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Dynamical randomness, information, and Landauer's principle

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Abstract. - New concepts from nonequilibrium thermodynamics are used to show that Landauer's principle can be understood in terms of time asymmetry in the dynamical randomness generated by the physical process of the erasure of digital information. In this way, Landauer's principle is generalized, showing that the dissipation associated with the erasure of a sequence of bits produces entropy at the rate $k_{\rm B} I$ per erased bit, where I is Shannon's information per bit.

Landauer's principle says that the minimal dissipation accompanying the erasure of one bit of information produces an entropy equal to $k_{\rm B} \ln 2$ where $k_{\rm B}$ is Boltzmann's constant [1]. This important result is based on the observation that the processing of digital information is a physical process among others and should thus obey the laws of thermodynamics. Landauer's result has therefore supported the idea that information processing devices working at temperature T should dissipate at least $k_{\rm B} T \ln 2$ of energy during an elementary act of information [2, 3]. Later, Bennett's work clarified the point that Landauer's dissipation is the feature of logically irreversible operations on data, i.e., operations whose inverse is not unique, such as data erasure [4]. Moreover, Bennett used Landauer's result to resolve Maxwell's demon paradox [5]. Since then, Landauer's principle has been explicitly verified in specific cases, for instance, in bistable potentials with white noise [6], for a bit in contact with a thermal reservoir [7], or from coarse graining in phase space [8].

Landauer's result has been verified using case studies and it is only recently that nonequilibrium statistical mechanics has been sufficiently advanced in order to reach its understanding in a general framework. Indeed, it has been recently established that the second law of thermodynamics finds its origin in the time asymmetry of the property of dynamical randomness, i.e., the temporal disorder that a physical process develops during its time evolution [9–12]. Dynamical randomness is characterized by the decay rate of the probabilities of the typical paths or histories followed by the fluctuating process. The time asymmetry of the process can be tested by comparing the decay rate of the path probabilities with the one of the corresponding time-reversed process. These decay rates are the so-called entropy per unit time

$$h = \lim_{n\to\infty} -\frac{1}{n} \sum_{\omega_1\omega_2\cdots\omega_n} \mu(\omega_1\omega_2\cdots\omega_n) \ln \mu(\omega_1\omega_2\cdots\omega_n) \quad (1)$$

and time-reversed entropy per unit time

$$h^{\rm R} = \lim_{n\to\infty} -\frac{1}{n} \sum_{\omega_1\omega_2\cdots\omega_n} \mu(\omega_1\omega_2\cdots\omega_n) \ln \bar{\mu}(\omega_n\cdots\omega_2\omega_1) \quad (2)$$

where $\omega_1\omega_2\cdots\omega_n$ denotes a path. μ is the invariant probability measure of the equilibrium or nonequilibrium process, whereas $\bar{\mu}$ is the invariant measure of the backward process with reversed nonequilibrium drives [9–11]. The paths are observed every time interval τ (considered as the unit time) and sampled into states ω_j (with $j = \cdots - 2, -1, 0, 1, 2, \cdots$). These dynamical entropies may be considered as rates of production of information generated by the fluctuations of the process during its time evolution. In this sense, they characterize the dynamical randomness of the process. It has been established [9–11] that the difference between these quantities gives the well-known thermodynamic entropy production according to

$$\Delta_{\rm i} S = k_{\rm B}(h^{\rm R} - h) \quad (3)$$

where $\Delta_{\rm i} S$ is the entropy produced during the unit time τ. The difference between the two entropies per unit time is the relative entropy between the forward and backward paths, which is known to be non-negative and to give the thermodynamic entropy production for instance in Markovian stochastic processes [9–11]. This result has been verified in experiments on driven Brownian motion and electric noise [11]. Equation (3) explicitly shows that the thermodynamic entropy production comes from the time asymmetry in the more microscopic property of dynamical randomness. The non-negativity of the entropy production leads to the principle of temporal ordering according to which, in nonequilibrium steady states, the typical paths are more ordered in time than their corresponding time reversals [12].

The remarkable result we will here report is that the aforementioned formula (3) allows us to relate the thermodynamic entropy production to the information as it is physically recorded *in space* inside some information processing device, generalizing in this way Landauer's principle. We here consider the process of erasure of *statistically correlated* random bits, that is a sequence of bits $\sigma_1\sigma_2\cdots\sigma_m\cdots$ (with $\sigma_i = 0$ or 1). We assume that this sequence is initially recorded on some spatially extended support such as a recording tape inside the device. The recorded data are described by some probability distribution $p(\sigma_1\sigma_2\cdots\sigma_m)$ giving the occurrence frequencies of the sequences $\sigma_1\sigma_2\cdots\sigma_m$ in the memory of the device. This probability distribution is general with possible spatial correlations among the bits σ_i. The information contained in the sequence is characterized by the quantity

$$I = \lim_{m\to\infty} -\frac{1}{m} \sum_{\sigma_1\sigma_2\cdots\sigma_m} p(\sigma_1\sigma_2\cdots\sigma_m)\ln p(\sigma_1\sigma_2\cdots\sigma_m). \quad (4)$$

This is an entropy per bit of information in the sense of Shannon, which we call the information I per bit of the sequence. In the case the bits are randomly distributed with equal probability and independently of each other, the information per bit is equal to $I = \ln 2$. In general, we have the inequality $I \leq \ln 2$.

The thermodynamic entropy produced during the process of erasure of the aforementioned sequence can be obtained using the formula (3) according to the following reasoning. We suppose that one bit is erased every unit time τ. Let us associate a state ω_i with the sequence of bits $(\cdots\sigma_m\cdots\sigma_{i+1}\sigma_i\cdot 000\cdots)$, as done in Fig. 1. When viewed forward in time, the erasure process transforms the state ω_i into ω_{i+1}. Therefore, the process of erasure does not generate dynamical randomness since the outcome is unique every time a bit is erased. Accordingly, the dynamical entropy per unit time (1) vanishes, $h = 0$, since the probability measure μ takes the unit value for the unique path followed during erasure and vanishes for all the other paths. On the other hand, the backward process corresponds to the generation of a sequence of bits. This reversed process is not unique and generates dynamical randomness at the rate $h^{\rm R} = I$ given by the information contained in the sequence of bits (see Fig. 1). Indeed, the sequence of bits $(\sigma_1\sigma_2\cdots\sigma_m)$ now appears at random with the probability $p(\sigma_1\sigma_2\cdots\sigma_m)$ and the probability measure $\bar{\mu}$ of this reversed process is distributed among several possible time-reversed paths. This observation can be understood by the following example. Let us consider a particle in a bistable potential, where the left and right wells correspond to the bits 0 and 1. We can slowly deform the potential in order to force the particle to end in the left well, which is equivalent to the erasure of the initial bit. However, since any such deformation must pass through a potential with a single minimum, undoing the latter transformation will result in the particle being in the left or right well with equal probability. This is the analog of our backward process, where the sequences of bits are now generated according to their respective probabilities. In this way, we can understand how dissipation is closely related to logical irreversibility [1]. Finally, we infer from Eq. (3) that the thermodynamic entropy production of the erasure is given by

$$\Delta_{\rm i} S = k_{\rm B}(h^{\rm R} - h) = k_{\rm B} I \text{ per bit.} \quad (5)$$

Landauer's principle is recovered in the particular case of statistically independent random bits of equal probability for which $\Delta_{\rm i} S = k_{\rm B} I = k_{\rm B}\ln 2$ [9]. This shows that Landauer's principle can be understood from concepts uniquely based on dynamical randomness. Therefore, it is completely model independent and $k_{\rm B} I$ is the *minimal* dissipation one can achieve for correlated random bits. Another way to obtain this result is to note that there exist universal coding schemes that will asymptotically compress any ergodic sequence of length m to its maximal possible value mI_2 (the subscript 2 indicates that the information is calculated with logarithms in base 2) [13]. Once compressed, the sequence is composed of uncorrelated random bits with relative probability one half (otherwise it would be possible to further compress the sequence, leading to a compression smaller than I_2 in contradiction with Shannon's bound on data compression). The usual Landauer principle can then be applied to this compressed sequence which leads to a dissipation of $k_{\rm B} I_2 \ln 2 = k_{\rm B} I$ per bit.

In the case where the exact probability distribution p of the bits is unknown, we can still obtain the minimal entropy production (5) as follows. The extra cost of compressing data using an *a priori* distribution q instead of the correct distribution p is the relative entropy $D(p\|q)$, given by [13]

$$D(p\|q) = \lim_{m\to\infty} \frac{1}{m} \sum_{\sigma_1\sigma_2\cdots\sigma_m} p(\sigma_1\sigma_2\cdots\sigma_m) \ln \frac{p(\sigma_1\sigma_2\cdots\sigma_m)}{q(\sigma_1\sigma_2\cdots\sigma_m)}. \quad (6)$$

It has been proven in Ref. [14] that there exists a stationary ergodic process $\bar{q}$ with the property to have a vanishing relative entropy, $D(p\|\bar{q}) = 0$, with respect to any other stationary ergodic process with probability distribution p. Therefore, the erasure process can be done assuming this particular distribution, which results in no extra cost with respect to the minimal entropy production (5).

As long as Landauer's principle is equivalent to the second law of thermodynamics [15], the agreement between

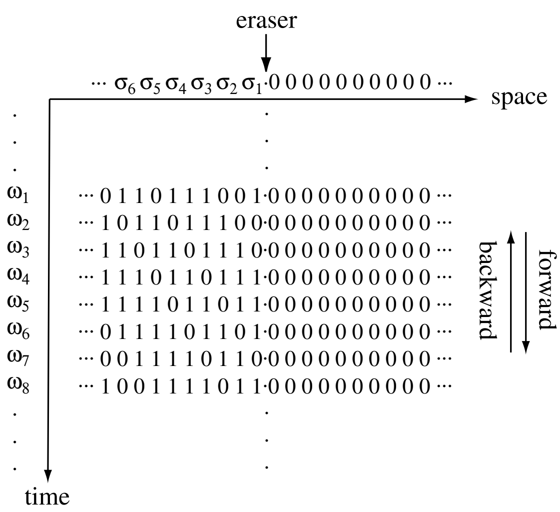

Fig. 1: Space-time plot of the physical process of erasure of a sequence of bits $\sigma_1\sigma_2\cdots\sigma_m\cdots$ of information. The bits are distributed along the space axis on the recording tape of the information processing device. The eraser is located somewhere along the recording tape and transforms each bit into a zero in this illustrative example. At every instant of time, the state of the system is given by the current sequence of bits: $\omega_j = \cdots\sigma_{j+2}\sigma_{j+1}\sigma_j\cdot 00000\cdots$ where the dot denotes the location of the eraser.

the argument based on dynamical randomness and the universal compression procedure shows that Eq. (3) is the appropriate measure of dissipation for general ergodic stationary stochastic processes.

We notice that the information I is positive if the probabilities $p(\sigma_1\sigma_2\cdots\sigma_m)$ characterizing the sequence of bits decay exponentially. Nevertheless, information can be sporadically distributed along the sequence of bits, in which case the probabilities $p(\sigma_1\sigma_2\cdots\sigma_m)$ decay as stretched exponentials so that the information per bit vanishes $I\to 0$ in the long-sequence limit $m\to\infty$. The fact is that such sporadic sequences are not uncommon in complex systems [16–19]. Accordingly, the above considerations lead to the interesting result that dissipation can be arbitrarily small during the erasure of sequences of sporadically distributed information.

In conclusion, we have here obtained a generalization of Landauer's principle on the basis of recent advances in nonequilibrium statistical mechanics showing that the second law of thermodynamics can be understand in terms of time asymmetry in the property of dynamical randomness [9–12]. These advances allow us to relate the thermodynamic entropy production during the erasure of a sequence of bits to the information contained in this sequence, taking into account the possible statistical correlations among the bits. This result clearly demonstrates that the second law of thermodynamics has fundamental implications on the way information is processed in physical systems.

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REFERENCES

[1] R. Landauer, IBM J. Res. Dev. **5**, 183 (1961).
[2] L. Brillouin, *Science and information theory*, 2nd edition (Academic Press, London, 1962).
[3] J. von Neumann, *Theory of self-reproducing automata*, A. W. Burks, Editor (University of Illinois Press, Urbana and London, 1966) p. 66.
[4] C. H. Bennett, IBM J. Res. Dev. **17**, 525 (1973).
[5] C. H. Bennett, Int. J. Theor. Phys. **21**, 905 (1982).
[6] K. Shizume, Phys. Rev. E **52**, 3495 (1995).
[7] B. Piechocinska, Phys. Rev. A **61**, 062314 (2000).
[8] R. Kawai, J. M. R. Parrondo, and C. Van den Broeck, Phys. Rev. Lett. **98**, 080602 (2007).
[9] P. Gaspard, J. Stat. Phys. **117**, 599 (2004).
[10] P. Gaspard, New J. Phys. **7**, 77 (2005).
[11] D. Andrieux, P. Gaspard, S. Ciliberto, N. Garnier, S. Joubaud, and A. Petrosyan, Phys. Rev. Lett. **98**, 150601 (2007).
[12] P. Gaspard, C. R. Physique **8**, 598 (2007).
[13] T. Cover and J. Thomas, *Elements of Information Theory*, (Wiley & Sons, New York, 1991).
[14] S. Xu, J. Th. Prob. **11**, 181 (1998).
[15] C. H. Bennett, Studies in History and Philosophy of Modern Physics **34**, 501 (2003).
[16] P. Gaspard and X.-J. Wang, Proc. Natl. Acad. Sci. USA **85**, 4591 (1988).
[17] W. Ebeling and G. Nicolis, Europhys. Lett. **14**, 191 (1991).
[18] W. Ebeling and G. Nicolis, Chaos, Solitons, & Fractals **2**, 635 (1992).
[19] G. Nicolis and P. Gaspard, Chaos, Solitons, & Fractals **4**, 41 (1994).