Counting shortest paths in Cartesian product graphs

E. Camby[1,2], G. Caporossi[2], M. H. M. Paiva[3], M. R. N. Ribeiro[3], and M. E. V. Segatto[3]

[1]Université Libre de Bruxelles, Bruxelles, Belgium, ecamby@ulb.ac.be
[2]GERAD & HEC Montréal, Montréal, Canada, gilles.caporossi@hec.ca
[3]LabTel - Laboratory of Telecommunications, Federal University of Espírito Santo, Vitória (Espírito Santo), Brazil, marcia.paiva@ufes.br, moises@ele.ufes.br, segatto@ele.ufes.br

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Abstract

In this paper, we establish the maximum number of three kinds of shortest paths in Cartesian product graphs : the simple shortest paths, the vertex-disjoint shortest paths and the edge-disjoint shortest paths. Moreover, we investigate the impact of a vertex or an edge removal on these three graphs invariants.

Keywords: Cartesian product graph; simple shortest path; vertex-disjoint shortest path; edge-disjoint shortest path; edge removal impact; vertex removal impact.

1 Introduction

The Cartesian product of graphs was introduced by G. Sabidussi [7, 8] in 1957, and it has been studied since 1972 in the context of communication networks [1, 5]. Cartesian product graphs are well suited for network design and analysis, regarding scalability, performance, and fault-tolerance [6], due to the following properties. The Cartesian product $G \Box H$ of two connected graphs G and H provides a way of building a graph much larger than the first ones, while keeping relatively small diameter and maximum degree. That is, whereas the order of $G \Box H$ is given by the product of the orders of G and H, its diameter corresponds to the sum of the diameters of G and H, and its maximum degree corresponds to the sum of the maximum degrees of G and H. Besides this, the (edge) connectivity of $G \Box H$ is never less than the sum of the (edge) connectivities of G and H [7]. Thus, whatever the (edge) connectivities of two connected graphs G and H, $G \Box H$ will remain connected after the removal of any single edge or vertex. Such a removal can however impact the number of shortest paths in a Cartesian product graph. The number of shortest paths is an important feature from the telecommunication point of view, but, to the best of our knowledge, was not accessed in the litterature. The main contribution of this paper is to fill this gap. This paper investigates in Section 3 the number of three kinds of shortest paths in Cartesian product graphs : the simple[1] shortest paths, the vertex-disjoint shortest paths and the edge-disjoint shortest paths. Moreover, we study the impact of a vertex

[1]We say "simple" by opposition of vertex-disjoint and edge-disjoint.

or an edge removal on these three graph invariants. We provide the needed background of the Cartesian product of graphs in Section 2, and our results are summarized in Section 4. For an overview of Cartesian product graphs, we refer the reader to [2, 3, 4].

2 Background

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two connected graphs with the vertex set $V(G)$, respectively $V(H)$, and the edge set $E(G)$, respectively $E(H)$. An edge between vertices u and u' of G is denoted uu', similarly for the adjacency in H. In this paper, we assume that $|V(G)| \geq 2$ and $|V(H)| \geq 2$. A definition of the Cartesian product $G\Box H$ of these two graphs G and H is the following:

$$V(G\Box H) = \{(u,v) | u \in V(G) \text{ and } v \in V(H)\}$$

and

$$\begin{aligned} E(G\Box H) &= \{(u,v)(u',v') | (u = u' \text{ and } vv' \in E(H)) \text{ or } (uu' \in E(G) \text{ and } v = v')\} \\ &= \{(u,v)(u,v') | vv' \in E(H)\} \cup \{(u,v)(u',v) | uu' \in E(G)\} \end{aligned}$$

We denote by $G^{\bar{v}}$ the induced subgraph of $G\Box H$ on the vertex set $\{(u,\bar{v}) \mid u \in G\}$ and say that $G^{\bar{v}}$ is the copy of G associated to the vertex $\bar{v} \in H$. Conversely, $H^{\bar{u}}$ denotes the induced subgraph on the vertex set $\{(\bar{u},v) \mid v \in H\}$ and is called the copy of H associated to the vertex $\bar{u} \in G$. Obviously, $G^{\bar{v}}$ is isomorphic to G and $H^{\bar{u}}$ is isomorphic to H, and the different copies of G are connected only by edges in copies of H and vice-versa.

Let $d^G(u,u')$ denote the geodesic distance between vertices u and u' in G, and $d^H(v,v')$ be the one between v and v' in H. Similarly, $d^{G\Box H}((u,v),(u',v'))$ is the geodesic distance between vertices (u,v) and (u',v') in the Cartesian product $G\Box H$.

3 Main results

The outcome of the paper is subdivided in three subsections : one for each kind of shortest paths. Through the three subsections, we illustrate the theory by a well-know example : the cube (see Figure 1). Actually, it is the Cartesian product between a cycle on 4 vertices and a path on 2 vertices, i.e. take two copies of the cycle on 4 vertices and join two corresponding vertices from different copies by an edge.

3.1 Maximum number of simple shortest paths

We denote by $\nu^s_G(u,u')$ the maximum number of simple shortest paths between u and u' in G, similarly in H and in $G\Box H$. Following theorems establish the maximum number of simple shortest paths in a Cartesian product graph, and what happens after an edge removal or a vertex removal.

Theorem 1. *The maximum number of simple shortest paths in the Cartesian product graph $G\Box H$ between (u,v) and (u',v') is exactly*

$$\begin{aligned} \nu^s_{G\Box H}((u,v),(u',v')) &= \nu^s_G(u,u')\nu^s_H(v,v')\binom{d^G(u,u')+d^H(v,v')}{d^G(u,u')} \\ &= \nu^s_G(u,u')\nu^s_H(v,v')\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')}. \end{aligned}$$

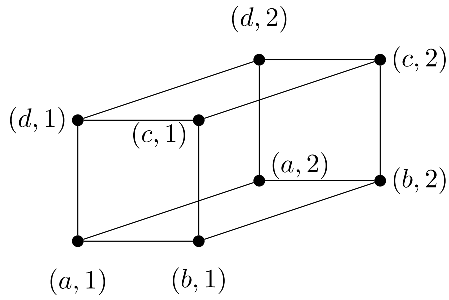

Figure 1: The cube is the Cartesian product between a cycle on vertices $\{a, b, c, d\}$ and a path on vertices $\{1, 2\}$.

Proof. Let $u, u_2, u_3, \ldots, u_k, u'$ be a sequence of vertices in a shortest path P_G in G and $v, v_2, v_3, \ldots, v_j, v'$ be a sequence of vertices in a shortest path P_H in H, where $k = d^G(u, u')$ and $j = d^H(v, v')$. Notice that by property of binomial coefficients, $\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')} = \binom{d^G(u,u')+d^H(v,v')}{d^G(u,u')}$. We prove that from these two paths P_G and P_H, we can build $\binom{d^G(u,u')+d^H(v,v')}{d^H(v,v')} = \binom{k+j}{j}$ paths from (u, v) to (u', v') in $G\Box H$. By property of the Cartesian product graph, we know that $d^{G\Box H}((u, v), (u', v')) = k + j$, i.e. the shortest paths from (u, v) to (u', v') contain $k + j$ edges. In $G\Box H$, these edges are alternatively from P_G in a copy of G and from P_H in a copy of H. For instance, if we take all the first edges from P_G and then from P_H, the corresponding path is: $(u, v)(u_2, v)(u_3, v)\ldots(u_k, v)(u', v)(u', v_2)(u', v_3)\ldots(u', v_j)(u', v')$, and in the other direction, if we take all the first edges from P_H and then from P_G, vertices $(u, v), (u, v_2), (u, v_3), \ldots, (u, v_j), (u, v'), (u_2, v'), (u_3, v'), \ldots, (u_k, v'), (u', v')$ form a shortest path from (u, v) to (u', v'). Another shortest path is $(u, v)(u_2, v)(u_2, v_2)(u_2, v_3)\ldots(u_2, v_j)(u_3, v_j)\ldots(u_k, v_j)(u', v_j)(u', v')$. By definition of the Cartesian product graph, all shortest paths can be built by this way. Actually, all such shortest path is well-defined by the paths P_G and P_H if and only if we know exactly which edge is in a copy of which graph : G or H. So, to count the maximum number of shortest paths, it is sufficient to count the number of manners to choose among the $k + j$ edges which ones will be in G, or which ones will be in H. In fact, there are $\binom{k+j}{j}$ such manners. All in all, by considering all shortest paths P_G between u and u' in G and all ones P_H between v and v' in H, we obtain $\nu^s_G(u, u')\nu^s_H(v, v')\binom{k+j}{j}$ shortest paths between (u, v) and (u', v') in $G\Box H$. $\square$

Let's go back to our example. If we count all possible simple shortest paths between vertices $(a, 1)$ and $(c, 2)$ in the cube (see Figure 1), we can construct this family by considering shortest paths in the cycle on 4 vertices and those in the path on 2 vertices, and by combining them. Indeed, in the cycle C_4 on vertices $\{a, b, c, d\}$ with the edge set $\{ab, bc, cd, ad\}$, the only 2 shortest paths between a and c is one going through the vertex b (noted by α) and another one going through the vertex d (noted by β). The length of both is 2. Naturally, there is only one direct shortest path in the path P_2 on 2 vertices. To construct shortest paths in the Cartesian product graph, it is sufficient to determine which path we use among $\{\alpha, \beta\}$ and then when we switch from the first copy of C_4 to the second one (at the beginning at the position "0", in the middle at the position "1" or at the end at the position "2"). All simple shortest paths of the cube are illustrated by Figure 2. There are exactly

$$\nu^s_{C_4}(a,c)\nu^s_{P_2}(1,2)\binom{d^{C_4}(a,c)+d^{P_2}(1,2)}{d^{C_4}(a,c)} = 2\times 1\times \binom{2+1}{2} = 2\times 1\times 3 = 6.$$

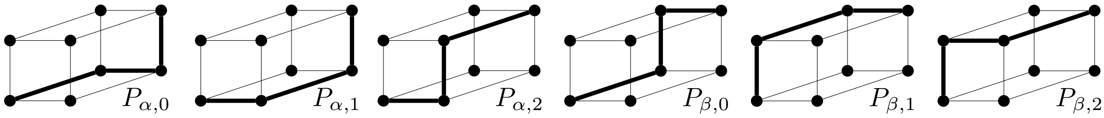

Figure 2: The cube and the only 6 simple shortest paths between $(a,1)$ and $(c,2)$, of length 3.

Theorem 2. *Let e be an arbitrary edge in $G\Box H$. Without loss of generality, we can suppose that e links vertices (u'',v'') and (u'',v'''). Then the number of shortest paths going through this edge from (u,v) to (u',v') in $G\Box H$ is exactly, if the vertex u'' appears in at least one shortest path P_G in G between u and u', and if the edge $v''v'''$ appears in at least one shortest path P_H in H between v and v',*

$$\sum_{P_G:V(P_G)\ni u''}\ \sum_{P_H:E(P_H)\ni v''v'''} \binom{d^G(u,u'')+d^H(v,v''v''',P_H)}{d^G(u,u'')}\binom{d^G(u'',u')+d^H(v',v''v''',P_H)}{d^G(u'',u')},$$

where $d^H(x,yz,P_H)=d^H(x,y)$ if y is the first vertex among $\{y,z\}$ encountered in P_H from x, $d^H(x,z)$ otherwise. In other cases, this number is 0.

Proof. Let P_G be a shortest path from u to u' in G and P_H be a shortest path from v to v' in H. Clearly if the vertex u'' does not appear in P_G or if the edge $v''v'''$ does not appear in P_H, no shortest path from (u,v) to (u',v') in $G\Box H$, built from P_G and P_H, goes through the edge e. So we may suppose that $u''\in V(P_G)$ and $v''v'''\in E(P_H)$. Without loss of generality, we can assume that v'' is the first vertex among $\{v'',v'''\}$ encountered in P_H from v. In that case, it is sufficient to count the number of shortest paths between (u,v) and (u'',v'') and the number of shortest paths between (u'',v''') and (u',v'), because a shortest path from (u,v) to (u',v'), going through e, begings with a shortest path from (u,v) to (u'',v''), goes through e and then finishes by a shortest path from (u'',v''') to (u',v'). Therefore, based on the previous result, we obtain

$$\sum_{P_G:V(P_G)\ni u''}\ \sum_{P_H:E(P_H)\ni v''v'''} \binom{d^G(u,u'')+d^H(v,v'')}{d^G(u,u'')}\binom{d^G(u'',u')+d^H(v',v''')}{d^G(u'',u')}$$

shortest paths from (u,v) to (u',v'), going through e, which completes the proof. $\square$

With a similar proof of the previous theorem, Theorem 3 counts the number of shortest simple paths going through a vertex.

Theorem 3. *Let (u'',v'') be an arbitrary vertex in $G\Box H$. Then the number of shortest paths going through this vertex from (u,v) to (u',v') in $G\Box H$ is exactly*

$$\sum_{P_G:V(P_G)\ni u''}\ \sum_{P_H:V(P_H)\ni v''} \binom{d^G(u,u'')+d^H(v,v'')}{d^H(v,v'')}\binom{d^G(u'',u')+d^H(v'',v')}{d^H(v'',v')},$$

if the vertex u'', respectively v'', appears in at least one shortest path P_G, respectively P_H, in G, respectively in H, between u and u', respectively between v and v', 0 otherwise.

3.2 Maximum number of vertex-disjoint shortest paths

We denote by $\nu^v_G(u,u')$ the maximum number of vertex-disjoint shortest paths between u and u' in G, similarly in H and in $G\Box H$. Following theorems establish the maximum number of vertex-disjoint shortest paths in a Cartesian product graph, and what happens after an edge removal or a vertex removal.

Theorem 4. *The maximum number of vertex-disjoint shortest paths in the Cartesian product graph $G\Box H$ between (u,v) and (u',v') is exactly*

$$\nu^v_{G\Box H}((u,v),(u',v')) = \nu^v_G(u,u') + \nu^v_H(v,v').$$

Proof. Because all shortest paths are made from shortest paths in G and shortest paths in H, the number of neighbors of (u,v) (or edges adjacent to (u,v)) used in a shortest path from (u,v) to (u',v') in $G\Box H$ is at most the sum between the number of neighbors of u (or edges adjacent to u) used in a shortest path from u to u' in G and the number of neighbors of v (or edges adjacent to v) used in a shortest path from v to v' in H. Thus,

$$\nu^v_{G\Box H}((u,v),(u',v')) \leq \nu^v_G(u,u') + \nu^v_H(v,v').$$

Now, we will describe $\nu^v_G(u,u') + \nu^v_H(v,v')$ shortest paths from (u,v) to (u',v'). Let $P^\star_G$ be a shortest path from u to u' in G, and let $P^\star_H$ be a shortest path from v to v' in H. For these two paths, we will consider the two shortest paths in $G\Box H$, by taking all the first edges from $P^\star_G$ and then from $P^\star_H$, and conversely, i.e. if $P^\star_G = uu^\star_2\dots u^\star_k u'$ and $P^\star_H = vv^\star_2\dots v^\star_j v'$, then one shortest path in $G\Box H$ is $(u,v)(u^\star_2,v)\dots(u^\star_k,v)(u',v)(u',v^\star_2)\dots(u',v^\star_j)(u',v')$ whereas the other one is $(u,v)(u,v^\star_2)\dots(u,v^\star_j)(u,v')(u^\star_2,v')\dots(u^\star_k,v')(u',v')$. Clearly these two paths are vertex-disjoint.

For any other possible shortest path P_G from u to u' in G (notice that if such path exists, then $d^G(u,u')\geq 2$), we proceed in the following way: if $P_G = uu_2\dots u_k u'$, then taking the first edge from P_G then following $P^\star_H$, and finally finishing P_G, i.e. $(u,v)(u_2,v)(u_2,v^\star_2)\dots(u_2,v^\star_j)(u_2,v')(u_3,v')\dots(u_k,v')(u',v')$ is a shortest path from (u,v) to (u',v'), which is vertex-disjoint from the other ones since it is also the case for paths in G. For any other possible shortest path P_H from v to v' in H, a similar argumentation holds, by symmetry of the Cartesian product graph. □

In our example of the cube, it is easy to see that the maximum number of vertex-disjoint shortest paths between $(a,1)$ and $(c,2)$ is $3 = \nu^v_{C_4}(a,c) + \nu^v_{P_2}(1,2)$. Moreover, an instance of family reaching this bound is $\{P_{\alpha,0}, P_{\alpha,2}, P_{\beta,1}\}$ from Figure 2.

The following theorem establishes results after the removal of an edge. Beforehand, we will prove two lemmas.

Lemma 1. *Let $e=(u,v)(u,v''')$ be an edge in $G\Box H$, which appears in a shortest path from (u,v) to (u',v'). If $\nu^v_{H\backslash vv'''}(v,v') = \nu^v_H(v,v')-1$ and $d_{H\backslash vv'''}(v,v') = d_H(v,v')$, or $d_{H\backslash vv'''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\backslash e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')) - 1,$$

otherwise

$$\nu^v_{G\Box H\backslash e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

Proof. The hypothesis mean that all maximum collection of vertex-disjoint shortest paths between v and v' in H necessary uses the edge vv'''. By construction of shortest paths in $G\Box H$, all maximum collection of vertex-disjoint shortest paths between (u,v) and (u',v') in $G\Box H$ necessary uses also the edge $(u,v)(u,v''')$, i.e. e. Accordingly,

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')) - 1.$$

Otherwise there exists a collection of $\nu^v_H(v,v')$ vertex-disjoint shortest paths between v and v' in H, avoiding the edge vv'''. With this collection, we can build the desired collection as we did in the proof of Theorem 4. Thus

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

□

Lemma 2. *Let $e = (u'',v'')(u'',v''')$ be an edge in $G\Box H$, which appears in a shortest path from (u,v) to (u',v') and is not adjacent to (u,v) and to (u',v'). Assume that $d^G(u,u') = 1$. If $\nu^v_H(v,v') = 1$ and $d_{H\setminus v''v'''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')) - 1,$$

otherwise

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

Proof. Let $P^\star_G$ be the shortest path between u and u' in G, i.e. $P^\star_G$ contains only the edge uu'. The hypothesis mean that all collection of vertex-disjoint shortest paths from v to v' in H contains only one path and always uses the edge $v''v'''$. Since $d^G(u,u') = 1$, then $\nu^v_G(u,u') = 1$, and so $\nu^v_{G\Box H}((u,v),(u',v')) = 2$. If $P^\star_H$ is a shortest path between v and v' in H, then the only one collection $\mathcal{C}$ of vertex-disjoint shortest paths between (u,v) and (u',v') made from $P^\star_H$ and from $P^\star_G$ is $P^\star_G$ followed by $P^\star_H$ and $P^\star_H$ followed by $P^\star_G$. Since $P^\star_H$ uses the edge $v''v'''$, then the collection $\mathcal{C}$ necessary uses edges $(u,v'')(u,v''')$ and $(u',v'')(u',v''')$. Accordingly, removing the edge e is equivalent to remove either $(u,v'')(u,v''')$ or $(u',v'')(u',v''')$, and so

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')) - 1.$$

Now, we consider the case when $\nu^v_H(v,v') = 1$ and $d_{H\setminus v''v'''}(v,v') = d_H(v,v')$. In that case, there exists a shortest path between v and v' in H, avoiding the edge $v''v'''$, say $P^\bullet_H$. Since $\nu^v_H(v,v') = 1$ and $d^G(u,u') = 1$, we have that $\nu^v_{G\Box H}((u,v),(u',v')) = 2$. The paths $P^\star_G$ followed by $P^\bullet_H$ and $P^\bullet_H$ followed by $P^\star_G$ are the two vertex-disjoint shortest paths from (u,v) to (u',v') in $G\Box H \setminus e$, thus

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

The last case is when $\nu^v_H(v,v') > 1$. Necessary, there exists at least one shortest path from v to v' in H, avoiding $v''v'''$. Choose one in a maximum collection of vertex-disjoint shortest paths, say $P^\oplus_H$. It remains to build a collection $\mathcal{C}$ of $\nu^v_{G\Box H}((u,v),(u',v'))$ vertex-disjoint shortest paths from (u,v) to (u',v') in $G\Box H$, avoiding e. First, we add the two shortest paths: $P^\star_G$ followed by $P^\oplus_H$ and $P^\oplus_H$ followed by $P^\star_G$. Clearly, they avoid e. We distinguish two subcases: $u'' = u$ and $u'' = u'$.

- If $u'' = u$, for any other vertex-disjoint shortest path P_H from v to v' in H, we build the following path P: we start with the first edge of P_H, then we take $P^\star_G$, finally we finish P_H. Since e is not adjacent to (u,v), P avoids e.

- If $u'' = u'$, we do a similar construction, except that we start with all edge from P_H excluding the last one, we follow $P_G^\star$ and then finish with the last edge of P_H. Since e is not adjacent to (u', v'), this path avoids e.

All in all, we proved that

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

$\square$

Theorem 5. *Let $e = (u'', v'')(u''', v''')$ be an arbitrary edge in $G\Box H$. We assume that e appears in a shortest path from (u,v) to (u',v') in $G\Box H$. If $d^G(u,u') = 0$ and $d_{H\setminus v''v'''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{H\setminus v''v'''}(v,v'),$$

otherwise if $d^H(v,v') = 0$ and $d_{G\setminus u''u'''}(u,u') > d_G(u,u')$ then

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\setminus u''u'''}(u,u'),$$

otherwise if one of the following conditions holds:

(i) $e = (u,v)(u,v''')$ *and* $\nu^v_{H\setminus vv'''}(v,v') = \nu^v_H(v,v') - 1$ *and* $d_{H\setminus vv'''}(v,v') = d_H(v,v')$

(ii) $e = (u,v)(u,v''')$ *and* $d_{H\setminus vv'''}(v,v') > d_H(v,v')$

(iii) $e = (u,v)(u''',v)$ *and* $\nu^v_{G\setminus uu'''}(u,u') = \nu^v_G(u,u') - 1$ *and* $d_{G\setminus uu'''}(u,u') = d_G(u,u')$

(iv) $e = (u,v)(u''',v)$ *and* $d_{G\setminus uu'''}(u,u') > d_G(u,u')$

(v) $e = (u',v'')(u',v')$ *and* $\nu^v_{H\setminus v''v'}(v,v') = \nu^v_H(v,v') - 1$ *and* $d_{H\setminus v''v'}(v,v') = d_H(v,v')$

(vi) $e = (u',v'')(u',v')$ *and* $d_{H\setminus v''v'}(v,v') > d_H(v,v')$

(vii) $e = (u'',v')(u',v')$ *and* $\nu^v_{G\setminus u''u'}(u,u') = \nu^v_G(u,u') - 1$ *and* $d_{G\setminus u''u'}(u,u') = d_G(u,u')$

(viii) $e = (u'',v')(u',v')$ *and* $d_{G\setminus u''u'}(u,u') > d_G(u,u')$

(ix) $d^G(u,u') = 0$ *and* $d_{H\setminus v''v'''}(v,v') = d_H(v,v')$ *and* $\nu^v_{H\setminus v''v'''}(v,v') = \nu^v_H(v,v') - 1$

(x) $d^G(u,u') = 1$ *and* $\nu^v_H(v,v') = 1$ *and* $d_{H\setminus v''v'''}(v,v') > d_H(v,v')$

(xi) $d^H(v,v') = 0$ *and* $d_{G\setminus u''u'''}(u,u') = d_G(u,u')$ *and* $\nu^v_{G\setminus u''u'''}(u,u') = \nu^v_G(u,u') - 1$

(xii) $d^H(v,v') = 1$ *and* $\nu^v_G(u,u') = 1$ *and* $d_{G\setminus u''u'''}(u,u') > d_G(u,u')$

then

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')) - 1,$$

otherwise

$$\nu^v_{G\Box H\setminus e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

Proof. Notice that $d^G(u,u') = 0$ means that we work only in the copy H^u and then computing $\nu^v_{G\Box H}((u,v),(u',v'))$, resp. $\nu^v_{G\Box H\backslash e}((u,v),(u',v'))$, is equivalent to compute $\nu^v_H(v,v')$, resp. $\nu^v_{H\backslash v''v'''}(v,v')$. If removing the edge $v''v'''$ in H does not change the distance between v and v' in H then in that case we can evaluate $\nu^v_{G\Box H\backslash e}((u,v),(u',v'))$. Indeed, if $\nu^v_{H\backslash v''v'''}(v,v') = \nu^v_H(v,v') - 1$ then

$$\begin{aligned}\nu^v_{G\Box H\backslash e}((u,v),(u',v')) &= \nu^v_{H\backslash v''v'''}(v,v') \\ &= \nu^v_H(v,v') - 1 \\ &= \nu^v_{G\Box H}((u,v),(u',v')) - 1.\end{aligned} \tag{1}$$

Otherwise $\nu^v_{H\backslash v''v'''}(v,v') = \nu^v_H(v,v')$, and so $\nu^v_{G\Box H\backslash e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v'))$ by a similar argumentation. Otherwise the distance between v and v' in H increases after removing $v''v'''$ and we can only say that $\nu^v_{G\Box H\backslash e}((u,v),(u',v')) = \nu^v_{H\backslash v''v'''}(v,v')$. The case $d^H(v,v') = 0$ is analogous by symmetry of the Cartesian product graph $G\Box H$.

By Lemma 1 and by symmetry, we have already the result when the edge $(u'',v'')(u''',v''')$ is adjacent to (u,v) or to (u',v'), especially proving cases from (i) to (viii). Now, we can suppose that the edge $(u'',v'')(u''',v''')$ is not adjacent to (u,v) and to (u',v').

If $d^G(u,u') = 1$ or $d^H(v,v') = 1$ then Lemma 2 deals these cases, especially cases (x) and (xii). From now, we can assume that $d^G(u,u') \geq 2$ and $d^H(v,v') \geq 2$.

We can assume that the collection $\mathcal{C}$ of shortest paths in proof of Theorem 4 is achieved. By symmetry, we may suppose that e links vertices (u'',v'') and (u'',v'''). Our goal is to prove that

$$\nu^v_{G\Box H\backslash e}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

If $u'' = u$ then the edge e is on the path from $P^\star_H$ following by $P^\star_G$. Instead of $(u,v)(u,v^\star_2)\ldots(u,v'')(u,v''')\ldots(u,v^\star_j)(u,v')(u^\star_2,v')\ldots(u^\star_k,v')(u',v')$, we consider $(u,v)(u,v^\star_2)\ldots(u,v'')(u^\star_2,v'')(u^\star_2,v''')\ldots(u^\star_2,v^\star_j)(u^\star_2,v')\ldots(u^\star_k,v')(u',v')$ in $\mathcal{C}$.

If $u'' = u'$, then two cases are possible: the path P is either $P^\star_G$ followed by $P^\star_H$, i.e. $(u,v)(u^\star_2,v)\ \ldots\ (u^\star_k,v)(u',v)(u',v^\star_2)\ldots(u',v'')(u',v''')\ldots(u',v^\star_j)(u',v')$, or made from $P^\star_G$ and another path P_H, i.e. $(u,v)(u,v_2)(u^\star_2,v_2)\ldots(u^\star_k,v_2)(u',v_2)\ldots(u',v'')(u',v''')\ldots(u',v_j)(u',v')$. In $\mathcal{C}$, it is so sufficient to consider $(u,v)(u^\star_2,v)(u^\star_2,v^\star_2)\ldots(u^\star_2,v''')\ldots(u^\star_k,v''')(u',v''')\ldots(u',v^\star_j)(u',v')$ for the first case, whereas $(u,v)(u,v_2)(u^\star_2,v_2)\ldots(u^\star_2,v''')\ldots(u^\star_k,v''')(u',v''')\ldots(u',v_j)(u',v')$ in the second case.

If $u \neq u'' \neq u'$, then the path P is made from another P_G and $P^\star_H$, i.e. $P = (u,v)(u_2,v)(u_2,v^\star_2)\ \ldots\ (u_2,v'')(u_2,v''')\ldots(u_2,v^\star_j)(u_2,v')\ldots(u_k,v')(u',v')$ and $u'' = u_2$. Then we can replace P and the path $P^\star_G$ followed by $P^\star_H$, and its converse, i.e. $P^\star_H$ followed by $P^\star_G$, by interchanging the role of P_G and $P^\star_G$ in the construction of these three paths, i.e.

$$(u,v)(u_2,v)\ldots(u_k,v)(u',v)(u',v^\star_2)\ldots(u',v^\star_j)(u',v'),$$

$$(u,v)(u,v^\star_2)\ldots(u,v^\star_j)(u,v')(u_2,v')\ldots(u_k,v')(u',v'),$$

and

$$(u,v)(u^\star_2,v)(u^\star_2,v^\star_2)\ldots(u^\star_2,v^\star_j)(u^\star_2,v')\ldots(u^\star_k,v')(u',v').$$

All in all, since e is not adjacent to (u,v) and to (u',v'), we create a new collection $\mathcal{C}'$ from $\mathcal{C}$, of vertex-disjoint shortest paths from (u,v) to (u',v') in $G\Box H$, avoiding e and with the same cardinality of $\mathcal{C}$. $\Box$

Despite the cases described by this theorem, where the number $\nu^v_{G\Box H}((u,v),(u',v'))$ changes after an edge removal, the theorem brings good news: overall, removing an arbitrary edge in $G\Box H$ does not change the maximum number of vertex-disjoint shortest paths, especially when (u,v) and (u',v') are far from each other as well as the pair u and u' in G and the pair v and v' in H. Indeed, in that case, we have many maximum collections of vertex-disjoint shortest paths, which is not surprising in view of the number of simple shortest paths between (u,v) and (u',v') in $G\Box H$.

Similarly to the removal of an edge, we will establish first two lemmas, and then we will set the general theorem about vertex removal. Since the following lemmas are quite the same than Lemma 1 and Lemma 2, their proof are left to the reader.

Lemma 3. *Let (u,v'') be a vertex in $G\Box H$, which appears in a shortest path from (u,v) to (u',v'), with v'' adjacent to v in H. If $\nu^v_{H\setminus v''}(v,v') = \nu^v_H(v,v') - 1$ and $d_{H\setminus v''}(v,v') = d_H(v,v')$, or $d_{H\setminus v''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\setminus(u,v'')}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')) - 1,$$

otherwise

$$\nu^v_{G\Box H\setminus(u,v'')}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

Lemma 4. *Let (u'',v'') be a vertex in $G\Box H$, which appears in a shortest path from (u,v) to (u',v') and is not adjacent to (u,v) and to (u',v'). Assume that $d^G(u,u') = 1$. If $\nu^v_H(v,v') = 1$ and $d_{H\setminus v''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\setminus(u'',v'')}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')) - 1,$$

otherwise

$$\nu^v_{G\Box H\setminus(u'',v'')}((u,v),(u',v')) = \nu^v_{G\Box H}((u,v),(u',v')).$$

Theorem 6. *Let (u'',v'') be an arbitrary vertex in $G\Box H$. We assume that (u'',v'') appears in a shortest path from (u,v) to (u',v') in $G\Box H$. If $d^G(u,u') = 0$ and $d_{H\setminus v''}(v,v') > d_H(v,v')$ then*

$$\nu^v_{G\Box H\setminus(u'',v'')}((u,v),(u',v')) = \nu^v_{H\setminus v''}(v,v'),$$

otherwise if $d^H(v,v') = 0$ and $d_{G\setminus u''}(u,u') > d_G(u,u')$ then

$$\nu^v_{G\Box H\setminus(u'',v'')}((u,v),(u',v')) = \nu^v_{G\setminus u''}(u,u'),$$

otherwise if one of the following conditions holds:

(i) $u'' = u$ and v'' adjacent to v in H and $\nu^v_{H\setminus v''}(v,v') = \nu^v_H(v,v') - 1$ and $d_{H\setminus v''}(v,v') = d_H(v,v')$

(ii) $u'' = u$ and v'' adjacent to v in H and $d_{H\setminus v''}(v,v') > d_H(v,v')$

(iii) $v'' = v$ and u'' adjacent to u in G and $\nu^v_{G\setminus u''}(u,u') = \nu^v_G(u,u') - 1$ and $d_{G\setminus u''}(u,u') = d_G(u,u')$

(iv) *$v'' = v$ and u'' adjacent to u in G and $d_{G\setminus u''}(u,u') > d_G(u,u')$*

(v) *$u'' = u'$ and v'' adjacent to v' in H and $\nu^v_{H\setminus v''}(v,v') = \nu^v_H(v,v') - 1$ and $d_{H\setminus v''}(v,v') = d_H(v,v')$*

(vi) *$u'' = u'$ and v'' adjacent to v' in H and $d_{H\setminus v''}(v,v') > d_H(v,v')$*

(vii) *$v'' = v'$ and u'' adjacent to u' in G and $\nu^v_{G\setminus u''}(u,u') = \nu^v_G(u,u') - 1$ and $d_{G\setminus u''}(u,u') = d_G(u,u')$*

(viii) *$v'' = v'$ and u'' adjacent to u' in G and $d_{G\setminus u''}(u,u') > d_G(u,u')$*

(ix) *$d^G(u,u') = 0$ and $d_{H\setminus v''}(v,v') = d_H(v,v')$ and $\nu^v_{H\setminus v''}(v,v') = \nu^v_H(v,v') - 1$*

(x) *$d^G(u,u') = 1$ and $d_{H\setminus v''}(v,v') > d_H(v,v')$ and $\nu^v_H(v,v') = 1$*

(xi) *$d^H(v,v') = 0$ and $d_{G\setminus u''}(u,u') = d_G(u,u')$ and $\nu^v_{G\setminus u''}(u,u') = \nu^v_G(u,u') - 1$*

(xii) *$d^H(v,v') = 1$ and $d_{G\setminus u''}(u,u') > d_G(u,u')$ and $\nu^v_G(u,u') = 1$*

then

$$\nu^v_{G\square H\setminus(u'',v'')}((u,v),(u',v')) = \nu^v_{G\square H}((u,v),(u',v')) - 1,$$

otherwise

$$\nu^v_{G\square H\setminus(u'',v'')}((u,v),(u',v')) = \nu^v_{G\square H}((u,v),(u',v')).$$

Proof. By a similar argumentation in proof of Theorem 5, the first equalities hold with Lemma 3 and Lemma 4. So we may suppose that (u'',v'') is not adjacent to (u,v) and to (u',v'), and that $d^G(u,u') \geq 2$ and $d^H(v,v') \geq 2$. Besides, we assume that the collection $\mathcal{C}$ of shortest paths in proof of Theorem 4 is achieved after the removal of vertex (u'',v''). We distinguish different cases, depending on the kind of the achieved path P. By symmetry of the Cartesian product graph, only two cases are considered: P is made either from $P^\star_G$ and $P^\star_H$ or from $P^\star_H$ and another P_G.

In the first case, we assume that $P = (u,v)(u^\star_2,v)\dots(u^\star_k,v)(u',v)(u',v^\star_2)\dots(u',v^\star_j)(u',v')$. Then it is sufficient to consider $(u,v)(u^\star_2,v)(u^\star_2,v^\star_2)\dots(u^\star_2,v^\star_j)\dots(u^\star_k,v^\star_j)(u',v^\star_j)(u',v')$, instead of P. This path and those in the remaining collection $\mathcal{C}\setminus\{P\}$ are vertex-disjoint since $d^G(u,u') > 1$ and $d^H(v,v') > 1$.

In the second case, we suppose that

$$P = (u,v)(u_2,v)(u_2,v^\star_2)\dots(u_2,v^\star_j)(u_2,v')\dots(u_k,v')(u',v'),$$

then replacing the three paths P,

$$(u,v)(u,v^\star_2)\dots(u,v^\star_j)(u,v')(u^\star_2,v')\dots(u^\star_k,v')(u',v')$$

and

$$(u,v)(u^\star_2,v)\dots(u^\star_k,v)(u',v)(u',v^\star_2)\dots(u',v^\star_j)(u',v')$$

by

$$(u,v)(u^\star_2,v)(u^\star_2,v^\star_2)\dots(u^\star_2,v^\star_j)(u^\star_2,v')\dots(u^\star_k,v')(u',v')$$

and

$$(u,v)(u,v^\star_2)\dots(u,v^\star_j)(u,v')(u_2,v')\dots(u_k,v')(u',v')$$

and

$$(u,v)(u_2,v)\dots(u_k,v)(u',v)(u',v_2^\star)\dots(u',v_j^\star)(u',v'),$$

i.e. interchanging the role of P_G and $P_G^\star$ in the construction of the three paths is sufficient to keep the same number of vertex-disjoint shortest paths. $\square$

Obviously, the previous theorem is inspired from Theorem 5 and the result is not surprisingly.

3.3 Maximum number of edge-disjoint shortest paths

We denote by $\nu_G^e(u,u')$ the maximum number of edge-disjoint shortest paths between u and u' in G, similarly in H and in $G\square H$. The whole theory about vertex-disjoint shortest paths in the previous subsection yields same results on the maximum number $\nu_{G\square H}^e((u,v),(u',v'))$ of edge-disjoint shortest paths between (u,v) and (u',v') in $G\square H$. Indeed, in this paper, we consider only simple[2] graphs and if $P = uu_2\dots u_k u'$ and $P^\bullet = uu_2^\bullet\dots u_k^\bullet u'$ are two edge-disjoint shortest paths from u to u' in G then it must be that $u_2 \neq u_2^\bullet$ and $u_k \neq u_k^\bullet$, which implies that paths in the collection $\mathcal{C}$ described in Theorem 4 are edge-disjoint if it is also the case for paths from u to u' in G and paths from v to v' in H.

Theorem 7. *The maximum number of edge-disjoint shortest paths in the Cartesian product graph $G\square H$ between vertices (u,v) and (u',v') is exactly*

$$\nu_{G\square H}^e((u,v),(u',v')) = \nu_G^e(u,u') + \nu_H^e(v,v').$$

As expected for our example, the maximum number of edge-disjoint shortest paths between $(a,1)$ and $(c,2)$ in the cube is exactly $3 = \nu_{C_4}^e(a,c) + \nu_{P_2}^e(1,2)$, and the same family in the vertex-disjoint case is reaching this bound.

Similarly, Theorem 5 and Theorem 6, i.e. theorems on the edge removal and the vertex removal, can be easly translated in terms of edge-disjoint paths. To avoid redundancy, we omit to formulate and to prove them here.

4 Conclusions

In this paper, we computed the maximum number of simple, vertex-disjoint and edge-disjoint shortest paths in Cartesian product graph in terms of those from the original graphs. For simple shortest paths, the maximum number in the Cartesian product graph is the product of the maximum numbers in the original graphs multiplied by a binomial coefficient depending on distances in the original graphs, i.e. for every $u,u' \in V(G)$ and $v,v' \in V(H)$,

$$\nu_{G\square H}^s((u,v),(u',v')) = \nu_G^s(u,u')\nu_H^s(v,v')\binom{d^G(u,u') + d^H(v,v')}{d^G(u,u')}.$$

However, for vertex-disjoint and edge-disjoint shortest paths, the results are quite different. Instead of a product, the maximum number of such paths in the Cartesian product graph is the sum of the corresponding maximum numbers in the original graphs, i.e. for every $u,u' \in V(G)$ and $v,v' \in V(H)$,

$$\nu_{G\square H}^v((u,v),(u',v')) = \nu_G^v(u,u') + \nu_H^v(v,v')$$

[2]The word "simple" means here "without loops or multiple edges".

and

$$\nu^e_{G\square H}((u,v),(u',v')) = \nu^e_G(u,u') + \nu^e_H(v,v').$$

Besides, we established the exact number of shortest paths going through a fixed vertex or a fixed edge. In the case of vertex-disjoint paths or edge-disjoint paths between two vertices (u,v) and (u',v'), the maximum number of shortest paths can not decrease by more than 1 but it remains mainly the same.

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