$ and $u'v'$. Removing an edge will therefore leave $d^{G\square H}(uv, u'v')$ unchanged. $\square$

**Theorem 7.** *The diameter of $G\square H$ remains unchanged after a vertex removal.*

*Proof.* This proof is very similar to that of Theorem 6. Consider $u'v'$ and $u''v''$ two vertices such that $d^{G\square H}(u'v', u''v'') = D(G\square H)$. As $n_G \geq 3$ and $n_H \geq 3$, we may suppose that $u' \neq u''$ and $v' \neq v''$. According to Property 3, there are at least two vertex-disjoint shortest paths between $u'v'$ and $u''v''$. Removing a vertex will therefore leave $d^{G\square H}(u'v', u''v'')$ unchanged, except maybe if that vertex is $u'v'$ or $u''v''$. Suppose now without loss of generality that the removed vertex is $u'v'$. According to Property 3, we have:

$$\begin{aligned} D(G\square H) &= d^{G\square H}(u'v', u''v'') \\ &= d^G(u', u'') + d^H(v', v'') \\ &= d^{G\square H}(u''v', u'v''), \end{aligned}$$

and the distance $d^{G\Box H}(u''v', u'v'')$ is not affected by the removal of $uv$, by considering a shortest path in the copy $H^{u''}$ between $u''v'$ and $u''v''$, following by a shortest path in the copy $G^{v''}$ between $u''v''$ and $u'v''$, and so avoinding the removed vertex. After all, the diameter of $G\Box H$ remains unchanged after a vertex removal. $\Box$

*3.4. Maximum number of simple shortest paths*

We denote by $\nu^s_G(u,u')$ the maximum number of simple[1] shortest paths between $u$ and $u'$ in $G$. Following theorems establish the maximum number of simple shortest paths in a Cartesian product graph, and what happens after an edge removal or a vertex removal.

**Theorem 8.** *The maximum number of simple shortest paths in the Cartesian product graph $G\Box H$ between $uv$ and $u'v'$ is exactly*

$$\begin{aligned}\nu^s_{G\Box H}(uv,u'v') &= \nu^s_G(u,u')\nu^s_H(v,v')\binom{d^G(u,u')+d^H(v,v')}{d^G(u,u')}\\ &= \nu^s_G(u,u')\nu^s_H(v,v')\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')}.\end{aligned}$$

*Proof.* Let $u,u_2,u_3,\ldots,u_k,u'$ be a sequence of vertices in a shortest path $P_G$ in $G$ and $v,v_2,v_3,\ldots,v_j,v'$ be a sequence of vertices in a shortest path $P_H$ in $H$, where $k=d^G(u,u')$ and $j=d^H(v,v')$. Notice that by property of binomial coefficients, $\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')}=\binom{d^G(u,u')+d^H(v,v')}{d^G(u,u')}$. We will prove that from these two paths, we can build $\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')}=\binom{k+j}{j}$ paths from $uv$ to $u'v'$ in $G\Box H$. We already know that $d^{G\Box H}(uv,u'v')=k+j$, i.e. the shortest paths from $uv$ to $u'v'$ contain $k+j$ edges. In $G\Box H$, these edges are alternatively from $P_G$ in a copy of $G$ and from $P_H$ in a copy of $H$. For instance, if we take all the first edges from $P_G$ and then from $P_H$, the corresponding path is: $uv{-}u_2v{-}u_3v{-}\cdots{-}u_kv{-}u'v{-}u'v_2{-}u'v_3{-}\cdots{-}u'v_j{-}u'v'$, and in the other direction, if we take all the first edges from $P_H$ and then from $P_G$, vertices $uv,uv_2,uv_3,\ldots,uv_j,uv',u_2v',u_3v',\ldots,u_kv',u'v'$ form a shortest path from $uv$ to $u'v'$. Another shortest path is $uv{-}u_2v{-}u_2v_2{-}u_2v_3{-}\cdots{-}u_2v_j{-}u_3v_j{-}\cdots{-}u_kv_j{-}u'v_j{-}u'v'$. By definition of the Cartesian product graph, all shortest paths can be built by this way. Actually, all such shortest path is well-defined by the paths $P_G$ and $P_H$ if and only if we know exactly which edge is in a copy of which graph : $G$ or $H$. So, to count the maximum number of shortest paths, it is sufficient to count the number of manners to choose among the $k+j$ edges which ones will be in $G$, or which ones will

[1] We say "simple" by opposition of vertex-disjoint and edge-disjoint.

be in $H$. In fact, there are $\binom{k+j}{j}$ such manners. All in all, by considering all shortest paths $P_G$ between $u$ and $u'$ in $G$ and all ones $P_H$ between $v$ and $v'$ in $H$, we obtain $\nu_G^s(u,u')\nu_H^s(v,v')\binom{k+j}{j}$ shortest paths between $uv$ and $u'v'$ in $G\Box H$. $\Box$

**Theorem 9.** *Let $e$ be an arbitrary edge in $G\Box H$. Without loss of generality, we can suppose that $e$ links vertices $u''v''$ and $u''v'''$. Then the number of shortest paths going through this edge from $uv$ to $u'v'$ in $G\Box H$ is exactly*

$$\sum_{P_G\ni u''}\sum_{P_H\ni v'-'v'''}\binom{d^G(u,u'')+d^H(v,v''-v''',P_H)}{d^G(u,u'')}\binom{d^G(u'',u')+d^H(v',v''-v''',P_H)}{d^G(u'',u')},$$

*where $d^H(x,y-z,P_H)=d^H(x,y)$ if $y$ is the first vertex among $\{y,z\}$ encountered in $P_H$ from $x$, $d^H(x,z)$ otherwise, if the vertex $u''$ appears in at least one shortest path $P_G$ in $G$ between $u$ and $u'$, and if the edge $v''-v'''$ appears in at least one shortest path $P_H$ in $H$ between $v$ and $v'$,* 0 *otherwise.*

*Proof.* Let $P_G$ be a shortest path from $u$ to $u'$ in $G$ and $P_H$ be a shortest path from $v$ to $v'$ in $H$. Clearly if the vertex $u''$ does not appear in $P_G$ or if the edge $v''-v'''$ does not appear in $P_H$, no shortest path from $uv$ to $u'v'$ in $G\Box H$, built from $P_G$ and $P_H$, goes through the edge $e$. So we may suppose that $u''\in V(P_G)$ and $v''-v'''\in E(P_H)$. Without loss of generality, we can assume that $v''$ is the first vertex among $\{v'',v'''\}$ encountered in $P_H$ from $v$. In that case, it is sufficient to count the number of shortest paths between $uv$ and $u''v''$ and the number of shortest paths between $u''v'''$ and $u'v'$, because a shortest path from $uv$ to $u'v'$, going through $e$, begings with a shortest path from $uv$ to $u''v''$, goes through $e$ and then finishes by a shortest path from $u''v'''$ to $u'v'$. Therefore, based on the previous result, we obtain

$$\sum_{P_G\ni u''}\sum_{P_H\ni v''v'''}\binom{d^G(u,u'')+d^H(v,v'')}{d^G(u,u'')}\binom{d^G(u'',u')+d^H(v',v''')}{d^G(u'',u')}$$

shortest paths from $uv$ to $u'v'$, going through $e$, which completes the proof. $\Box$

With a similar proof of the previous theorem, Theorem 10 counts the number of shortest simple paths going through a vertex.

**Theorem 10.** *Let $u''v''$ be an arbitrary vertex in $G\Box H$. Then the number of shortest paths going through this vertex from $uv$ to $u'v'$ in $G\Box H$ is exactly*

$$\sum_{P_G\ni u''}\sum_{P_H\ni v''}\binom{d^G(u,u'')+d^H(v,v'')}{d^H(v,v'')}\binom{d^G(u'',u')+d^H(v'',v')}{d^H(v'',v')},$$

*if the vertex $u''$, respectively $v''$, appears in at least one shortest path $P_G$, respectively $P_H$, in $G$, respectively in $H$, between $u$ and $u'$, respectively between $v$ and $v'$,* 0 *otherwise.*

*3.5. Maximum number of vertex-disjoint shortest paths*

We denote by $\nu^v_G(u,u')$ the maximum number of vertex-disjoint shortest paths between $u$ and $u'$ in $G$. Following theorems establish the maximum number of vertex-disjoint shortest paths in a Cartesian product graph, and what happens after an edge removal or a vertex removal.

**Theorem 11.** *The maximum number of vertex-disjoint shortest paths in the Cartesian product graph $G\Box H$ between $uv$ and $u'v'$ is exactly*

$$\nu^v_{G\Box H}(uv,u'v') = \nu^v_G(u,u') + \nu^v_H(v,v').$$

*Proof.* Because all shortest paths are made from shortest paths in $G$ and shortest paths in $H$, the number of neighbors of $uv$ (or edges adjacent to $uv$) used in a shortest path from $uv$ to $u'v'$ in $G\Box H$ is at most the sum between the number of neighbors of $u$ (or edges adjacent to $u$) used in a shortest path from $u$ to $u'$ in $G$ and the number of neighbors of $v$ (or edges adjacent to $v$) used in a shortest path from $v$ to $v'$ in $H$. Thus,

$$\nu^v_{G\Box H}(uv,u'v') \leq \nu^v_G(u,u') + \nu^v_H(v,v').$$

Now, we will describe $\nu^v_G(u,u') + \nu^v_H(v,v')$ shortest paths from $uv$ to $u'v'$. Let $P^\star_G$ be a shortest path from $u$ to $u'$ in $G$, and let $P^\star_H$ be a shortest path from $v$ to $v'$ in $H$. For these two paths, we will consider the two shortest paths in $G\Box H$, by taking all the first edges from $P^\star_G$ and then from $P^\star_H$, and conversely, i.e. if $P^\star_G = u - u^\star_2 - \cdots - u^\star_k - u'$ and $P^\star_H = v - v^\star_2 - \cdots - v^\star_j - v'$, then one shortest path in $G\Box H$ is $uv - u^\star_2 v - \cdots - u^\star_k v - u'v - u'v^\star_2 - \cdots - u'v^\star_j - u'v'$ whereas the other one is $uv - uv^\star_2 - \cdots - uv^\star_j - uv' - u^\star_2 v' - \cdots - u^\star_k v' - u'v'$. Clearly these two paths are vertex-disjoint.

For any other possible shortest path $P_G$ from $u$ to $u'$ in $G$ (notice that if such path exists, then $d^G(u,u') \geq 2$), we proceed in the following way: if $P_G = u - u_2 - \cdots - u_k - u'$, then taking the first edge from $P_G$ then following $P^\star_H$, and finally finishing $P_G$, i.e. $uv - u_2 v - u_2 v^\star_2 - \cdots - u_2 v^\star_j - u_2 v' - u_3 v' - \cdots - u_k v' - u'v'$, is a shortest path from $uv$ to $u'v'$, which is vertex-disjoint from the other ones since it is also the case for paths in $G$. For any other possible shortest path $P_H$ from $v$ to $v'$ in $H$, a similar argumentation holds, by symmetry of the Cartesian product graph. $\Box$

The following theorem establishes results after the removal of an edge. Beforehand, we will prove two lemmas.

**Lemma 1.** *Let $e = uv - uv'''$ be an edge in $G\Box H$, which appears in a shortest path from $uv$ to $u'v'$. If $\nu^v_{H\setminus v-v'''}(v,v') = \nu^v_H(v,v') - 1$ and $d_{H\setminus v-v'''}(v,v') = d_H(v,v')$, or $d_{H\setminus v-v'''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\setminus e}(uv,u'v') = \nu^v_{G\Box H}(uv,u'v') - 1,$$

*otherwise*

$$\nu^v_{G\Box H\setminus e}(uv,u'v') = \nu^v_{G\Box H}(uv,u'v').$$

*Proof.* The hypothesis mean that all maximum collection of vertex-disjoint shortest paths between $v$ and $v'$ in $H$ necessary uses the edge $v-v'''$. By construction of shortest paths in $G\Box H$, all maximum collection of vertex-disjoint shortest paths between $uv$ and $u'v'$ in $G\Box H$ necessary uses also the edge $uv - uv'''$, i.e. $e$. Accordingly,

$$\nu^v_{G\Box H\setminus e}(uv,u'v') = \nu^v_{G\Box H}(uv,u'v') - 1.$$

Otherwise there exists a collection of $\nu^v_H(v,v')$ vertex-disjoint shortest paths between $v$ and $v'$ in $H$, avoiding the edge $v - v'''$. With this collection, we can build the desired collection as we did in the proof of Theorem 11. Thus

$$\nu^v_{G\Box H\setminus e}(uv,u'v') = \nu^v_{G\Box H}(uv,u'v').$$

□

**Lemma 2.** *Let $e = u''v'' - u''v'''$ be an edge in $G\Box H$, which appears in a shortest path from $uv$ to $u'v'$ and is not adjacent to $uv$ and to $u'v'$. Assume that $d^G(u,u') = 1$. If $\nu^v_H(v,v') = 1$ and $d_{H\setminus v''-v'''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\setminus e}(uv,u'v') = \nu^v_{G\Box H}(uv,u'v') - 1,$$

*otherwise*

$$\nu^v_{G\Box H\setminus e}(uv,u'v') = \nu^v_{G\Box H}(uv,u'v').$$

*Proof.* Let $P^\star_G$ be the shortest path between $u$ and $u'$ in $G$, i.e. $P^\star_G$ contains only the edge $u - u'$. The hypothesis mean that all collection of vertex-disjoint shortest paths from $v$ to $v'$ in $H$ contains only one path and always uses the edge $v'' - v'''$. Since $d^G(u,u') = 1$, then $\nu^v_G(u,u') = 1$, and so $\nu^v_{G\Box H}(uv,u'v') = 2$. If $P^\star_H$ is a shortest path between $v$ and $v'$ in $H$, then the only one collection $\mathcal{C}$ of vertex-disjoint shortest paths between $uv$ and $u'v'$ made from $P^\star_H$ and from $P^\star_G$ is $P^\star_G$ followed by $P^\star_H$ and $P^\star_H$ followed by $P^\star_G$. Since $P^\star_H$ uses the edge $v'' - v'''$, then the collection $\mathcal{C}$ necessary uses edges $uv'' - uv'''$ and $u'v'' - u'v'''$. Accordingly, removing the edge $e$ is equivalent to remove either $uv'' - uv'''$ or $u'v'' - u'v'''$, and so

$$\nu^v_{G\square H\setminus e}(uv, u'v') = \nu^v_{G\square H}(uv, u'v') - 1.$$

Now, we will consider the case when $\nu^v_H(v, v') = 1$ and $d_{H\setminus v''-v'''}(v, v') = d_H(v, v')$. In that case, there exists a shortest path between $v$ and $v'$ in $H$, avoiding the edge $v''-v'''$, say $P^\bullet_H$. Since $\nu^v_H(v, v') = 1$ and $d^G(u, u') = 1$, we have that $\nu^v_{G\square H}(uv, u'v') = 2$. The paths $P^\star_G$ followed by $P^\bullet_H$ and $P^\bullet_H$ followed by $P^\star_G$ are the two vertex-disjoint shortest paths from $uv$ to $u'v'$ in $G\square H \setminus e$, thus

$$\nu^v_{G\square H\setminus e}(uv, u'v') = \nu^v_{G\square H}(uv, u'v').$$

The last case is when $\nu^v_H(v, v') > 1$. Necessary, there exists at least one shortest path from $v$ to $v'$ in $H$, avoiding $v'' - v'''$. Choose one in a maximum collection of vertex-disjoint shortest paths, say $P^\oplus_H$. It remains to build a collection $\mathcal{C}$ of $\nu^v_{G\square H}(uv, u'v')$ vertex-disjoint shortest paths from $uv$ to $u'v'$ in $G\square H$, avoiding $e$. First, we add the two shortest paths: $P^\star_G$ followed by $P^\oplus_H$ and $P^\oplus_H$ followed by $P^\star_G$. Clearly, they avoid $e$. We distinguish two subcases: $u'' = u$ and $u'' = u'$.

- If $u'' = u$, for any other vertex-disjoint shortest path $P_H$ from $v$ to $v'$ in $H$, we build the following path $P$: we start with the first edge of $P_H$, then we take $P^\star_G$, finally we finish $P_H$. Since $e$ is not adjacent to $uv$, $P$ avoids $e$.

- If $u'' = u'$, we do a similar construction, except that we start with all edge from $P_H$ excluding the last one, we follow $P^\star_G$ and then finish with the last edge of $P_H$. Since $e$ is not adjacent to $u'v'$, this path avoids $e$.

All in all, we proved that

$$\nu^v_{G\square H\setminus e}(uv, u'v') = \nu^v_{G\square H}(uv, u'v').$$

□

**Theorem 12.** *Let $e = u''v'' - u'''v'''$ be an arbitrary edge in $G\square H$. We assume that $e$ appears in a shortest path from $uv$ to $u'v'$ in $G\square H$. If $d^G(u, u') = 0$ and $d_{H\setminus v''-v'''}(v, v') > d_H(v, v')$ then*

$$\nu^v_{G\square H\setminus e}(uv, u'v') = \nu^v_{H\setminus v''-v'''}(v, v'),$$

*otherwise if $d^H(v, v') = 0$ and $d_{G\setminus u''-u'''}(u, u') > d_G(u, u')$ then*

$$\nu^v_{G\square H\setminus e}(uv, u'v') = \nu^v_{G\setminus u''-u'''}(u, u'),$$

*otherwise if one of the following conditions holds:*

(i) $e = uv - uv'''$ *and* $\nu^v_{H\backslash v-v'''}(v,v') = \nu^v_H(v,v') - 1$ *and* $d_{H\backslash v-v'''}(v,v') = d_H(v,v')$
(ii) $e = uv - uv'''$ *and* $d_{H\backslash v-v'''}(v,v') > d_H(v,v')$
(iii) $e = uv - u'''v$ *and* $\nu^v_{G\backslash u-u'''}(u,u') = \nu^v_G(u,u') - 1$ *and* $d_{G\backslash u-u'''}(u,u') = d_G(u,u')$
(iv) $e = uv - u'''v$ *and* $d_{G\backslash u-u'''}(u,u') > d_G(u,u')$
(v) $e = u'v'' - u'v'$ *and* $\nu^v_{H\backslash v''-v'}(v,v') = \nu^v_H(v,v') - 1$ *and* $d_{H\backslash v''-v'}(v,v') = d_H(v,v')$
(vi) $e = u'v'' - u'v'$ *and* $d_{H\backslash v''-v'}(v,v') > d_H(v,v')$
(vii) $e = u''v' - u'v'$ *and* $\nu^v_{G\backslash u''-u'}(u,u') = \nu^v_G(u,u') - 1$ *and* $d_{G\backslash u''-u'}(u,u') = d_G(u,u')$
(viii) $e = u''v' - u'v'$ *and* $d_{G\backslash u''-u'}(u,u') > d_G(u,u')$
(ix) $d^G(u,u') = 0$ *and* $d_{H\backslash v''-v'''}(v,v') = d_H(v,v')$ *and* $\nu^v_{H\backslash v''-v'''}(v,v') = \nu^v_H(v,v') - 1$
(x) $d^G(u,u') = 1$ *and* $\nu^v_H(v,v') = 1$ *and* $d_{H\backslash v''-v'''}(v,v') > d_H(v,v')$
(xi) $d^H(v,v') = 0$ *and* $d_{G\backslash u''-u'''}(u,u') = d_G(u,u')$ *and* $\nu^v_{G\backslash u''-u'''}(u,u') = \nu^v_G(u,u') - 1$
(xii) $d^H(v,v') = 1$ *and* $\nu^v_G(u,u') = 1$ *and* $d_{G\backslash u''-u'''}(u,u') > d_G(u,u')$

*then*

$$\nu^v_{G\Box H\backslash e}(uv, u'v') = \nu^v_{G\Box H}(uv, u'v') - 1,$$

*otherwise*

$$\nu^v_{G\Box H\backslash e}(uv, u'v') = \nu^v_{G\Box H}(uv, u'v').$$

*Proof.* Notice that $d^G(u,u') = 0$ means that we work only in the copy $H^u$ and then computing $\nu^v_{G\Box H}(uv, u'v')$, resp. $\nu^v_{G\Box H\backslash e}(uv, u'v')$, is equivalent to compute $\nu^v_H(v,v')$, resp. $\nu^v_{H\backslash v''-v'''}(v,v')$. If removing the edge $v''-v'''$ in $H$ does not change the distance between $v$ and $v'$ in $H$ then in that case we can evaluate $\nu^v_{G\Box H\backslash e}(uv, u'v')$. Indeed, if $\nu^v_{H\backslash v''-v'''}(v,v') = \nu^v_H(v,v') - 1$ then

$$\begin{aligned} \nu^v_{G\Box H\backslash e}(uv, u'v') &= \nu^v_{H\backslash v''-v'''}(v,v') \\ &= \nu^v_H(v,v') - 1 \\ &= \nu^v_{G\Box H}(uv, u'v') - 1. \end{aligned} \tag{3}$$

Otherwise $\nu^v_{H\backslash v''-v'''}(v,v') = \nu^v_H(v,v')$, and so $\nu^v_{G\Box H\backslash e}(uv, u'v') = \nu^v_{G\Box H}(uv, u'v')$ by a similar argumentation. Otherwise the distance between $v$ and $v'$ in $H$ increases after removing $v''-v'''$ and we can only say that $\nu^v_{G\Box H\backslash e}(uv, u'v') = \nu^v_{H\backslash v''-v'''}(v,v')$. The case $d^H(v,v') = 0$ is analogous by symmetry of the Cartesian product graph $G\Box H$.

By Lemma 1 and by symmetry, we have already the result when the edge $u''v'' - u'''v'''$ is adjacent to $uv$ or to $u'v'$, especially proving cases from (i) to (viii). Now, we can suppose that the edge $u''v'' - u'''v'''$ is not adjacent to $uv$ and to $u'v'$.

If $d^G(u,u')=1$ or $d^H(v,v')=1$ then Lemma 2 deals these cases, especially cases (x) and (xii). From now, we can assume that $d^G(u,u')\geq 2$ and $d^H(v,v')\geq 2$.

We can assume that the collection $\mathcal{C}$ of shortest paths in proof of Theorem 11 is achieved. By symmetry, we may suppose that $e$ links vertices $u''v''$ and $u''v'''$. Our goal is to prove that

$$\nu^v_{G\Box H\setminus e}(uv,u'v')=\nu^v_{G\Box H}(uv,u'v').$$

If $u''=u$ then the edge $e$ is on the path from $P^\star_H$ following by $P^\star_G$. Instead of $uv-uv^\star_2-\cdots-uv''-uv'''-\cdots-uv^\star_j-uv'-u^\star_2v'-\cdots-u^\star_kv'-u'v'$, we consider $uv-uv^\star_2-\cdots-uv''-u^\star_2v''-u^\star_2v'''-\cdots-u^\star_2v^\star_j-u^\star_2v'-\cdots-u^\star_kv'-u'v'$ in $\mathcal{C}$.

If $u''=u'$, then two cases are possible: the path $P$ is either $P^\star_G$ followed by $P^\star_H$, i.e. $uv-u^\star_2v-\cdots-u^\star_kv-u'v-u'v^\star_2-\cdots-u'v''-u'v'''-\cdots-u'v^\star_j-u'v'$, or made from $P^\star_G$ and another path $P_H$, i.e. $uv-uv_2-u^\star_2v_2-\cdots-u^\star_kv_2-u'v_2-\cdots-u'v''-u'v'''-\cdots-u'v_j-u'v'$. In $\mathcal{C}$, it is so sufficient to consider $uv-u^\star_2v-u^\star_2v^\star_2-\cdots-u^\star_2v'''-\cdots-u^\star_kv'''-u'v'''-\cdots-u'v^\star_j-u'v'$ for the first case, whereas $uv-uv_2-u^\star_2v_2-\cdots-u^\star_2v'''-\cdots-u^\star_kv'''-u'v'''-\cdots-u'v_j-u'v'$ in the second case.

If $u\neq u''\neq u'$, then the path $P$ is made from another $P_G$ and $P^\star_H$, i.e. $P=uv-u_2v-u_2v^\star_2-\cdots-u_2v''-u_2v'''-\cdots-u_2v^\star_j-u_2v'-\cdots-u_kv'-u'v'$ and $u''=u_2$. Then we can replace $P$ and the path $P^\star_G$ followed by $P^\star_H$, and its converse, i.e. $P^\star_H$ followed by $P^\star_G$, by interchanging the role of $P_G$ and $P^\star_G$ in the construction of these three paths, i.e.

$$uv-u_2v-\cdots-u_kv-u'v-u'v^\star_2-\cdots-u'v^\star_j-u'v',$$

$$uv-uv^\star_2-\cdots-uv^\star_j-uv'-u_2v'-\cdots-u_kv'-u'v',$$

and

$$uv-u^\star_2v-u^\star_2v^\star_2-\cdots-u^\star_2v^\star_j-u^\star_2v'-\cdots-u^\star_kv'-u'v'.$$

All in all, since $e$ is not adjacent to $uv$ and to $u'v'$, we create a new collection $\mathcal{C}'$ from $\mathcal{C}$, of vertex-disjoint shortest paths from $uv$ to $u'v'$ in $G\Box H$, avoiding $e$ and with the same cardinality of $\mathcal{C}$. $\square$

Despite the cases described by this theorem, where the number $\nu^v_{G\Box H}(uv, u'v')$ changes after an edge removal, the theorem brings good news: overall, removing an arbitrary edge in $G\Box H$ does not change the maximum number of vertex-disjoint shortest paths, especially when $uv$ and $u'v'$ are far from each other as well as the pair $u$ and $u'$ in $G$ and the pair $v$ and $v'$ in $H$. Indeed, in that case, we have many maximum collections of vertex-disjoint shortest paths, which is not surprising in view of the number of simple shortest paths between $uv$ and $u'v'$ in $G\Box H$.

Similarly to the removal of an edge, we will establish first two lemmas, and then we will set the general theorem about vertex removal. Since the following lemmas are quite the same than Lemma 1 and Lemma 2, their proof are left to the reader.

**Lemma 3.** *Let $uv''$ be a vertex in $G\Box H$, which appears in a shortest path from $uv$ to $u'v'$, with $v''$ adjacent to $v$ in $H$. If $\nu^v_{H\backslash v''}(v, v') = \nu^v_H(v, v') - 1$ and $d_{H\backslash v''}(v, v') = d_H(v, v')$, or $d_{H\backslash v''}(v, v') > d_H(v, v')$ then*

$$\nu^v_{G\Box H\backslash uv''}(uv, u'v') = \nu^v_{G\Box H}(uv, u'v') - 1,$$

*otherwise*

$$\nu^v_{G\Box H\backslash uv''}(uv, u'v') = \nu^v_{G\Box H}(uv, u'v').$$

**Lemma 4.** *Let $u''v''$ be a vertex in $G\Box H$, which appears in a shortest path from $uv$ to $u'v'$ and is not adjacent to $uv$ and to $u'v'$. Assume that $d^G(u, u') = 1$. If $\nu^v_H(v, v') = 1$ and $d_{H\backslash v''}(v, v') > d_H(v, v')$ then*

$$\nu^v_{G\Box H\backslash u''v''}(uv, u'v') = \nu^v_{G\Box H}(uv, u'v') - 1,$$

*otherwise*

$$\nu^v_{G\Box H\backslash u''v''}(uv, u'v') = \nu^v_{G\Box H}(uv, u'v').$$

**Theorem 13.** *Let $u''v''$ be an arbitrary vertex in $G\Box H$. We assume that $u''v''$ appears in a shortest path from $uv$ to $u'v'$ in $G\Box H$. If $d^G(u, u') = 0$ and $d_{H\backslash v''}(v, v') > d_H(v, v')$ then*

$$\nu^v_{G\Box H\backslash u''v''}(uv, u'v') = \nu^v_{H\backslash v''}(v, v'),$$

*otherwise if $d^H(v, v') = 0$ and $d_{G\backslash u''}(u, u') > d_G(u, u')$ then*

$$\nu^v_{G\Box H\backslash u''v''}(uv, u'v') = \nu^v_{G\backslash u''}(u, u'),$$

*otherwise if one of the following conditions holds:*

(i) $u'' = u$ *and* $v''$ *adjacent to* $v$ *in* $H$ *and* $\nu^v_{H\setminus v''}(v,v') = \nu^v_H(v,v') - 1$ *and* $d_{H\setminus v''}(v,v') = d_H(v,v')$

(ii) $u'' = u$ *and* $v''$ *adjacent to* $v$ *in* $H$ *and* $d_{H\setminus v''}(v,v') > d_H(v,v')$

(iii) $v'' = v$ *and* $u''$ *adjacent to* $u$ *in* $G$ *and* $\nu^v_{G\setminus u''}(u,u') = \nu^v_G(u,u')-1$ *and* $d_{G\setminus u''}(u,u') = d_G(u,u')$

(iv) $v'' = v$ *and* $u''$ *adjacent to* $u$ *in* $G$ *and* $d_{G\setminus u''}(u,u') > d_G(u,u')$

(v) $u'' = u'$ *and* $v''$ *adjacent to* $v'$ *in* $H$ *and* $\nu^v_{H\setminus v''}(v,v') = \nu^v_H(v,v') - 1$ *and* $d_{H\setminus v''}(v,v') = d_H(v,v')$

(vi) $u'' = u'$ *and* $v''$ *adjacent to* $v'$ *in* $H$ *and* $d_{H\setminus v''}(v,v') > d_H(v,v')$

(vii) $v'' = v'$ *and* $u''$ *adjacent to* $u'$ *in* $G$ *and* $\nu^v_{G\setminus u''}(u,u') = \nu^v_G(u,u') - 1$ *and* $d_{G\setminus u''}(u,u') = d_G(u,u')$

(viii) $v'' = v'$ *and* $u''$ *adjacent to* $u'$ *in* $G$ *and* $d_{G\setminus u''}(u,u') > d_G(u,u')$

(ix) $d^G(u,u') = 0$ *and* $d_{H\setminus v''}(v,v') = d_H(v,v')$ *and* $\nu^v_{H\setminus v''}(v,v') = \nu^v_H(v,v') - 1$

(x) $d^G(u,u') = 1$ *and* $d_{H\setminus v''}(v,v') > d_H(v,v')$ *and* $\nu^v_H(v,v') = 1$

(xi) $d^H(v,v') = 0$ *and* $d_{G\setminus u''}(u,u') = d_G(u,u')$ *and* $\nu^v_{G\setminus u''}(u,u') = \nu^v_G(u,u') - 1$

(xii) $d^H(v,v') = 1$ *and* $d_{G\setminus u''}(u,u') > d_G(u,u')$ *and* $\nu^v_G(u,u') = 1$

*then*

$$\nu^v_{G\square H\setminus u''v''}(uv,u'v') = \nu^v_{G\square H}(uv,u'v') - 1,$$

*otherwise*

$$\nu^v_{G\square H\setminus u''v''}(uv,u'v') = \nu^v_{G\square H}(uv,u'v').$$

*Proof.* By a similar argumentation in proof of Theorem 12, the first equalities hold with Lemma 3 and Lemma 4. So we may suppose that $u''v''$ is not adjacent to $uv$ and to $u'v'$, and that $d^G(u,u') \geq 2$ and $d^H(v,v') \geq 2$. Besides, we assume that the collection $\mathcal{C}$ of shortest paths in proof of Theorem 11 is achieved after the removal of vertex $u''v''$. We distinguish different cases, depending on the kind of the achieved path $P$. By symmetry of the Cartesian product graph, only two cases are considered: $P$ is made either from $P^\star_G$ and $P^\star_H$ or from $P^\star_H$ and another $P_G$.

In the first case, we assume that $P = uv - u^\star_2 v - \cdots - u^\star_k v - u'v - u'v^\star_2 - \cdots - u'v^\star_j - u'v'$. Then it is sufficient to consider $uv - u^\star_2 v - u^\star_2 v^\star_2 - \cdots - u^\star_2 v^\star_j - \cdots - u^\star_k v^\star_j - u'v^\star_j - u'v'$, instead of $P$. This path and those in the remaining collection $\mathcal{C}\setminus\{P\}$ are vertex-disjoint since $d^G(u,u') > 1$ and $d^H(v,v') > 1$.

In the second case, we suppose that

$$P = uv - u_2 v - u_2 v^\star_2 - \cdots - u_2 v^\star_j - u_2 v' - \cdots - u_k v' - u'v',$$

then replacing the three paths $P$,

$$uv - uv_2^\star - \cdots - uv_j^\star - uv' - u_2^\star v' - \cdots - u_k^\star v' - u'v'$$

and

$$uv - u_2^\star v - \cdots - u_k^\star v - u'v - u'v_2^\star - \cdots - u'v_j^\star - u'v'$$

by

$$uv - u_2^\star v - u_2^\star v_2^\star - \cdots - u_2^\star v_j^\star - u_2^\star v' - \cdots - u_k^\star v' - u'v'$$

and

$$uv - uv_2^\star - \cdots - uv_j^\star - uv' - u_2 v' - \cdots - u_k v' - u'v'$$

and

$$uv - u_2 v - \cdots - u_k v - u'v - u'v_2^\star - \cdots - u'v_j^\star - u'v',$$

i.e. interchanging the role of $P_G$ and $P_G^\star$ in the construction of the three paths is sufficient to keep the same number of vertex-disjoint shortest paths. □

Obviously, the previous theorem is inspired from Theorem 12 and the result of is not surprisingly.

*3.6. Maximum number of edge-disjoint shortest paths*

We denote by $\nu_G^e(u,u')$ the maximum number of edge-disjoint shortest paths between $u$ and $u'$ in $G$. The whole theory about vertex-disjoint shortest paths in the previous subsection yields same results on the maximum number $\nu_{G\Box H}^e(uv,u'v')$ of edge-disjoint shortest paths between $uv$ and $u'v'$ in $G\Box H$. Indeed, in this paper, we consider only simple[2] graphs and if $P = u - u_2 - \cdots - u_k - u'$ and $P^\bullet = u - u_2^\bullet - \cdots - u_k^\bullet - u'$ are two edge-disjoint shortest paths from $u$ to $u'$ in $G$ then it must be that $u_2 \neq u_2^\bullet$ and $u_k \neq u_k^\bullet$, which implies that paths in the collection $\mathcal{C}$ described in Theorem 11 are edge-disjoint if it is also the case for paths from $u$ to $u'$ in $G$ and paths from $v$ to $v'$ in $H$.

**Theorem 14.** *The maximum number of edge-disjoint shortest paths in the Cartesian product graph $G\Box H$ between $uv$ and $u'v'$ is exactly*

$$\nu_{G\Box H}^e(uv,u'v') = \nu_G^e(u,u') + \nu_H^e(v,v').$$

Similarly, Theorem 12 and Theorem 13, theorems on the edge removal and the vertex removal, can be easly translated in terms of edge-disjoint paths. To avoid redundancy, we omit to formulate and to prove them here.

[2]The word "simple" means here "without loops or multiple edges".

## 4. Conclusions

Bounds were obtained for the impact of one edge or vertex removal on distance-based invariants for the Cartesian product graph $G \square H$ of two connected graphs $G$ and $H$ of order greater than 2. In particular, it was shown that, after the removal of any single edge or vertex, the distance between any pair of vertices of $G \square H$ cannot increase by more than 2, and the diameter of $G \square H$ remains unchanged. Besides, removing any single edge or vertex in a Cartesian product graph has no significant impact on the maximum number of (simple, vertex-disjoint or edge-disjoint) shortest paths between two distant vertices. These results apply to fault tolerance analysis of communication networks modeled by Cartesian product graphs.

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