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POWER LAW OR POWER FLAW?

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ABSTRACT

Since its introduction by Svensson in 1959, the power law curve $y=ax^b$ (where x and y are horizontal and vertical direction, respectively) has been widely used in morphological analysis of glacial trough cross-profiles. The numerical constants a and b are obtained by a linear regression analysis of the logarithmic form of the power law curve ($\ln y = \ln a + b \ln x$). The value b then gives a measure for the form of the cross-section. However, over the years this form of the power law has endured a lot of criticism. This criticism is well founded, since this particular form of the power law is not suitable for curve fitting in morphological analyses. In this paper a general power law is proposed, of the form $y-y_0=a|x-x_0|^b$ (where x_0, y_0 are the coordinates of the origin of the cross-profile). A unique and unbiased solution for this equation is obtained with a general least-squares method, thereby minimizing the error between the cross-profile data and the curve, and not between the logarithmic transform of the data and its regression line. This provides a robust way to characterize trough cross-section forms. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: glacial trough; least squares; morphometry; power law

THE POWER FLAW

When studying the morphology of glacial troughs and ultimately glacial trough development, the trough cross-profile is often approximated by a mathematical function, allowing a quantitative analysis. Glacial and/or glaciated trough cross-sections are generally referred to as being U-shaped or parabolic, while troughs that underwent little glacial modification tend to be V-shaped, characterized by more or less linear slopes (Hirano and Aniya, 1988, 1990; Harbor, 1990; Harbor and Wheeler, 1992). A common argument is that under sustained glacial erosion, cross-profile morphologies progress from V-shaped to U-shaped forms (Harbor and Wheeler, 1992).

It was Svensson (1959) who first introduced the power law equation as a mathematical function to represent the glacial trough cross-profile:

$$y=ax^b \tag{1}$$

with x the horizontal and y the vertical direction. The numerical constants of Equation 1 are obtained by a linear regression analysis in its logarithmic form. Since Svensson (1959), the power law equation has become a common tool in analysing the morphology of glacial trough cross-profiles (Doornkamp and King, 1971), but was later subjected to considerable criticism. The deficiencies associated with the use of the power law equation were addressed by Wheeler (1984) and further analysed by Harbor and Wheeler (1992). Three main problems were encountered:

- the datum problem, i.e. the fitted curve is forced to pass through the origin of the coordinate system used ($y=ax^b$, at $x=0$, $y=0$);
- the curve can only be fitted to one side of the trough, since x in Equation 1 cannot take negative values;
- because of the logarithmic transformation, the 'best fit' curve is biased in favour of points close to the origin of the coordinate system.

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Aniya and Welch (1981) approached the first (i.e. datum) problem by repeating the analysis for a large number of points of origin in search for the best possible fit. As a solution to the second and partly also to the third problem, James (1996) considered each trough slope separately, and simply left out the data points in the trough centre close to the origin, so that they did not bias the logarithmic transformation. Jansson (1985), on the other hand, introduced a correction factor in the logarithmic detransformation to reduce the bias.

Although the deficiencies of the power law have been well known for some time and many remedies have been proposed, the real problem lies in the fact that this power law equation (Equation 1) is not suited for use in a curve-fitting algorithm. A solution to the power law equation is obtained by transforming Equation 1 in its logarithmic form:

$$1\mathrm{n}\, y = 1\mathrm{n}\, a + b\, 1\mathrm{n}\, x \tag{2}$$

A linear regression applied to Equation 2 then gives ln a and b. However, there is a snag in the logarithmic detransformation or the transformation of ln y to y. The fact that a 'best fit' was found between ln x and ln y does not imply that one obtains the 'best fit' between x and y. All the problems associated with the datum and the bias in favour of the points close to the origin can be traced back to this 'best fit' fallacy. We are interested in a curve that describes the form of the observed trough cross-section, not the form of a logarithmic trough.

On the other hand, fitting a parabolic equation [$y=A+Bx+Cx^2$ or $y-y_0=a(x-x_0)^2$] to the observational data is a correct curve-fitting approach, since a solution to this equation is obtained by minimizing the error between the observational data and the curve. However, the interpretation of the parabolic function for morphometric analyses is different compared to the power law equation. While the power law seeks the form of the cross-profile through its power coefficient b (V-shape [$b=1$], parabolic [$b=2$], U-shaped [$b>2$]), the parabolic function defines to what degree the profile resembles a parabola through its correlation coefficient. In principle, however, the power law equation has much more potential in understanding the form of the cross-profile than has the parabolic function.

In this paper a more general power law equation is presented, better adapted for curve fitting, which was first suggested by Pattyn and Decleir (1995).

A FLAWLESS POWER LAW

A well suited power law should (i) take into account the determination of the coordinates of the origin, (ii) be symmetrical around the origin, so that the two trough walls can be considered at the same time, (iii) perform a curve fitting on the original coordinates $x^{\circ\circ}, y^{\circ\circ}$ and not on any (logarithmic) transform, whereby an equal weight is given to all coordinates (not biased for the centre coordinates). Such a power law is given by:

$$y - y_0 = a\,|\,x - x_0\,|^{\,b} = a\,\exp\{b1\mathrm{n}\,|\,x - x_0\,|\,\} \tag{3}$$

A unique and unbiased solution of this non-linear equation is obtained by the method of general least-squares adjustment.

General least squares solution

Each measured point of the cross-section will give rise to a measurement vector consisting of two observations, i.e. an x and y coordinate. The observation vector for the k^{th} measurement will thus be:

$$\overset{\circ\circ}{\mathbf{I}}_k = \left[x^{\circ\circ} \;\; y^{\circ\circ}\right]_k^T \tag{4}$$

The parameter vector will consist of the unknowns, i.e. the slope a and the coordinates of the origin x_0 and y_0. In order to stabilize the calculations, the parameter b was not considered as an unknown. Consequently, the least-squares analysis is repeated for a large number of b values (see below). The parameter vector thus becomes:

$$\bar{\mathbf{p}}_k^{\circ} = \left[a^{\circ}\ x_0^{\circ}\ y_0^{\circ}\right]^T \tag{5}$$

where the index ° stands for iteratively updated estimates. The least-squares analysis starts with a set of estimated values for the parameter vector:

$$\bar{p}_k^{\circ\circ} = \left[1\cdot 0\ \ \frac{x_{\max}^{\circ\circ} - x_{\min}^{\circ\circ}}{2}\ \ y_{\min}^{\circ\circ}\right]^T \tag{6}$$

where $x_{\max}^{\circ\circ}$ and $x_{\min}^{\circ\circ}$ are the maximum and minimum x values and $y_{\min}^{\circ\circ}$ the lowest y value of the original data set. The measurements and parameters are related to each other by Equation 3. As this relationship is non-linear it has to be linearized by Taylor series expansion before the condition equations of adjustment can be formulated and solved by iteration. Therefore, Equation 3 is rewritten in such a way that:

$$\begin{aligned} F &\equiv y - y_0 - a\left|x - x_0\right|^b &&= 0 \\ &\equiv y - y_0 - a\exp\{b\ln\left|x - x_0\right|\} &&= 0 \end{aligned} \tag{7}$$

F is expanded into Taylor series around approximate values $\mathbf{I}^{\circ\circ}$ and $\mathbf{p}^{\circ}$ for the measurements and parameters, using $\mathbf{I}^{\circ\circ}$ and $\mathbf{p}^{\circ\circ}$ in the first iteration. Second and higher order terms are neglected, and the first order derivatives are collected in the matrices:

$$\mathbf{A}_k = \left[\frac{\partial F}{\partial x}\ \frac{\partial F}{\partial y}\right] \tag{8}$$

$$\mathbf{B}_k = \left[\frac{\partial F}{\partial a}\ \frac{\partial F}{\partial x_0}\ \frac{\partial F}{\partial y_0}\right] \tag{9}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix} \tag{10}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_n \end{bmatrix} \tag{11}$$

where

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{-ab}{x - x_0}\left|x - x_0\right|^b = -\frac{\partial F}{\partial x_0} \\ \frac{\partial F}{\partial y} &= 1 = -\frac{\partial F}{\partial y_0} \\ \frac{\partial F}{\partial a} &= -\left|x - x_0\right|^b \end{aligned} \tag{12}$$

The least-squares solution to this system then becomes (e.g. Mikhail and Gracie, 1981)

$$\Delta = \left(\mathbf{B}^{\mathrm{T}}(\mathbf{AQA}^{T})^{-1}\mathbf{B}\right)^{-1}\mathbf{B}^{T}\left(\mathbf{AQA}^{T}\right)^{-1}\mathbf{f}^{\circ} \tag{13}$$

where $\mathbf{Q}$ is the cofactor matrix, which is taken as the unity matrix (hence considering all measurements of equal weight), and $\mathbf{f}^{\circ} = -F(\mathbf{I}^{\circ}, \mathbf{p}^{\circ})$ the solution vector. Δ contains the corrections to the unknowns a, x_0 and y_0. The solution has to be properly iterated until it converges or $\Delta \equiv 0$.

Determining the optimal b

For any given value of b, a unique least-squares solution or best-fitted curve is obtained with Equation 13 on the original x and y coordinates and not on their logarithmic values. For example, the solution for $b = 2{\cdot}0$ perfectly coincides with a parabolic curve, thus corroborating the reliability of the proposed technique. In order to measure how well the curve fits the observational data, the root mean square (RMS) error is calculated:

$$\text{RMS error} = \sqrt{\frac{\sum_{i=1}^{n}(y_i^{\circ\circ} - y_i^{\circ})^2}{n}} \tag{14}$$

where $y_i^{\circ\circ}$ and y_i° are the measured and fitted y values respectively.

Repeating the least-squares analysis for a large number of b values results in a set of RMS errors. The optimal b value is the one corresponding to the lowest RMS error. Experiments showed that for each cross-sectional profile only one optimal b value, corresponding to the lowest RMS error, is obtained. Therefore, a more efficient iteration was devised, starting from $b = 2$, and working its way progressively towards the optimal b value, through knowledge of the RMS errors. The values for a, x_0 *and* y_0 of the former step were used as first estimate to the new step, instead of the values given by Equation 6. The major advantage of this computation scheme is that less iterations need to be carried out. Moreover, by decreasing the step size whenever the vicinity of the minimum RMS error is reached, a higher precision in determining the optimal b value is obtained.

DISCUSSION OF RESULTS

In order to test the accuracy and robustness of the general power law (GPL) of Equation 3, the Tenaya Canyon cross-section of James (1996) was used. The cross-sectional profile was digitized from the main Yosemite Valley (USA) (James, 1996). Only the part in between the two trim lines at either side of the trough was considered. The reason for studying this cross-profile is that James found b values ranging from 0·2 to 3·0 according to the old power law (OPL) of Equation 2. This large discrepancy in b values resulted from the different positions of the chosen datum, i.e. a datum set at sea level, at the bottom of the channel, or at the origin of the bench. The latter is situated at the upper valley wall were lateral moraines, trim lines or erratics can provide a clear limit to the glacial extent based on field evidence.

Choosing an arbitrary datum should be avoided. The origin of the curve $y = ax^b$ is situated at the lowest part of the curve, i.e. the point of inflection near the bottom of the trough. This is the case when fitting a parabola ($y - y_0 = a(x - x_0)^2$) or the GPL solution where x_0, y_0 refer to the coordinates of the origin or the point of inflection. Since a solution to the OPL is obtained by fitting a straight line to its logarithmic form, the origin then becomes the origin of this regression line (1n x = 0; 1n y=1n a), which has no affiliation with the original coordinate system. Carrying out experiments with different datum points based on geomorphological characteristics (bottom of trough, origin of bench) will get one nowhere, as will be proven below.

The GPL curve fitting was applied on the whole trough cross-section of Tenaya Valley (Figure 1). A b value of 0·92 was obtained, which means that the cross-section is V-shaped (Table I, Exp. 1). The right panel of

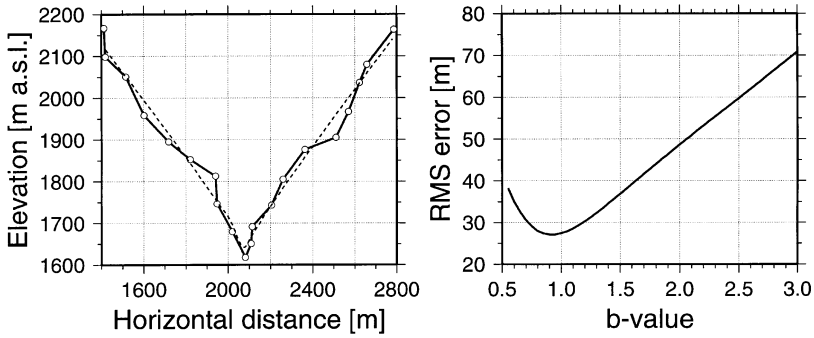

Figure 1. Left panel: curve fitting with GPL (dotted line) on Tenaya Valley cross-profile. Right panel: RMS error as a function of *b* value for GPL fitting. The optimal *b* value = 0·92. See also Table I, Exp. 1

Table I. Power law experiments (1–5) with the general power law of Equation 3 (GPL), the general power law with fixed coordinates of the origin using Equation 15 (GPLF), and with the old power law equation of Equation 2 (OPF). For the GPL model type x_0 and y_0 were determined automatically. In all other experiments these coordinates were fixed

Exp.	Model*	*b*	RMSE (m)	x_0*(m)*	y_0*(m)*	*a*
1	GPL all	0·92	27·09	2081·08000	1642·38800	1·2024E+00
2	GPL no centre	1·96	19·41	2080·95000	1792·09500	9·8922E–04
	OPL left	0·75	31·01	2081·00000	1617·00000	3·6830E+00
	OPL right	0·74	27·92	2081·00000	1617·00000	3·8374E+00
3	GPLF left	0·81	28·37	2081·00000	1617·00000	2·5094E+00
	GPLF right	0·82	26·11	2081·00000	1617·00000	2·3460E+00
	GPLF all	0·82	27·93	2081·00000	1617·00000	2·4292E+00
	OPL left	3·29	1364·42	2081·00000	1617·61905	1·7527E–06
	OPL right	3·09	1773·61	2081·00000	1617·61905	8·3039E–06
4	GPLF left	0·82	28·35	2081·00000	1617·61905	2·4719E+00
	GPLF right	0·83	26·06	2081·00000	1617·61905	2·3025E+00
	GPLF all	0·82	27·88	2081·00000	1617·61905	2·3871E+00
	OPL left	0·21	116·56	2080·91286	1617·00000	7·6024E+01
	OPL right	0·21	124.08	2080·91286	1617·00000	6·7792E+01
5	GPLF left	0·81	23·38	2080·91286	1617·00000	2·5159E+00
	GPLF right	0·82	26·11	2080·91286	1617·00000	2·3428E+00
	GPLF all	0·82	27·93	2080·91286	1617·00000	2·4292E+00

*‘Left’ and ‘right’ mean application of the power laws to the left and right trough wall, respectively; and ‘all’ means on both sides at once; ‘no centre’ means that the central gorge data are omitted

Figure 1 displays the RMS error of the curve fitting for different *b* values. A unique minimum RMS error is observed corresponding to the optimum *b* value of 0·92. A second experiment consisted of leaving out the data in the centre of the trough (the gorge). In that case a *b* value of 1·96 is obtained, i.e. a curve of parabolic shape (Figure 2 and Table I, Exp. 2).

In order to compare these results with the OPL, a datum needs to be defined. For this line of experiments different datum points were considered, all lying close to the trough bottom. The results are listed in Table I (Exp. 3 to 5; see also Figure 3). Three different coordinates x_0, y_0 were chosen as datum points. Applying the OPL led to *b* values ranging from 0·2 to 3·3. To demonstrate the robustness of the GPL and its complicated least-squares solution, the same experiment was carried out, i.e. by fixing the coordinates of the origin instead of determining them automatically. Equation 9 then reduces to:

$$\mathbf{B}_k = \left[\frac{\partial F}{\partial a}\right] \tag{15}$$

Applying this special case of the GPL with the three sets of coordinates x_0, y_0, one obtains a *b* value around 0·82 in *all* cases. Furthermore, the RMS error for these curves is smaller than the RMS error of the OPL experiments.

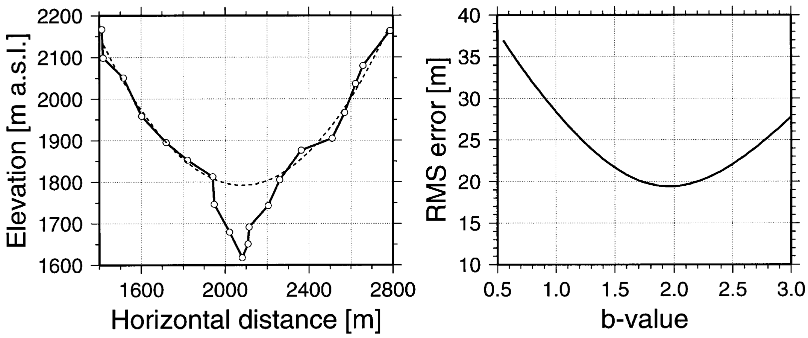

Figure 2. Same as Figure 1, but leaving out the central data points (gorge). An optimal *b* value of 1·96 is obtained. See also Table I, Exp. 2

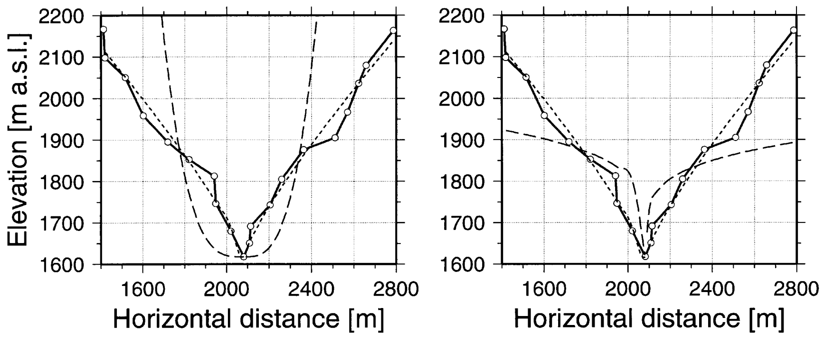

Figure 3. Application of GPLF (dotted line) and OPL (dashed line) with fixed x_0, y_0 on Tenaya Valley cross-section according to experiment 4 (left panel) and experiment 5 (right panel). See also Table I

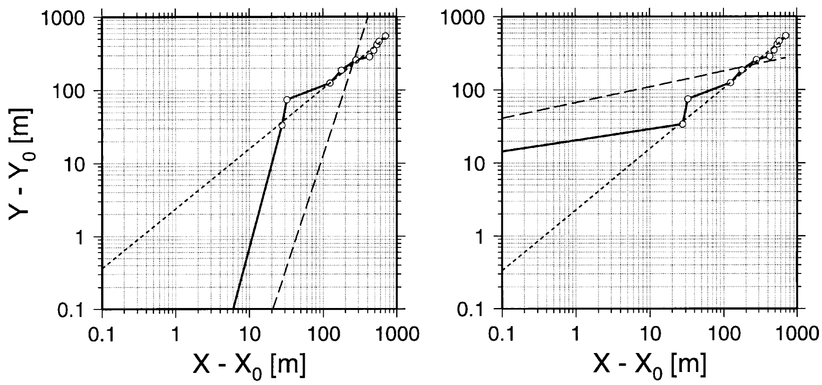

Figure 4. Same as Figure 3, but log–log plot of the results in local coordinate system (x–x_0), (y–y_0)

The reason for this is that the best fit with the *observations* is found and not the best fit of the logarithmic transformation. This is observed in Figure 4, which shows a log–log plot of x–x_0 versus y–y_0 for experiments 3 and 4 on the right trough wall. The log-linear fitted curve obtained with the OPL is in both cases strongly biased by the centre trough point. The GPL result is the same in both cases and fits close to all trough points, except the central trough point, lying close to the origin of the chosen coordinate system The difference between this line and the central trough point might seem large; in real-world coordinates (x, y) it is of the order of 1 m. Fixing x_0, y_0 in the GPL means that the curve is forced to go through this origin which is regarded as the point of inflection. This is certainly not the case for the OPL, where the datum has no affiliation with the real origin of the trough.

PROGRAM SOURCE CODE

The program source code offering a solution to the power law equation by means of a general least-squares technique is available free of charge on http://www.vub.ac.be/DGGF/PowerLaw/. All technical details concerning program compilation, input/output, etc. are found on this web site.

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NOTES

1. Both ‘General least squares solution’ and ‘determining the optimal b’ are subsections of ‘A flawless power law’.
2. It is important that the web-site address is spelled correctly:

http://www.vub.ac.be/DGGF/Powerlaw/

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