

## Pattern Formation in the Fractional Quantum Hall Effect

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The possibility of pattern formation in the fractional quantum Hall effect is investigated. A Ginzburg-Landau free-energy functional is defined in which an order parameter is coupled to the electromagnetic gauge field as well as to a Chern-Simons gauge field. It is shown that this free energy functional admits spatially non-uniform extremal solutions under certain conditions. These spatial structures are array of vortices, reminiscent of the vortex states of superconductivity or superfluidity.

**KEYWORDS:** Chern-Simons gauge field, equilibrium structure, fractional quantum Hall effect, Ginzburg-Landau free energy, vortex state

### 1. Introduction

Vortex states have been observed in many quantum phases such as type-II superconductivity,<sup>1-4)</sup> the different superfluidities<sup>5,6)</sup> and, more recently, in Bose-Einstein condensates.<sup>7,8)</sup> Vortices may form regular patterns such as triangular or rectangular lattices, as well as disordered lattices. These patterns are equilibrium structures generated by the spontaneous breaking of rotational or translational symmetries. In this regard, vortex states are similar to solid crystals.

On the other hand, new quantum phases have been discovered in the eighties with the fractional quantum Hall effect (FQHE).<sup>9-12)</sup> The quantum phases here arise in low-temperature two-dimensional systems of electrons in high magnetic fields. The two-dimensional character of the electron system allows the emergence of fractional charges by a topological effect. As a consequence, the Hall conductivity turns out to be quantized not only at integer values of  $e^2/h$  as in the integral quantum Hall effect,<sup>13)</sup> but also at fractional values. This strange effect has been explained as a collective effect among the electrons due to the interplay between the high magnetic field and the Coulomb interaction. The collective effect modifies the ground state of the many-electron system into a Laughlin wave function:<sup>14)</sup>

$$\Psi(z_1, \dots, z_N) = \prod_{j < k} (z_j - z_k)^q \exp \left( -\frac{1}{4l^2} \sum_j |z_j|^2 \right) \quad (1)$$

where  $z_j = x_j + iy_j$  and  $q = 3, 5, 7, \dots$ . This microscopic wave function describes the quantum phase corresponding to the Hall conductivity  $\sigma_{xy} = e^2/qh$ . An energy gap  $E_{\text{gap}}$  of the order of a few

Kelvin has been identified for these quantum phases. Besides, there exists a macroscopic description of quantum phases in terms of some local order parameter  $\phi$  which is a spatially varying function minimizing a free-energy functional. Such a Ginzburg-Landau approach has been developed for the FQHE as well.<sup>15–17)</sup> We may therefore wonder if vortex states could not exist in the FQHE as in other quantum phases.

The purpose of the present paper is to investigate the possibility of pattern formation in the FQHE. For this purpose, we use a Ginzburg-Landau approach. In the FQHE, the local order parameter  $\phi$  is coupled not only to the electromagnetic gauge field but also to a Chern-Simons gauge field. The Chern-Simons field is an effective gauge field which is specific to two-dimensional systems. This field represents frustration due to deviations with respect to the Laughlin state.<sup>15)</sup> The Ginzburg-Landau free-energy functional is thus modified by the addition of a Chern-Simons term. We show that this modified free-energy functional is minimized for vortex states under certain conditions.

The plan of the paper is the following. In Sec. 2, we review the formation of patterns in quantum phases. The Ginzburg-Landau approach for the FQHE is summarized in Sec. 3. The solutions of the Ginzburg-Landau-Chern-Simons free energy are described in Sec. 4 where the vortex states are displayed. Conclusions are drawn in Sec. 5.

## 2. Pattern Formation in Quantum Phases

A Ginzburg-Landau approach has been developed to describe the formation of patterns, and especially of vortex states, in the different known quantum phases. The description is macroscopic in terms of a local order parameter  $\phi(\mathbf{r})$ . The vortex states are equilibrium patterns which thus minimize the free energy of the system at non-zero temperature. At zero temperature, the free energy reduces to the energy of the system. The Ginzburg-Landau approach is phenomenological in the sense that the free energy functional depends on coefficients which should be determined either by comparison with experimental observation or by statistical mechanics based on the microscopic dynamics. The Ginzburg-Landau approach is therefore not an *ab initio* theory and it is limited by the range of validity of the phenomenological assumptions. Nevertheless, the Ginzburg-Landau approach turned out to be very efficient to understand large-scale structures such as vortex states. In this section, we review the free energy functionals describing the known quantum phases.

### 2.1 Type-II Superconductivity

The Ginzburg-Landau approach was first developed to understand superconductivity. The local order parameter describes the amplitude of the condensate of Cooper pairs of electric charge  $e^* = -2e$  and effective mass  $m$ . The Ginzburg-Landau free-energy density for superconductivity is

given by

$$\mathcal{F} = \frac{1}{2m} \left\| -i\hbar\nabla\phi - e^*\mathbf{A}\phi \right\|^2 - \alpha|\phi|^2 + \frac{\beta}{2}|\phi|^4 + \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 \quad (2)$$

where  $\mathbf{A}$  is the electromagnetic vector potential while  $\alpha$  and  $\beta$  are two phenomenological coefficients depending on temperature. The coefficient  $\alpha$  vanishes as  $T_c - T$  at the critical temperature  $T_c$  between the superconducting and normal states. The condensate is characterized by two lengths:

- (1) the coherence length:  $\xi \equiv \frac{\hbar}{\sqrt{2m\alpha}}$ ;
- (2) the magnetic length:  $\lambda \equiv \sqrt{\frac{m\beta}{e^{*2}\mu_0\alpha}}$ .

The ratio between these two lengths

$$\kappa \equiv \frac{\lambda}{\xi} \quad (3)$$

turned out to be a crucial parameter. Indeed, Abrikosov shown in 1957 the existence of two types of superconductivity:<sup>1)</sup>

- type-I superconductors which are uniform if  $\kappa < \frac{1}{\sqrt{2}}$ ;
- type-II superconductors which are non-uniform if  $\kappa > \frac{1}{\sqrt{2}}$ . In this case, there is a spontaneous breaking of the invariance under rotations and spatial translations in mean-field theory, leading to the formation of a triangular (or square) lattice of vortices known as the Abrikosov vortex state. The vortex state is an equilibrium structure similar to a crystal which forms at the fluid-solid transition (see Table I). These equilibrium structures should not be confused with the nonequilibrium patterns which require energy dissipation to be sustained. Abrikosov vortex states have been experimentally observed in standard and high- $T_c$  superconductivities.<sup>2-4)</sup>

Table I. Analogy between the formation of a vortex state in superconductivity and the fluid-solid transition. All these phases are equilibrium thermodynamic states minimizing some free energy.

superconductor	matter	invariance group
type-I phase	fluid phase	continuous rotations and translations
type-II phase (vortex state)	solid phase (crystal)	discrete rotations or translations

## 2.2 Rotating $^4\text{He}$

In the fifties, Onsager and Feynman showed that vortices should also appear in a rotating helium-4 superfluid. Helium-4 atoms are bosons which condensate at low temperature and form a

superfluid. When this superfluid rotates arrays of vortices were indeed observed.<sup>5)</sup>

### 2.3 Rotating $^3\text{He-A}$ and $^3\text{He-B}$

Arrays of vortices were also observed in helium-3 superfluidity.<sup>6)</sup> Here also the superfluid becomes non-uniform under rotation. Helium-3 atoms are fermions as the electrons so that helium-3 superfluidity has some analogy with superconductivity. However, the local order parameter of helium-3 superfluidity is no longer a scalar so that the free-energy functional and the vortex states are more complicated in helium-3 superfluidity than in superconductivity.

### 2.4 Rotating Bose-Einstein Condensates

Very recently, vortex states have been experimentally observed in rotating Bose-Einstein condensates (BEC) as well.<sup>7,8)</sup> A rotating BEC is described in a frame rotating at the angular velocity  $\Omega$  by the following Gross-Pitaevskii free-energy density:

$$\begin{aligned} \mathcal{F} = & \frac{1}{2m} \| -i\hbar\nabla\phi - m(\Omega \times \mathbf{r})\phi \|^2 \\ & + V_{\text{ext}}(\mathbf{r})|\phi|^2 + \frac{g}{2}|\phi|^4 - \frac{m}{2}(\Omega \times \mathbf{r})^2|\phi|^2 \end{aligned} \quad (4)$$

In rotating BEC, the angular velocity plays a similar role as the magnetic field in superconductivity. Vortices are nucleated beyond some critical angular velocity  $\Omega_c$ .

## 3. Ginzburg-Landau theory of the FQHE

### 3.1 Lagrangian action functional

The FQHE has been discovered in 2D electron systems in high magnetic field.<sup>9)</sup> Since its discovery in 1982, the FQHE has been studied in detail in GaAs heterostructures.<sup>10-12)</sup> Recently, the noise from fractional charges  $e/3$  has been experimentally observed.<sup>18,19)</sup>

At the macroscopic level, the FQHE can be described by a Ginzburg-Landau approach in terms of the following fields:

- the local order parameter  $\phi(x, y)$  describing the 2D electron system of thickness  $\ell_z$  and extending in plane  $(x, y)$ ;
- the Maxwell electromagnetic gauge field  $A_\mu = (A_0, \mathbf{A})$ ;
- the Chern-Simons gauge field  $a_\mu = (a_0, \mathbf{a})$  with  $\mu = 0, 1, 2 = t, x, y$ .

A free-energy functional can be inferred for the FQHE by starting from the Lagrangian formulation of the Hamiltonian interaction between the aforementioned fields. The Lagrangian action functional is given by<sup>16,17,20)</sup>

$$\begin{aligned} W = & \int (\mathcal{L}_{\text{Ginzburg-Landau}} + \mathcal{L}_{\text{Maxwell}} \\ & + \mathcal{L}_{\text{Chern-Simons}}) dt d^2r \end{aligned} \quad (5)$$

with the Ginzburg-Landau Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{Ginzburg-Landau}} &= \phi^*(i\hbar\partial_t - eA_0 - ea_0)\phi \\ &\quad - \frac{1}{2m} \|\ -i\hbar\nabla\phi - e\mathbf{A}\phi - e\mathbf{a}\phi\|^2 \\ &\quad + \alpha|\phi|^2 - \frac{\beta}{2}|\phi|^4 \end{aligned} \quad (6)$$

the Maxwell Lagrangian:

$$\mathcal{L}_{\text{Maxwell}} = \frac{\varepsilon_0\ell_z}{2}(\nabla A_0 + \partial_t\mathbf{A})^2 - \frac{\ell_z}{2\mu_0}(\nabla \times \mathbf{A})^2 \quad (7)$$

and the Chern-Simons Lagrangian:

$$\mathcal{L}_{\text{Chern-Simons}} = \frac{e^2}{4\hbar\theta} \epsilon^{\lambda\mu\nu} a_\lambda \partial_\mu a_\nu \quad (8)$$

with  $\epsilon^{012} = +1$ ,  $\epsilon^{021} = -1, \dots$  and the parameter  $\theta = (2p+1)\pi$  with the interger  $p = 0, 1, 2, \dots$  for the condensates corresponding to the Hall conductivity  $\sigma_{xy} = \frac{e^2}{2\theta\hbar} = \frac{e^2}{(2p+1)h}$ .<sup>16)</sup>

### 3.2 Free-energy functional

The energy density of the system can be derived from the action functional (5)-(8). This energy density would describe the system at zero temperature and should therefore be equal to the limit of the free energy as the temperature vanishes. Accordingly, a free-energy density can be assumed which as the same expression as the energy density but with coefficients replaced by effective values depending on the temperature.

The same characteristic lengths as in superconductivity can here also be introduced:

- (1) the coherence length:  $\xi \equiv \frac{\hbar}{\sqrt{2m\alpha}}$ ;
- (2) the magnetic length:  $\lambda \equiv \sqrt{\frac{m\beta\ell_z}{e^2\mu_0\alpha}}$ .

In the FQHE, there are two relevant parameters:

$$\kappa \equiv \frac{\lambda}{\xi} \quad (9)$$

and

$$\eta \equiv \frac{\hbar^2\theta}{m\beta} \quad (10)$$

The variables and fields are rescaled according to  $\mathbf{r} \rightarrow \xi \mathbf{r}$ ,  $\phi \rightarrow \sqrt{\frac{\alpha}{\beta}} \phi$ ,  $\mathcal{F} \rightarrow \frac{\alpha^2}{\beta} \mathcal{F}$ ,  $A_0 \rightarrow \frac{\alpha}{e} A_0$ ,  $a_0 \rightarrow \frac{\alpha}{e} a_0$ ,  $\mathbf{A} \rightarrow \frac{\hbar}{e\xi} \mathbf{A}$ ,  $\mathbf{a} \rightarrow \frac{\hbar}{e\xi} \mathbf{a}$ . We suppose that the electric charges of the condensate are well screened by the background. This corresponds to the limit  $\alpha \rightarrow \infty$  where the electrostatic potential  $A_0$  becomes constant. We thus obtain the Ginzburg-Landau-Chern-Simons (GLCS) free-

energy density:

$$\begin{aligned} \mathcal{F} = & \| -i \nabla \phi - \mathbf{A} \phi - \mathbf{a} \phi \|^2 - |\phi|^2 + \frac{1}{2} |\phi|^4 \\ & + a_0 |\phi|^2 - \frac{1}{\eta} a_0 (\partial_x a_y - \partial_y a_x) \\ & + \kappa^2 (\nabla \times \mathbf{A})^2 \end{aligned} \quad (11)$$

The fields minimizing this free energy should satisfy the Ginzburg-Landau equation

$$(-i \nabla - \mathbf{A} - \mathbf{a})^2 \phi - \phi + |\phi|^2 \phi + a_0 \phi = 0 \quad (12)$$

the Maxwell equation

$$\nabla \times \nabla \times \mathbf{A} = \frac{1}{2\kappa^2} \mathbf{j} \quad (13)$$

with the electric current

$$\mathbf{j} = \phi^* (-i \nabla - \mathbf{A} - \mathbf{a}) \phi + \text{c.c.} \quad (14)$$

and the further equations

$$\partial_x a_y - \partial_y a_x = \eta |\phi|^2 \quad (15)$$

$$\partial_x a_0 = +\eta j_y \quad (16)$$

$$\partial_y a_0 = -\eta j_x \quad (17)$$

These equations are invariant under the gauge transformation:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \Omega \quad (18)$$

$$\mathbf{a} \rightarrow \mathbf{a} + \nabla \omega \quad (19)$$

$$\phi \rightarrow \phi \exp(i\Omega + i\omega) \quad (20)$$

which allows the gauge fixing:  $A_x = a_x = 0$ . We consider solutions in a rectangular domain of sizes  $L_x$  and  $L_y$  and satisfying the periodic boundary conditions:

$$A_y(x, L_y) = A_y(x, 0) \quad (21)$$

$$a_y(x, L_y) = a_y(x, 0) \quad (22)$$

$$\phi(x, L_y) = \phi(x, 0) \quad (23)$$

$$A_y(L_x, y) = A_y(0, y) + \partial_y \Omega(y) \quad (24)$$

$$a_y(L_x, y) = a_y(0, y) + \partial_y \omega(y) \quad (25)$$

$$\phi(L_x, y) = \phi(0, y) \exp[i\Omega(y) + i\omega(y)] \quad (26)$$

Since  $\phi$  is single valued, we furthermore have that

$$\int \nabla \times \mathbf{A} \, d^2r + \int \nabla \times \mathbf{a} \, d^2r = \oint \mathbf{A} \cdot d\mathbf{r} + \oint \mathbf{a} \cdot d\mathbf{r} = 2\pi N \quad (27)$$

where  $N$  is the number of vortices in the area  $L_x \times L_y$ .

### 3.3 Thermodynamics

We can define the mean free energy per unit area as

$$f(B) \equiv \frac{1}{L_x L_y} \int \mathcal{F} d^2 r \quad (28)$$

which is a function of the mean magnetic induction

$$B \equiv \frac{1}{L_x L_y} \int \kappa \partial_x A_y d^2 r \quad (29)$$

The magnetic intensity is then given by

$$H = +\frac{1}{2} \frac{\partial f}{\partial B} \quad (30)$$

Alternatively, we can introduce the Gibbs free energy per unit area

$$g(H) \equiv f - 2 H B \quad (31)$$

which is a function of the magnetic intensity  $H$ . The magnetic induction is given in terms of the Gibbs free energy as

$$B = -\frac{1}{2} \frac{\partial g}{\partial H} \quad (32)$$

## 4. The solutions of the GLCS problem

The solutions of Eqs. (12)-(17) giving the extremal functions of the GLCS free energy (11) are the normal state, the pure state, and the vortex state.

### 4.1 The Normal State (NS)

The condensate does not exist in the normal state so that

$$A_y = \frac{B}{\kappa} x \quad (33)$$

$$a_y = 0 \quad (34)$$

$$\phi = 0 \quad (35)$$

$$f = B^2 \quad (36)$$

$$B = H \quad (37)$$

$$g = -H^2 \quad (38)$$

#### 4.2 The Pure State (PS)

The condensate is uniform in the pure state so that

$$A_y = \frac{B}{\kappa} x = -\eta |\phi|^2 x \quad (39)$$

$$a_y = -A_y = -\frac{B}{\kappa} x = \eta |\phi|^2 x \quad (40)$$

$$|\phi| = \sqrt{-\frac{B}{\kappa\eta}} \quad (41)$$

$$f = \frac{B}{\kappa\eta} + \left(1 + \frac{1}{2\kappa^2\eta^2}\right) B^2 \quad (42)$$

$$B = \frac{H - \frac{1}{2\kappa\eta}}{1 + \frac{1}{2\kappa^2\eta^2}} \quad (43)$$

$$g = -\frac{\left(H - \frac{1}{2\kappa\eta}\right)^2}{1 + \frac{1}{2\kappa^2\eta^2}} \quad (44)$$

The pure state corresponds to the following Landau-level filling fraction:

$$\nu = \frac{h|\phi_{\text{ph.}}|^2}{eB_{\text{ph.}}} = \frac{\pi}{\theta} = \frac{1}{2p+1} \quad (45)$$

#### 4.3 The Vortex State (VS)

Spontaneous breaking of the invariance under rotations and spatial translations happens in the vortex state, which can be constructed by a Monte-Carlo method of minimization of the free energy (11) with the periodic boundary conditions (21)-(26).

Figure 1 depicts an example of vortex state in the form of a triangular lattice. Because of the Chern-Simons field, the magnetic field does not vanish between the vortices contrary to the vortex states of type-II superconductivity. The vortex state exists between two critical values  $B_{c1} < B_{c2}$  of the magnetic induction. Near  $B_{c1}$ , the distance between the vortices diverges as

$$d \sim \frac{1}{\sqrt{B - B_{c1}}} \quad (46)$$

(see Fig. 2). Near  $B_{c2}$ , the mean density of the condensate vanishes as

$$\langle |\phi|^2 \rangle \sim B_{c2} - B \quad (47)$$

(see Fig. 3).

The mean free energy  $f(B)$  is depicted in Fig. 4 and the Gibbs free energy  $g(H)$  in Fig. 5, showing that the vortex state may exist as an intermediate state between the pure and normal states.

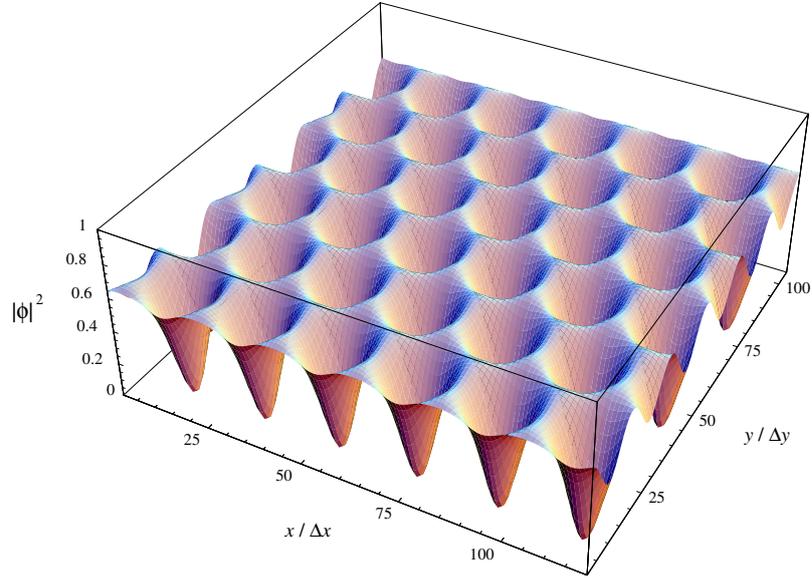


Fig. 1. Density  $|\phi(x, y)|^2$  versus  $(x, y)$  of a condensate with  $\kappa = 5$ ,  $\eta = -0.02$ , and  $B = 1.098$ . The mean free energy is  $f = 0.95$  and the mean density  $\langle |\phi|^2 \rangle = 0.51$ . The spacing between the grid points is  $\Delta x = \Delta y = 0.29$ .

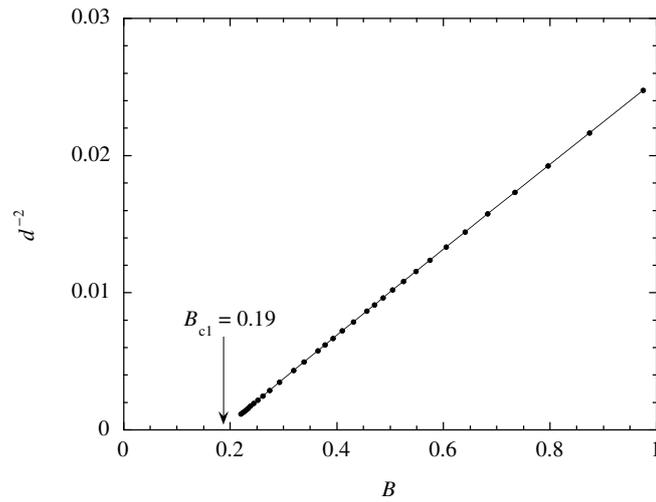


Fig. 2. Square of the inverse of the distance  $d$  between nearest-neighboring vortices versus the mean magnetic induction  $B$  for vortex states with  $\kappa = 5$  and  $\eta = -0.04$ .

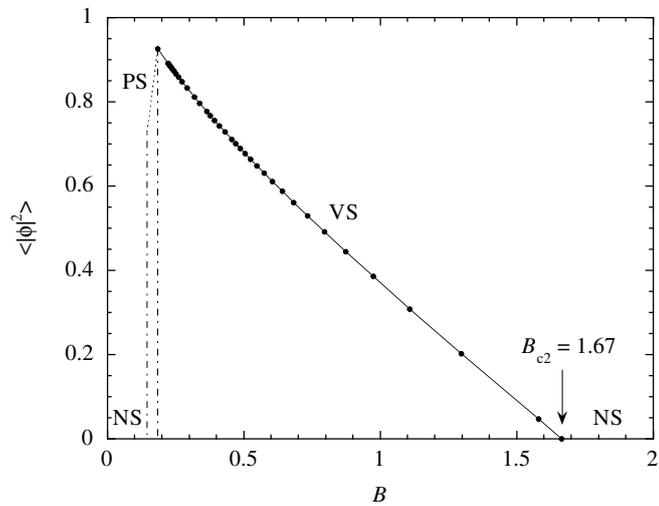


Fig. 3. Mean density  $\langle |\phi|^2 \rangle$  versus the mean magnetic induction  $B$  for pure states (PS, short-dashed line) and vortex states (VS, continuous line with dots) with  $\kappa = 5$  and  $\eta = -0.04$ .

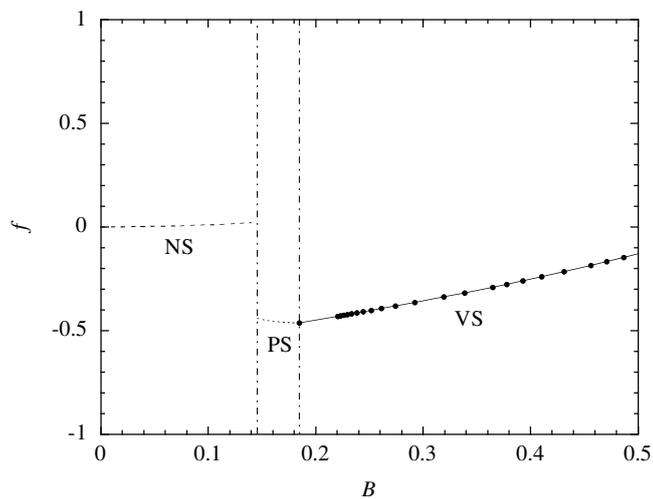


Fig. 4. Mean free energy  $f(B)$  versus the mean magnetic induction  $B$  for normal states (NS, long-dashed lines), pure states (PS, short-dashed line), and vortex states (VS, continuous line with dots) with  $\kappa = 5$  and  $\eta = -0.04$ .

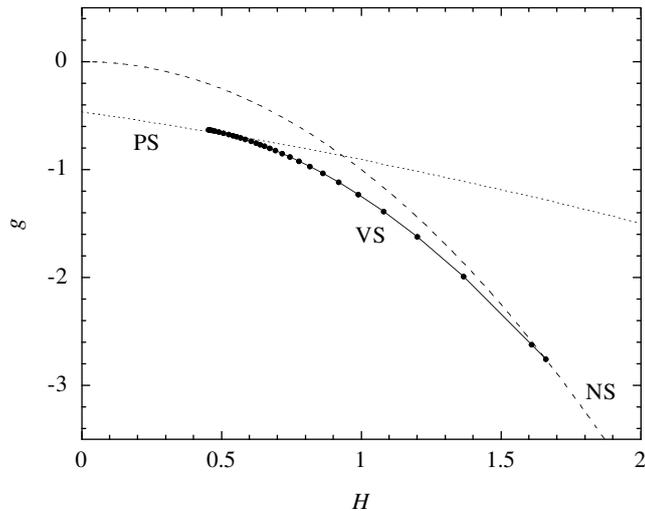


Fig. 5. Gibbs free energy  $g(H)$  versus the mean magnetic intensity  $H$  for normal states (NS, long-dashed lines), pure states (PS, short-dashed line), and vortex states (VS, continuous line with dots) with  $\kappa = 5$  and  $\eta = -0.04$ .

A condition of existence of the vortex state is that  $\kappa \gg 1$ , i.e., the magnetic length  $\lambda$  should be larger than the coherence length  $\xi$ . The coherence length can be estimated by

$$\xi \sim \frac{\hbar v_F}{E_{\text{gap}}} \quad (48)$$

and the magnetic length by

$$\lambda \sim \frac{1}{k_F} \quad (49)$$

with the Fermi wavenumber  $k_F = \frac{mv_F}{\hbar} = \sqrt{4\pi n_e}$ . Since the energy gap of the FQHE is of the order of

$$E_{\text{gap}} \simeq \frac{0.02 e^2}{4\pi\epsilon\lambda} \quad (50)$$

we find the condition

$$n_e \ll \frac{1}{4\pi} \left( \frac{0.02 e^2 m}{4\pi\epsilon\hbar^2} \right)^2 \quad (51)$$

The electron density should satisfy  $n_e \ll 3 \cdot 10^5 \text{ cm}^{-2}$  for  $m = 0.07 m_e$  and  $\epsilon = 12.9 \epsilon_0$ , or  $n_e \ll 10^{10} \text{ cm}^{-2}$  for  $m = m_e$  and  $\epsilon = \epsilon_0$ .

## 5. Conclusions

In this paper, we have investigated the possibility of formation of vortex states in the FQHE. For this purpose, we have used the Ginzburg-Landau theory of the FQHE.<sup>15–17,20)</sup> In this approach, the local order parameter is coupled not only to the Maxwell electromagnetic field but also to a Chern-Simons gauge field, which modifies the standard Ginzburg-Landau equations of superconductivity.

This approach leads to the prediction of vortex states in the FQHE under the condition that the magnetic length should be larger than the coherence length. According to this condition, the

vortex states should exist for systems with a low enough density of electrons. With the simplest Ginzburg-Landau functional, the vortices form a triangular lattice. Our analysis is based on the use of periodic boundary conditions in mean-field theory so that the melting of the two-dimensional lattice over large spatial scales because of thermal or quantum fluctuations leaving a disordered vortex state with rotational order is possible. The results obtained in the previous sections concern a sole condensate corresponding to one plateau of the Hall conductivity. Similar results should hold around each plateau of the Hall conductivity. Moreover, the vortex state should appear on the high-field side of each plateau.

Such vortex states could perhaps be experimentally observed in the FQHE by electron microscopy or neutron diffraction.

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