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Trace Formula for Noisy Flows

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A trace formula is derived for the Fokker-Planck equation associated with Itô stochastic differential equations describing noisy time-continuous nonlinear dynamical systems. In the weak-noise limit, the trace formula provides estimations of the eigenvalues of the Fokker-Planck operator on the basis of the Pollicott-Ruelle resonances of the noiseless deterministic system, which is assumed to be non-bifurcating. At first order in the noise amplitude, the effect of noise on a periodic orbit is given in terms of the period and the derivative of the period with respect to the pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme.

Key words: noise, stochastic process, nonequilibrium oscillation, chaotic attractor, semiclassical method, Hamilton-Jacobi method, correlation time, trace formula, zeta function

I. INTRODUCTION

Oscillation is a ubiquitous phenomenon observed in macroscopic far-from-equilibrium systems, such as autocatalytic chemical reactions, fluids beyond convective instabilities, or lasers [1]. The oscillations of such systems are described in terms of macroscopic variables which obey deterministic nonlinear equations. The oscillations may be periodic if the attractor is a limit cycle or chaotic if the motion is unstable on the attractor. In the case of a periodic attractor, the oscillations are characterized by the precise periodicity of the same motion as time evolves.

In many circumstances, the macroscopic deterministic description does not take into account of all the aspects of the time evolution. Indeed, the macroscopic dynamics ignores the random fluctuations of different kinds, which can then be described by extra stochastic forces [2–7]. In the case of periodic oscillations, such noises affect the periodicity of the time evolution and cause a diffusion of the phase of the oscillations. Much work has been devoted to the effect of noise on nonequilibrium oscillators, especially, on chemical clocks [8–15]. As a consequence of phase diffusion, the time-correlation function of the signal presents damped oscillations [12, 13]. The damping is often exponential which leads to the introduction of the lifetime τ of the correlations between successive oscillations. Therefore, a noisy oscillator is characterized not only by its period T but also by the *quality factor* which is proportional to the ratio of the lifetime of the correlations to the period:

$$\mathcal{Q} \equiv 2\,\pi\,\frac{\tau}{T} \tag{1}$$

The quality factor gives an estimation of the number of oscillations over which the periodicity is maintained. The larger the quality factor is, the longer the oscillator keeps the time. Accordingly, oscillators with a large quality factor can be used as clocks. On the other hand, if the quality factor is less than one, $\mathcal{Q} < 1$, the oscillations cease to be correlated and the oscillator can no longer keep the time.

The purpose of the present paper is to develop a general theory for the quality factor of oscillations in nonlinear dynamical systems submitted to a weak noise. Such systems may be described by Itô stochastic differential equations or, equivalently, by a general Fokker-Planck equation [3–7]. The Fokker-Planck equation rules the time evolution of the probability density in the phase space of the system. The time evolution of the probability density can be used to obtain the time-correlation functions of the different variables. If the time-correlation function has oscillations of period T which are damped with a lifetime τ the Laplace transform, $\int_0^\infty C(t)\,\mathrm{e}^{-st}dt$, of the time-correlation function $C(t)$ should have poles at the complex values $s = -(1/\tau) \pm \mathrm{i}(2\pi/T)$. Such complex poles can be obtained as the eigenvalues of the Fokker-Planck operator, for instance, by computing the trace of the time-evolution operator.

In the weak-noise limit, the time evolution of the probability density can be expressed in terms of the Onsager-Machlup Lagrangian [16–18] or, equivalently, of the Fredlin-Wentzell Hamiltonian [19–21]. This Hamiltonian description involves the macroscopic variables of the nonlinear system augmented by canonically conjugated variables which are often referred to as the thermodynamic forces. The solution of the Fokker-Planck equation can be expanded semiclassically around the trajectories in this augmented phase space. We can then use the semiclassical methods developed since Gutzwiller's work [22–24] in order to calculate the trace of the time-evolution operator. A trace formula for noisy flows has already been anticipated in Ref. [25] by arguments based on phase diffusion and on the trace formula for the deterministic classical time evolution [26], but the quality factor of a periodic orbit was unknown at that time. Since then, trace formulas have also been derived for noisy mappings in Refs. [30–35]. Noisy mappings provide a time-discrete representation of the stochastic evolution. Because we are here interested by time-continuous stochastic evolutions – and, especially, by the phenomenon of phase diffusion which happens in the direction parallel to the flow – we need a trace formula for noisy flows. For the purpose of the present paper, we shall thus carry out a complete derivation of a trace formula for the Fokker-Planck time-continuous evolution in the limit of weak noise. In this way, a very simple expression will be obtained for the quality factor of a periodic orbit. The obtained result applies not only to stable periodic orbits of a periodic attractor, but also to unstable periodic orbits of a chaotic attractor or repeller. For such systems, we shall obtain a zeta function which can be used to calculate the different damping rates of the nonlinear system submitted to a weak noise. In the noiseless limit, we show in detail how the eigenvalues of the Fokker-Planck operator converge to the Pollicott-Ruelle resonances of the deterministic dynamics [27–29]. Moreover, the noise is shown to reduce the escape out of a repeller at first order in the noise amplitude, which corresponds to a lengthening of the lifetime on the repeller due to the noise.

The plan of the paper is the following. In Sec. II, we define the class of stochastic systems of interest for us and we pose the problem of the eigenvalues of the Fokker-Planck operator and the trace formula. In Sec. III, we consider the weak-noise limit, we summarize the Onsager-Machlup and Freidlin-Wentzell theories [16–21], and we obtain a semiclassical expression for the propagator and the Green function of the Fokker-Planck equation. In Sec. IV, we perform the semiclassical expansion of the trace formula. In Sec. V, we obtain the damping rates of the time-correlation functions for stationary, periodic, and chaotic attractors, as well as the expression for the quality factor of a periodic orbit. Conclusions are drawn in Sec. VI. Appendix A contains an effective method to calculate the correlation time of a periodic orbit.

II. GAUSSIAN STOCHASTIC SYSTEMS

A. Itô and Fokker-Planck equations

In the present paper, we consider systems described by Itô stochastic differential equations of the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) + \varepsilon\ \mathbf{G}(\mathbf{x}) + \varepsilon^{1/2} \sum_{\alpha=1}^{\nu} \mathbf{C}_\alpha(\mathbf{x})\ \xi^\alpha(t) \tag{2}$$

where the variables $\mathbf{x} = (x^1, x^2, ..., x^d)$ belong to the d-dimensional phase space of the system, while $\mathbf{F}(\mathbf{x})$, $\mathbf{G}(\mathbf{x})$, and $\mathbf{C}_\alpha(\mathbf{x})$ are vector fields defined in the phase space [3–7, 20, 21]. We assume that these vector fields do not depend on time. The quantities $\xi^\alpha(t)$ are Gaussian white noises such that

$$\langle \xi^\alpha(t) \rangle = 0 \qquad \text{and} \qquad \langle \xi^\alpha(t)\xi^\beta(t') \rangle = \delta^{\alpha\beta}\ \delta(t-t') \tag{3}$$

for $\alpha, \beta = 1, 2, ..., \nu$. The parameter ε controls the amplitude of the noise. The weak-noise limit corresponds to the formal limit $\varepsilon \to 0$.

In order to be as general as possible we have also included the term $\varepsilon\mathbf{G}(\mathbf{x})$ in the deterministic part of the differential equation, which thus also depends on the noise amplitude. Indeed, such terms may appear if the Itô stochastic differential equation arises in some approximation from a more microscopic kinetic master equation such as the chemical master equation by Nicolis and coworkers [2, 8–10].

The stochastic differential equation (2) should be understood in the sense of Itô, which means that the approximate discrete time evolution is of the form:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + [\mathbf{F}(\mathbf{x}_n) + \varepsilon\ \mathbf{G}(\mathbf{x}_n)]\,\Delta t + \varepsilon^{1/2} \sum_{\alpha=1}^{\nu} \mathbf{C}_\alpha(\mathbf{x}_n)\ \Xi_n^\alpha\ \Delta t^{1/2} \tag{4}$$

where $\mathbf{x}_n \simeq \mathbf{x}(n\Delta t)$ and $\Xi_n^\alpha \simeq \xi^\alpha(n\Delta t)\Delta t^{1/2}$ are independent random variables with a Gaussian distribution of zero mean and unit variance. Eq. (4) means that the values of the variables $\mathbf{x}$ at the next time step are determined by the coefficients $\mathbf{C}_\alpha(\mathbf{x})$ evaluated at the current time step [3].

In the stochastic scheme of Eq. (2), the variables evolve in time along individual random trajectories $\mathbf{x}(t)$. An alternative description of the time evolution can be formulated in terms of the probability density $\mathcal{P}(\mathbf{x}, t)$ for the trajectory to be found at the phase-space point $\mathbf{x}$ at the current time t. This probability density is known to evolve in time according to a Fokker-Planck equation which, for the Itô stochastic differential equation (2), is given by

$$\partial_t\ \mathcal{P}(\mathbf{x}, t) = -\partial_i \left\{ \left[F^i(\mathbf{x}) + \varepsilon\ G^i(\mathbf{x}) \right] \mathcal{P}(\mathbf{x}, t) \right\} + \varepsilon\ \partial_i \partial_j \left[Q^{ij}(\mathbf{x})\ \mathcal{P}(\mathbf{x}, t) \right] \equiv \hat{L}\ \mathcal{P}(\mathbf{x}, t) \tag{5}$$

where

$$Q^{ij}(\mathbf{x}) = \frac{1}{2}\sum_{\alpha=1}^{\nu} C^i_\alpha(\mathbf{x})\; C^j_\alpha(\mathbf{x}) \tag{6}$$

is supposed to be a positive real symmetric matrix of diffusion in phase space [3, 20]. We have here adopted a convention of summation over repeated indices and $\partial_i = \partial/\partial x^i$. The Fokker-Planck equation (5) is a linear partial differential equation of first order in time and second order in the phase-space variables x^i. The operator $\hat{L}$ is called the Fokker-Planck operator.

B. Eigenvalue problem

Since we assume that the system is time-independent, the solution of the Fokker-Planck equation from the initial density $\mathcal{P}_0(\mathbf{x})$ is

$$\mathcal{P}(\mathbf{x},t) = \mathrm{e}^{\hat{L}t}\mathcal{P}_0(\mathbf{x}) = \int K(\mathbf{x},\mathbf{x}_0,t)\; \mathcal{P}_0(\mathbf{x}_0)\; d\mathbf{x}_0 \tag{7}$$

where $K(\mathbf{x},\mathbf{x}_0,t)$ is called the propagator. The propagator is the conditional probability density for the system to be in the state $\mathbf{x}$ at time t given that it was initially in the state $\mathbf{x}_0$. The propagator is a special solution of the Fokker-Planck equation with the initial condition

$$\lim_{t\to 0} K(\mathbf{x},\mathbf{x}_0,t) = \delta(\mathbf{x}-\mathbf{x}_0) \tag{8}$$

Since the system is time-independent, the Fokker-Planck operator admits eigenvalues and associated right- and left-eigenfunctions

$$\hat{L}\;\psi_l \;=\; s_l\;\psi_l \tag{9}$$

$$\hat{L}^\dagger\;\tilde{\psi}_l \;=\; s_l^*\;\tilde{\psi}_l \tag{10}$$

with $s_l \in \mathbb{C}$ and possible other root functions in the case of the formation of Jordan-block structures. Indeed, the Fokker-Planck operator is not hermitian so that its eigenvalues are not all real, the left-eigenfunction differs in general from the right-eigenfunction, and its complete diagonalization may not be possible whereupon Jordan-block structures would appear. To fix the ideas, we have here supposed that the spectrum is discrete. The eigenfunctions (10) should obey the biorthonormality condition

$$\int \tilde{\psi}_k^*(\mathbf{x})\;\psi_l(\mathbf{x})\;d\mathbf{x} \;=\; \delta_{kl} \tag{11}$$

The general solution of the Fokker-Planck equation can thus be expressed in terms of a spectral decomposition

$$\mathcal{P}(\mathbf{x},t) = \sum_l c_l\;\mathrm{e}^{s_l t}\;\psi_l(\mathbf{x}) + (\mathrm{Jb}) \tag{12}$$

where (Jb) denotes the possible Jordan-block terms and where the coefficients c_l are determined by the initial density according to

$$c_l = \int \tilde{\psi}_l^*(\mathbf{x}_0)\;\mathcal{P}_0(\mathbf{x}_0)\;d\mathbf{x}_0 \tag{13}$$

Hence, the propagator takes the form

$$K(\mathbf{x},\mathbf{x}_0,t) = \sum_l \psi_l(\mathbf{x})\;\mathrm{e}^{s_l t}\;\tilde{\psi}_l^*(\mathbf{x}_0) + (\mathrm{Jb}) \tag{14}$$

The eigenvalues s_l give the different frequencies and damping rates of the system. Indeed, these eigenvalues have in general both a real and an imaginary parts

$$s_l = \mathrm{Re}\;s_l + \mathrm{i}\;\mathrm{Im}\;s_l = -\gamma_l + \mathrm{i}\;\omega_l \tag{15}$$

The real part is non-positive because the probability density should evolve in time toward an invariant solution, $\mathcal{P}_\infty(\mathbf{x}) = \lim_{t\to\infty} \mathcal{P}(\mathbf{x},t)$, which is non-vanishing if the stochastic process is stationary as here assumed. The real part thus gives the damping rate γ_l of a relaxation mode of exponential type. The lifetime of this relaxation mode is

$$\tau_l = \frac{1}{\gamma_l} \tag{16}$$

The imaginary part gives the angular frequency or pulsation ω_l, which is related to the period of that particular relaxation mode by

$$T_l = \frac{2\pi}{|\omega_l|} \tag{17}$$

The eigenvalues control the time evolution of the average quantities as well as of the time-correlation functions. If $A(\mathbf{x})$ is an observable quantity, its average over a statistical ensemble of trajectories issued from the initial probability density $\mathcal{P}_0$ evolves in time according to

$$\langle A_t\rangle_0 = \int A(\mathbf{x})\ \mathcal{P}(\mathbf{x},t)\ d\mathbf{x} = \int A(\mathbf{x})\ K(\mathbf{x},\mathbf{x}_0,t)\ \mathcal{P}_0(\mathbf{x}_0)\ d\mathbf{x}\ d\mathbf{x}_0 = \sum_l a_l\ \mathrm{e}^{s_l t}\ c_l + (\mathrm{Jb}) \tag{18}$$

where $\langle\cdot\rangle_0$ denotes the average over the initial density $\mathcal{P}_0$, c_l is the coefficient (13), and

$$a_l = \int A(\mathbf{x})\ \psi_l(\mathbf{x})\ d\mathbf{x} \tag{19}$$

A similar result holds for the time-correlation function between two observable quantities $A(\mathbf{x})$ and $B(\mathbf{x})$ defined by

$$C_{AB}(t) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T d\tau \int A(\mathbf{x}_{t+\tau})\ B(\mathbf{x}_\tau)\ \mathcal{P}_0(\mathbf{x}_0)\ d\mathbf{x}_0 \tag{20}$$

where $\mathbf{x}_t$ is a stochastic trajectory of initial condition $\mathbf{x}_0$ distributed according to the initial probability density $\mathcal{P}_0(\mathbf{x}_0)$. We have here to assume that the stochastic process is stationary and admits an asymptotic invariant density $\mathcal{P}_\infty(\mathbf{x})$ in terms of which the time-correlation function (20) can be expressed as

$$C_{AB}(t) = \int A(\mathbf{x}')\ K(\mathbf{x}',\mathbf{x},t)\ B(\mathbf{x})\ \mathcal{P}_\infty(\mathbf{x})\ d\mathbf{x}'\ d\mathbf{x} = \langle A_t B_0\rangle_\infty \tag{21}$$

Using the spectral decomposition (14) we find that

$$C_{AB}(t) = \sum_l a_l\ \mathrm{e}^{s_l t}\ b_l + (\mathrm{Jb}) \tag{22}$$

where a_l is given by (19) and

$$b_l = \int \tilde{\psi}_l^*(\mathbf{x})\ B(\mathbf{x})\ \mathcal{P}_\infty(\mathbf{x})\ d\mathbf{x} \tag{23}$$

We notice that a Jordan block of multiplicity m_l would contribute by a power of time multiplying the exponential damping as $t^{m_l-1}\exp(s_l t)$. The power-law factors t^{m_l-1} have slower time behavior than the exponential damping so that the knowledge of the exponential damping rates remains essential if $\mathrm{Re}\ s_l \neq 0$.

Over long times, the evolution will be dominated by the relaxation modes with the longest lifetime. Beside the invariant density $\mathcal{P}_\infty(\mathbf{x})$ which has an infinite lifetime, we find one or several relaxation modes with a finite lifetime, which are of prime importance for us. Indeed, according to the previous reasoning, the quality factor of a noisy oscillator would then be given by

$$\mathcal{Q} = 2\pi\,\frac{\tau_l}{T_l} = \frac{|\omega_l|}{\gamma_l} = \frac{|\mathrm{Im}\ s_l|}{|\mathrm{Re}\ s_l|} \tag{24}$$

if l refers to the dominant relaxation modes.

C. Trace formula

The trace of the evolution operator can be defined as

$$\mathrm{tr}\ \mathrm{e}^{\hat{L}t} \equiv \int K(\mathbf{x},\mathbf{x},t)\ d\mathbf{x} = \sum_l\ m_l\ \mathrm{e}^{s_l t} \tag{25}$$

where m_l is the multiplicity of the eigenvalue s_l. We notice that this multiplicity takes into account of the possible terms associated with the Jordan-block structures so that no extra term should appear in Eq. (25) contrary to Eqs. (12), (14), or (22). We observe that the knowledge of the trace function allows us to determine the different damping rates $\{s_l\}$ of the system.

The eigenvalues can be determined by considering the Laplace transform of the trace (25), which gives the trace of the resolvent of the Fokker-Planck operator as

$$\int_0^\infty \mathrm{e}^{-st}\ \mathrm{tr}\ \mathrm{e}^{\hat{L}t}\ dt = \mathrm{tr}\ \frac{1}{s-\hat{L}} = \sum_l\ \frac{m_l}{s-s_l} \tag{26}$$

with $s \in \mathbb{C}$. The eigenvalues are determined as the poles of this trace function obtained by continuation toward the complex values of the variable s. Therefore, the trace formula contains enough information in order to determine the eigenvalues of the Fokker-Planck operator and, in particular, the quality factor (24).

III. WEAK-NOISE LIMIT

In this section, we consider the weak-noise limit in which the noise amplitude decreases as $\varepsilon \to 0$. We use the Hamilton-Jacobi method to solve the Fokker-Planck equation of the stochastic system. This method has been much developed during the last decades [20, 21, 36–38].

The probability density entering the Fokker-Planck equation (5) is taken in the following form

$$\mathcal{P}\ =\ \exp\left(-\frac{\phi}{\varepsilon}\right) \tag{27}$$

The function ϕ can be interpreted as a negentropy associated with the state $\mathbf{x}$, while ε can be considered as the Boltzmann constant $k_{\mathrm{B}} = 1.38\times 10^{-23}$ J K^{-1}, which smallness characterizes the scale where the molecular fluctuations described by the probability density $\mathcal{P}(\mathbf{x},t)$ become important.

We replace the expression (27) in the Fokker-Planck equation (5) to get:

$$\begin{aligned} 0 \ &=\ \partial_t\phi + F^i\ \partial_i\phi + Q^{ij}\ \partial_i\phi\ \partial_j\phi \\ &+\ \varepsilon\ \left(G^i\ \partial_i\phi - \partial_i F^i - 2\ \partial_i Q^{ij}\ \partial_j\phi - Q^{ij}\ \partial_i\partial_j\phi\right) \\ &+\ \varepsilon^2\ \left(-\partial_i G^i + \partial_i\partial_j Q^{ij}\right) \end{aligned} \tag{28}$$

The function ϕ is expanded in powers of ε as

$$\phi\ =\ \phi_0\ +\ \varepsilon\ \phi_1\ +\ \varepsilon^2\ \phi_2\ +\ \cdots \tag{29}$$

Substituting in Eq. (28) and identifying the terms with the same power in ε, we obtain the successive equations:

$$0\ =\ \partial_t\phi_0 + F^i\ \partial_i\phi_0 + Q^{ij}\ \partial_i\phi_0\ \partial_j\phi_0 \tag{30}$$

$$0\ =\ \partial_t\phi_1 + F^i\ \partial_i\phi_1 + 2\ Q^{ij}\ \partial_i\phi_0\ \partial_j\phi_1 + G^i\ \partial_i\phi_0 - \partial_i F^i - 2\ \partial_i Q^{ij}\ \partial_j\phi_0 - Q^{ij}\ \partial_i\partial_j\phi_0 \tag{31}$$

$$\vdots$$

A. The Hamilton-Jacobi equation

The leading equation (30) is nothing else than the Hamilton-Jacobi equation of a deterministic Lagrangian or Hamiltonian system which has been identified for the first time by Onsager and Machlup [16, 17]:

$$0\ =\ \partial_t W + F^i\ \partial_i W + Q^{ij}\ \partial_i W\ \partial_j W \tag{32}$$

where $W = \phi_0$ is a function of the time t and of the state variables $\mathbf{x}$. By using methods from classical mechanics the solutions of Eq. (32) can be written in terms of the action

$$\phi_0 \;=\; W \;\equiv\; \int L\; dt \;=\; \int \mathbf{p}\cdot d\mathbf{x} - H\; dt \tag{33}$$

The function L is the Onsager-Machlup Lagrangian given by [16–18]

$$L(\mathbf{x},\dot{\mathbf{x}}) \;\equiv\; \frac{1}{4}\, Q_{ij}(\mathbf{x})\; (\dot{x}^i - F^i)\; (\dot{x}^j - F^j) \tag{34}$$

where Q_{ij} is the inverse of the matrix Q^{ij}:

$$Q_{ij}(\mathbf{x})\; Q^{jk}(\mathbf{x}) \;=\; \delta_i^k \tag{35}$$

Since the matrix Q_{ij} is positive the action W is non-negative according to Eq. (33) and (34).

Momenta $\mathbf{p} = \{p_i\}_{i=1}^d$ which are canonically conjugated to the state variables $\mathbf{x} = \{x^i\}_{i=1}^d$ can be introduced as

$$p_i \;\equiv\; \frac{\partial L}{\partial \dot{x}^i} \;=\; \frac{\partial W}{\partial x^i} \tag{36}$$

which leads to the definition of the Freidlin-Wentzell Hamiltonian [19, 21]

$$H(\mathbf{x},\mathbf{p}) \;\equiv\; Q^{ij}(\mathbf{x})\; p_i\; p_j \;+\; F^i(\mathbf{x})\; p_i \tag{37}$$

Accordingly, the solutions of the Hamilton-Jacobi equation (32) can be expressed as the action integrals (33) along the trajectories of a Hamiltonian dynamical system defined in a phase space $\{x^i, p_i\}_{i=1}^d$ with a *doubled dimension* $2d$, as compared with the dimension of the original phase space of the state variables $\{x^i\}_{i=1}^d$. The trajectories are given by the extremal curves of the action integrals (33). According to Hamilton's variational principle, the trajectories are thus solutions of Hamilton's equations:

$$\begin{cases} \dot{x}^i = +\dfrac{\partial H}{\partial p_i} = F^i(\mathbf{x}) + 2\; Q^{ij}(\mathbf{x})\; p_j \\[2ex] \dot{p}_i = -\dfrac{\partial H}{\partial x^i} = -\dfrac{\partial F^j(\mathbf{x})}{\partial x^i}\; p_j - \dfrac{\partial Q^{jk}(\mathbf{x})}{\partial x^i}\; p_j\; p_k \end{cases} \tag{38}$$

This Hamiltonian system has the special property to leave invariant the subspace $\mathbf{p} = 0$ which is the original phase space of dimension d. Indeed, Hamilton's equations (38) admit as special solutions those of the macroscopic and deterministic equations

$$\dot{x}^i = F^i(\mathbf{x}) \qquad \text{with} \quad \mathbf{p} = 0 \tag{39}$$

corresponding to the Itô equations (2) in the absence of noise when $\varepsilon = 0$.

Furthermore, if the fields $F^i(\mathbf{x})$ and $Q^{ij}(\mathbf{x})$ are time-independent, the Hamiltonian (37) is also time-independent so that the Hamiltonian itself is a constant of motion: $H = E$. We notice that this is not the case if the system is driven by an external time-dependent force for instance.

Around the thermodynamic equilibrium and in the weak-noise limit $W \gg \varepsilon = k_{\rm B}$, the action $W = \phi_0$ can be identified with the negentropy due to the fluctuations: $\phi_0 = -S$. In this case, we observe that each momentum p^i can be interpreted as minus the thermodynamic force which is conjugated to the state variable x^i:

$$p_i = \frac{\partial W}{\partial x^i} \simeq_{\varepsilon\to 0} -\frac{\partial S}{\partial x^i} = -\mathcal{X}_i \tag{40}$$

Table I gives the correspondence with the notations used by Onsager and Machlup in their paper about nonequilibrium fluctuations around an equilibrium state. In the present paper, we shall be concerned by general fluctuations described by the Itô stochastic differential equation (2) which may apply to far-from-equilibrium systems as well.

Table I. Correspondence between the notations of the present paper on nonequilibrium fluctuations described by Itô stochastic differential equations and of Onsager-Machlup's papers [16, 17] on the fluctuations around the thermodynamic equilibrium.

quantity	*present case*	*Onsager-Machlup's case*
small parameter	ε	$k_{\rm B}$
negentropy function	$\phi_0 = W$	$-S$
state variable	x^i	α^i
conjugate variable	$p_i = \frac{\partial W}{\partial x^i}$	$-\mathcal{X}_i = -\frac{\partial S}{\partial \alpha^i}$
dissipation matrix	Q_{ij}	R_{ij}
reciprocal dissipation matrix	Q^{ij}	L^{ij}
deterministic field	F^i	$L^{ij}\frac{\partial S}{\partial \alpha^j}$
Lagrangian function	$4L = Q_{ij}(\dot{x}^i - F^i)(\dot{x}^j - F^j)$	$\mathcal{L} = \underbrace{R_{ij}\dot{\alpha}^i\dot{\alpha}^j}_{2\Phi} + \underbrace{L^{ij}\frac{\partial S}{\partial \alpha^i}\frac{\partial S}{\partial \alpha^j}}_{2\Psi} - 2\dot{S}(\{\alpha^i\})$

B. The transport equation

The further equation (31) for ϕ_1 is known as the transport equation which rules the change of the amplitude along the Hamiltonian trajectories of the Hamilton-Jacobi equation (30) for ϕ_0.

In order to interpret the quantities appearing in the transport equation (31) we should first introduce two auxiliary quantities which have a time evolution directly determined by the Hamilton-Jacobi equation (32) for the action function $W(\mathbf{x}, \mathbf{x}_0, t)$ associated with the Hamiltonian trajectory from $\mathbf{x}_0$ to $\mathbf{x}$ during the time interval t. These auxiliary quantities are the curvature matrix

$$\mathsf{C} = [C_{ij}]_{i,j=1}^{d} \equiv \left[\frac{\partial^2 W}{\partial x^i \partial x^j}\right]_{i,j=1}^{d} \tag{41}$$

the Morette-Van Hove matrix

$$\mathsf{D} = [D_{ij}]_{i,j=1}^{d} \equiv \left[-\frac{\partial^2 W}{\partial x^i \partial x_0^j}\right]_{i,j=1}^{d} \tag{42}$$

and the corresponding determinants

$$C = \det \mathsf{C} = \det \left[\frac{\partial^2 W}{\partial x^i \partial x^j}\right]_{i,j=1}^{d} \tag{43}$$

$$D = \det \mathsf{D} = \det \left[-\frac{\partial^2 W}{\partial x^i \partial x_0^j}\right]_{i,j=1}^{d} \tag{44}$$

(see e.g. Ref. [29]).

The time evolution of the matrices C and D along the Hamiltonian trajectories of (38) can be derived from the Hamilton-Jacobi equation (32)

$$\partial_t W + H(\mathbf{x}, \partial_{\mathbf{x}} W) = 0 \tag{45}$$

by taking successive and appropriate derivatives $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial x_0^i}$ since the above matrices are functions of the variables t, $\mathbf{x}$, and $\mathbf{x}_0$ (and *not* of momenta). The total time derivative which has to be considered here is thus defined for the projection of the Hamiltonian flow (38) onto the space of the state variables $\mathbf{x}$:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} \tag{46}$$

We obtain the system

$$\frac{d\mathsf{C}}{dt} = -\mathsf{C}\cdot\frac{\partial^2 H}{\partial \mathbf{p}^2}\cdot\mathsf{C} - \mathsf{C}\cdot\frac{\partial^2 H}{\partial \mathbf{p}\partial \mathbf{x}} - \frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{p}}\cdot\mathsf{C} - \frac{\partial^2 H}{\partial \mathbf{x}^2} \tag{47}$$

$$\frac{d\mathsf{D}}{dt} = -\mathsf{C}\cdot\frac{\partial^2 H}{\partial \mathbf{p}^2}\cdot\mathsf{D} - \frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{p}}\cdot\mathsf{D} \tag{48}$$

which, together with Hamilton's equations (38), form a closed set of coupled differential equations [29].

The Morette-Van Hove determinant can then be obtained from

$$\frac{d}{dt}\ln D = \mathrm{tr}\ \mathsf{D}^{-1}\cdot\frac{d\mathsf{D}}{dt} = -\mathrm{tr}\left(\mathsf{C}\cdot\frac{\partial^2 H}{\partial \mathbf{p}^2} + \frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{p}}\right) \tag{49}$$

The solution of this equation is given by

$$D = D_0\ \exp\left[-\int_0^t \mathrm{tr}\left(\mathsf{C}\cdot\frac{\partial^2 H}{\partial \mathbf{p}^2} + \frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{p}}\right) d\tau\right] \tag{50}$$

For the Freidlin-Wentzell Hamiltonian (37), Eq. (49) for the Morette-Van Hove determinant becomes

$$\frac{d}{dt}\ln D = -\frac{\partial^2 H}{\partial x^i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial p_j}\ \partial_i\partial_j W = -\partial_i F^i - 2\ \partial_i Q^{ij}\ \partial_j W - 2\ Q^{ij}\ \partial_i\partial_j W \tag{51}$$

We observe that we have here an expression which is very close to the last three terms of the transport equation (31). This transport equation can indeed be rewritten as

$$\begin{aligned}\frac{d\phi_1}{dt} &= -G^i\ \partial_i W + \partial_i F^i + 2\ \partial_i Q^{ij}\ \partial_j W + Q^{ij}\ \partial_i\partial_j W \\ &= -G^i\ p_i - \frac{d}{dt}\ln D - \frac{1}{2}\ \frac{\partial^2 H}{\partial p_i \partial p_j}\ C_{ij}\end{aligned} \tag{52}$$

If we define the amplitude

$$A \equiv \mathrm{e}^{-\phi_1} \tag{53}$$

the following equation is obtained for this amplitude

$$\frac{d}{dt}\ln A = \frac{d}{dt}\ln D + \mathbf{G}\cdot\mathbf{p} + \frac{1}{2}\ \mathrm{tr}\frac{\partial^2 H}{\partial \mathbf{p}^2}\cdot\mathsf{C} \tag{54}$$

with the field $\mathbf{G} = \mathbf{G}(\mathbf{x})$ of the Itô stochastic differential equation (2). Integrating along the Hamiltonian trajectories, we get

$$A = \frac{A_0}{D_0}\ D\ \mathrm{e}^{\int_0^t \mathbf{G}\cdot\mathbf{p}\, d\tau}\ \mathrm{e}^{\frac{1}{2}\int_0^t \mathrm{tr}\frac{\partial^2 H}{\partial \mathbf{p}^2}\cdot\mathsf{C}\, d\tau} \tag{55}$$

By rewriting the result (50) in the following form

$$\mathrm{e}^{\frac{1}{2}\int_0^t \mathrm{tr}\frac{\partial^2 H}{\partial \mathbf{p}^2}\cdot\mathsf{C}\, d\tau} = D_0^{\frac{1}{2}}\ D^{-\frac{1}{2}}\ \mathrm{e}^{-\frac{1}{2}\int_0^t \mathrm{tr}\frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{p}}\, d\tau} \tag{56}$$

we can eliminate the curvature matrix C from Eq. (55) to finally obtain

$$A = \frac{A_0}{D_0^{\frac{1}{2}}}\ D^{\frac{1}{2}}\ \mathrm{e}^{\int_0^t \mathbf{G}\cdot\mathbf{p}\, d\tau}\ \mathrm{e}^{-\frac{1}{2}\int_0^t \mathrm{tr}\frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{p}}\, d\tau} \tag{57}$$

The solution of the transport equation (31) is thus given by $\phi_1 = -\ln A$ in terms of the Morette-Van Hove determinant (44). The constant amplitude $A_0 D_0^{-\frac{1}{2}}$ can be fixed by the initial conditions.

Higher orders in the ε-expansion (29) can be obtained systematically with the same method.

C. The propagator in the weak-noise limit

By gathering the results of the two preceding subsections we can obtain an approximation for the kernel of the evolution operator of the Fokker-Planck equation in the weak-noise limit $\varepsilon \to 0$.

If there is only one Hamiltonian trajectory from the initial state $\mathbf{x}_0$ to the final state $\mathbf{x}$ over the time interval t, the constant amplitude in Eq. (57) can be fixed as $A_0 D_0^{-\frac{1}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}}$ for the propagator (7) satisfying the initial condition (8). A semiclassical approximation of the propagator of the Fokker-Planck equation is thus given for $\varepsilon \to 0$ by

$$K(\mathbf{x},\mathbf{x}_0,t) \simeq \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} \left| \det\left(-\frac{\partial^2 W}{\partial\mathbf{x}\partial\mathbf{x}_0}\right)\right|^{\frac{1}{2}} \mathrm{e}^{\int_0^t \mathbf{G}\cdot\mathbf{p}\, d\tau}\ \mathrm{e}^{-\frac{1}{2}\int_0^t \mathrm{tr}\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{p}}\, d\tau}\ \mathrm{e}^{-\frac{1}{\varepsilon}W(\mathbf{x},\mathbf{x}_0,t)} \tag{58}$$

where $W(\mathbf{x},\mathbf{x}_0,t)$ is the action of the Hamiltonian trajectory from $\mathbf{x}_0$ to $\mathbf{x}$ during the time interval t.

The approximate propagator (58) of the Fokker-Planck equation is very similar to the Morette-Van Hove semiclassical propagator of the quantum Schrödinger equation but with significant differences [39]. The first difference is the presence of the factor involving $\mathrm{tr}\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{p}}$. This factor does not appear in the WKB treatment of the Schrödinger equation essentially because the Schrödinger equation is a complex equation for a quantum amplitude Ψ while the Fokker-Planck equation is a real equation for a probability density $\mathcal{P}$ so that they do not have the same transport equation. The continuity equation associated with each one of these equations concerns different quantities, $\mathcal{P}$ itself for the Fokker-Planck equation but the quadratic expression $|\Psi|^2$ for the Schrödinger equation. The second difference is due to the factor involving the ε-correction $\mathbf{G}$ to the vector field $\mathbf{F}$. This difference comes because we have assumed to include this extra correction which is not an essential point.

We point out that, for the orbits in the invariant subspace $\mathbf{p} = 0$, we have that $\mathbf{G}\cdot\mathbf{p} = 0$ and

$$\mathrm{tr}\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{p}}\Big|_{\mathbf{p}=0} = \partial_i F^i = \mathrm{div}\mathbf{F} \tag{59}$$

so that the extra factors take into account the possible phase-space contraction (or dilatation) due to the generally dissipative vector field $\mathbf{F}$ [1].

A remark is that, if several trajectories join the initial state $\mathbf{x}_0$ to the final state $\mathbf{x}$, we have to consider the sum over all these trajectories, the Fokker-Planck equation being linear. We may then have to reajust the normalization constant accordingly.

D. The Laplace transform of the propagator

Assuming that the system is time-independent, we can introduce the resolvent of the Fokker-Planck operator, which is given by the Laplace transform of the evolution operator as

$$\hat{R}(s) = \frac{1}{s-\hat{L}} = \int_0^\infty dt\ \mathrm{e}^{-st}\ \mathrm{e}^{\hat{L}t} \tag{60}$$

The kernel of the resolvent operator (60) is a Green function which is obtained as the Laplace transform of the propagator:

$$G(\mathbf{x},\mathbf{x}_0,s) = \int_0^\infty dt\ \mathrm{e}^{-st}\ K(\mathbf{x},\mathbf{x}_0,t) \tag{61}$$

In the weak-noise limit $\varepsilon \to 0$, this Laplace transform can be carried out for the semiclassical approximation (58) of the propagator by using the steepest-descent method. The Laplace transform takes the form

$$G(\mathbf{x},\mathbf{x}_0,s) \simeq \int_0^\infty dt\ A(\mathbf{x},\mathbf{x}_0,t)\ \mathrm{e}^{-\frac{1}{\varepsilon}[\varepsilon st + W(\mathbf{x},\mathbf{x}_0,t)]} \tag{62}$$

In the steepest-descent approximation, a Legendre transform is performed between the action $W(\mathbf{x},\mathbf{x}_0,t)$ and a new function $V(\mathbf{x},\mathbf{x}_0,s)$ of the Laplace variable s

$$V(\mathbf{x},\mathbf{x}_0,s) = \varepsilon\ s\ t + W(\mathbf{x},\mathbf{x}_0,t) \tag{63}$$

with a time $t = t(\mathbf{x}, \mathbf{x}_0, s)$ such that

$$0 = \varepsilon\, s + \partial_t W(\mathbf{x}, \mathbf{x}_0, t) \tag{64}$$

which expresses the independence of V on the time t. Reciprocally, the independence of W on the Laplace variable s yields

$$\partial_s V(\mathbf{x}, \mathbf{x}_0, s) = \varepsilon\, t \tag{65}$$

which solves for $s = s(\mathbf{x}, \mathbf{x}_0, t)$. The theorem of implicit functions is used to determine the special time $t(\mathbf{x}, \mathbf{x}_0, s)$ from Eq. (64) or the special value $s(\mathbf{x}, \mathbf{x}_0, t)$ of the Laplace variable from Eq. (65), which eliminates either the time or the Laplace variable in Eq. (63).

The Laplace variable s appears to be canonically conjugated to the time t. Since the system is supposed to be time-independent we can introduce a pseudo-energy $E = H$. Because of the Hamilton-Jacobi equation (45), Eq. (64) can be rewritten as

$$H(\mathbf{x}, \mathbf{p}) = E = \varepsilon\, s \tag{66}$$

Since $(-s)$ is a relaxation rate for the system, the pseudo-energy E can thus be interpreted as minus the relaxation rate multiplied by the noise amplitude ε.

By using Gutzwiller's theory [22–24], the Green function takes the following form in the weak-noise limit $\varepsilon \to 0$:

$$G(\mathbf{x}, \mathbf{x}_0, s) \simeq \frac{1}{(2\pi\varepsilon)^{\frac{d-1}{2}}}\, |\Delta(\mathbf{x}, \mathbf{x}_0, s)|^{\frac{1}{2}}\; \mathrm{e}^{\int_0^{t(\mathbf{x},\mathbf{x}_0,s)} \mathbf{G}\cdot\mathbf{p}\, d\tau}\; \mathrm{e}^{-\frac{1}{2}\int_0^{t(\mathbf{x},\mathbf{x}_0,s)} \mathrm{tr}\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{p}}\, d\tau}\; \mathrm{e}^{-\frac{1}{\varepsilon}V(\mathbf{x},\mathbf{x}_0,s)} \tag{67}$$

where

$$\Delta = \frac{1}{\varepsilon^2}\left(\frac{\partial^2 V}{\partial s^2}\right)^{1-d} \det\left(\frac{\partial^2 V}{\partial s^2}\frac{\partial^2 V}{\partial\mathbf{x}\partial\mathbf{x}_0} - \frac{\partial^2 V}{\partial\mathbf{x}\partial s}\frac{\partial^2 V}{\partial\mathbf{x}_0\partial s}\right) \tag{68}$$

In the general case, we should expect a sum over all the trajectories connecting $\mathbf{x}_0$ to $\mathbf{x}$ at the pseudo-energy $E = \varepsilon s$.

IV. SEMICLASSICAL TRACE FORMULA

We consider the trace of the evolution operator associated with the Fokker-Planck equation. This trace selects all the Hamiltonian trajectories which are closed on themselves in the doubled phase space $(\mathbf{x}, \mathbf{p})$. These closed trajectories are of two types:

(1) the stationary states $\dot{\mathbf{x}} = \dot{\mathbf{p}} = 0$ which are orbits of arbitrary period;

(2) the periodic orbits which have a well defined prime period $\mathcal{T}_{\mathrm{p}}(E)$.

For the periodic orbits, recurrences to the initial conditions happen at each multiple of the prime period $t = r\mathcal{T}_{\mathrm{p}}(E)$. The integer $r \in \mathbb{Z}$ is called the repetition number.

Typically, the stationary states occur for a critical value of the pseudo-energy while the periodic orbits can be deformed continuously with the pseudo-energy (except at bifurcations) as shown in Fig. 1.

We notice that the different contributions add since the Fokker-Planck equation is linear.

Since the Freidlin-Wentzell Hamiltonian flow (38) leaves invariant the subspace $\mathbf{p} = 0$, we can classify the stationary and periodic orbits into the ones which lie in the subspace $\mathbf{p} = 0$ and the other ones for which $\mathbf{p} \neq 0$. We notice that, for the orbits with zero momenta $\mathbf{p} = 0$, both the pseudo-energy and the action vanish: $E = 0$ and $W = 0$. Since the orbits contribute to the propagator (58) by a factor $\exp(-W/\varepsilon)$, we can conclude that the orbits in the subspace $\mathbf{p} = 0$ have a dominant contribution as compared with the orbits with a nonvanishing action $W \neq 0$ which are exponentially suppressed in the weak-noise limit $\varepsilon \to 0$.

A. Contribution from a stationary state

Stationary states can exist in the invariant subspace $\mathbf{p} = 0$ or outside this subspace for $\mathbf{p} \neq 0$. For the aforementioned reason, we shall here focus on the dominant stationary states such that

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}_{\mathrm{s}}) = 0 \qquad \text{and} \qquad \mathbf{p} = 0 \tag{69}$$

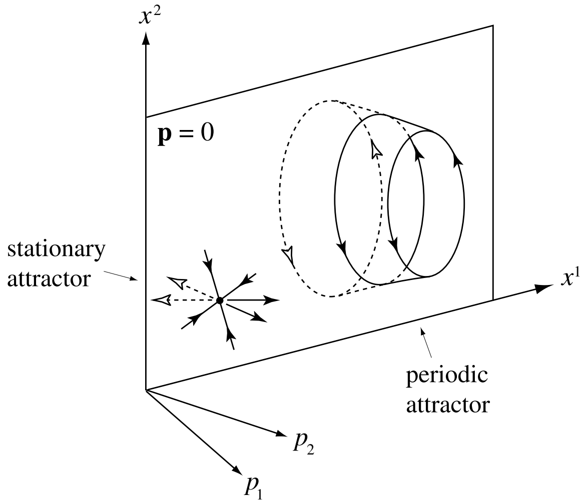

FIG. 1: Phase portrait of a stationary state and a periodic orbit in the doubled phase space of variables $(\mathbf{x}, \mathbf{p})$ near the invariant subspace $\mathbf{p} = 0$.

Furthermore, we assume that this stationary point is not undergoing a bifurcation so that the $d \times d$ Jacobian matrix of the macroscopic vector field $\mathbf{F}(\mathbf{x})$ at the stationary point

$$\mathsf{L} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}_{\rm s}) \tag{70}$$

does not have eigenvalues $\{\xi_k\}_{k=1}^d$ with a vanishing real part. The eigenvalues are given as the d roots of the characteristic determinant

$$\det(\mathsf{L} - \xi\,\mathsf{I}) = 0 \tag{71}$$

In general, these eigenvalues fall in two sets: d_s eigenvalues associated with the stable directions with Re $\xi_i < 0$ and d_u other eigenvalues associated with the unstable directions with Re $\xi_j > 0$. We have that $d = d_s + d_u$. The Lyapunov exponents of the stationary state are given by $\lambda_k = \text{Re}\ \xi_k$.

In the weak-noise limit $\varepsilon \to 0$, we expect a Gaussian distribution around a stationary point which is away from a bifurcation. This expectation can be confirmed by calculating the propagator in the vicinity of the stationary point.

For this purpose, the Hamiltonian vector field (38) can be approximated by the linearized vector field:

$$\begin{cases} \dot{\mathbf{y}} = \mathsf{L}\cdot\mathbf{y} + 2\,\mathsf{Q}\cdot\mathbf{p} \\ \dot{\mathbf{p}} = -\mathsf{L}^{\rm T}\cdot\mathbf{p} \end{cases} \tag{72}$$

where $\mathbf{y} = \mathbf{x} - \mathbf{x}_{\rm s}$ and $\mathsf{Q} = \left[Q^{ij}(\mathbf{x}_{\rm s})\right]$. The Hamiltonian of this linearized system is given by

$$H = \mathbf{p}^{\rm T}\cdot\mathsf{Q}\cdot\mathbf{p} + \mathbf{p}^{\rm T}\cdot\mathsf{L}\cdot\mathbf{y} \tag{73}$$

The solutions of Eqs. (72) are of the form:

$$\begin{cases} \mathbf{y} = {\rm e}^{\mathsf{L}t}\cdot\mathbf{y}_0 + 2\,\mathsf{R}(t)\cdot{\rm e}^{-\mathsf{L}^{\rm T}t}\cdot\mathbf{p}_0 \\ \mathbf{p} = {\rm e}^{-\mathsf{L}^{\rm T}t}\cdot\mathbf{p}_0 \end{cases} \tag{74}$$

with

$$\mathsf{R}(t) = \int_0^t d\tau\ {\rm e}^{\mathsf{L}\tau}\cdot\mathsf{Q}\cdot{\rm e}^{\mathsf{L}^{\rm T}\tau} = \mathsf{R}(t)^{\rm T} \geq 0 \tag{75}$$

Accordingly, the action of the trajectory (74) is given by

$$W(\mathbf{y}, \mathbf{y}_0, t) = \int_0^t \mathbf{p}^{\rm T}\cdot d\mathbf{y} - H\,d\tau = \int_0^t \mathbf{p}^{\rm T}\cdot\mathsf{Q}\cdot\mathbf{p}\,d\tau = \mathbf{p}_0^{\rm T}\cdot{\rm e}^{-\mathsf{L}t}\cdot\mathsf{R}(t)\cdot{\rm e}^{-\mathsf{L}^{\rm T}t}\cdot\mathbf{p}_0 \tag{76}$$

Since

$$\mathbf{p}_0 = \frac{1}{2}\, \mathrm{e}^{\mathsf{L}^{\mathrm{T}} t} \cdot \mathsf{R}(t)^{-1} \cdot \big(\mathbf{y} - \mathrm{e}^{\mathsf{L}t} \cdot \mathbf{y}_0\big) \tag{77}$$

the action becomes

$$W(\mathbf{y},\mathbf{y}_0,t) = \frac{1}{4}\, \big(\mathbf{y} - \mathrm{e}^{\mathsf{L}t} \cdot \mathbf{y}_0\big)^{\mathrm{T}} \cdot \mathsf{R}(t)^{-1} \cdot \big(\mathbf{y} - \mathrm{e}^{\mathsf{L}t} \cdot \mathbf{y}_0\big) \tag{78}$$

By Eqs. (58) and (59), the propagator near the stationary point is therefore obtained as

$$K(\mathbf{y},\mathbf{y}_0,t) \simeq \frac{1}{(4\pi\varepsilon)^{\frac{d}{2}}}\, \frac{1}{\sqrt{\det \mathsf{R}(t)}}\; \mathrm{e}^{-\frac{1}{4\varepsilon}\left(\mathbf{y}-\mathrm{e}^{\mathsf{L}t}\cdot\mathbf{y}_0\right)^{\mathrm{T}}\cdot\mathsf{R}(t)^{-1}\cdot\left(\mathbf{y}-\mathrm{e}^{\mathsf{L}t}\cdot\mathbf{y}_0\right)} \tag{79}$$

in the limit $\varepsilon \to 0$, as it should.

We notice that, if the stationary state is linearly stable so that $\mathrm{Re}\,\xi_i < 0$ for $i = 1, 2, ..., d$, the propagator approaches the invariant probability density in the long-time limit $t \to +\infty$:

$$\mathcal{P}_\infty(\mathbf{x}) = \lim_{t\to+\infty} K(\mathbf{y},\mathbf{y}_0,t) \simeq \frac{1}{(4\pi\varepsilon)^{\frac{d}{2}}}\, \frac{1}{\sqrt{\det \mathsf{R}(\infty)}}\; \mathrm{e}^{-\frac{1}{4\varepsilon}\, \mathbf{y}^{\mathrm{T}}\cdot\mathsf{R}(\infty)^{-1}\cdot\mathbf{y}} \tag{80}$$

In the weak-noise limit $\varepsilon \to 0$, the invariant probability density is thus Gaussian around a stationary attractor. In systems with several stable stationary states (which do not bifurcate), we notice that the invariant probability density is a sum of such Gaussian functions in the weak-noise limit.

We infer from Eq. (79) that the stationary state contributes to the trace of the evolution operator by the term

$$\mathrm{tr}\; \mathrm{e}^{\hat{L}t}\Big|_{\mathrm{s}} = \int d\mathbf{y}\; K(\mathbf{y},\mathbf{y},t) \simeq \frac{1}{\vert \det(\mathsf{I} - \mathrm{e}^{\mathsf{L}t})\vert} + \mathcal{O}(\varepsilon) \tag{81}$$

As a consequence, we observe that the decay of this contribution is a sum of exponential functions of the time t. At leading order, the decay is independent of the noise amplitude ε in the weak-noise limit $\varepsilon \to 0$, which is a consequence of the assumption that the stationary state does not bifurcate. At the leading zeroth order in the noise amplitude, the result (81) is nothing else than the contribution of a stationary state to the trace formula of the Frobenius-Perron operator of the deterministic dynamics of the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ [29].

We can express the contribution (81) in terms of the eigenvalues (71) of the linear stability of the stationary state as [29]

$$\mathrm{tr}\; \mathrm{e}^{\hat{L}t}\Big|_{\mathrm{s}} \simeq \sum_{\mathbf{l},\mathbf{m}=0}^{\infty} \exp\left[\sum_{\mathrm{Re}\,\xi_i<0} l_i\; \xi_i\; t - \sum_{\mathrm{Re}\,\xi_j>0} (m_j+1)\; \xi_j\; t\right] + \mathcal{O}(\varepsilon) \tag{82}$$

We observe that this contribution decays to zero in the limit $t \to +\infty$ if the stationary state is unstable, i.e., as soon as there is an eigenvalue with $\mathrm{Re}\,\xi_j > 0$. If all the eigenvalues have $\mathrm{Re}\,\xi_i < 0$ the stationary state is linearly stable and the contribution (82) tends to the unit value for $t \to +\infty$. We shall discuss the consequences of Eq. (82) on the spectrum of eigenvalues in Sec. V.

B. Contribution from a periodic orbit

The contribution from a periodic orbit can be calculated by using Gutzwiller's theory [22, 23]. We assume that the periodic orbit is isolated and non-bifurcating. Near a periodic orbit, the trace of the propagator can be calculated again by using the steepest-descent method. This kind of calculation is not new and has been carried out in several papers and books [22–24, 39, 40]. The result is given by:

$$\mathrm{tr}\; \mathrm{e}^{\hat{L}t}\Big|_{\mathrm{p}} = \int d\mathbf{x}\; K(\mathbf{x},\mathbf{x},t)\Big|_{\mathrm{p}} \simeq \sum_{r:\, r\mathcal{T}_{\mathrm{p}}(E)=t} \frac{\mathcal{T}_{\mathrm{p}}(E)\; \mathrm{e}^{\int_0^t \mathbf{G}\cdot\mathbf{p}\; d\tau}\; \mathrm{e}^{-\frac{1}{2}\int_0^t \mathrm{tr}\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{p}}\; d\tau}\; \mathrm{e}^{-\frac{1}{\varepsilon}W(t)}}{\vert 2\,\pi\,\varepsilon\, r\, \partial_E \mathcal{T}_{\mathrm{p}}(E)\; \det(\mathsf{J}_{\mathrm{p}}^r - \mathsf{I})\vert^{\frac{1}{2}}} \tag{83}$$

where the sum is carried out over all the repetition numbers such that

$$r\; \mathcal{T}_{\mathrm{p}}(E) = t \tag{84}$$

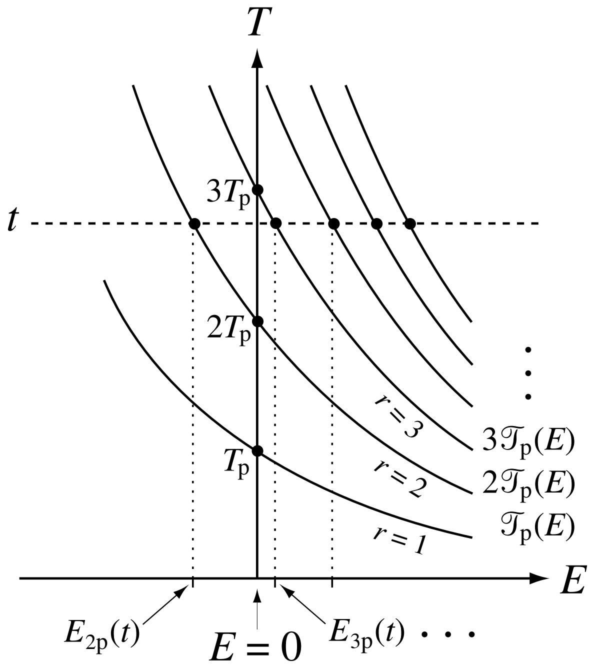

FIG. 2: Typical diagram of the prime period of a periodic orbit and its repetitions $r\mathcal{T}_{\rm p}(E)$ as a function of the pseudo-energy E. The dots at the pseudo-energy $E = 0$ are the periods $rT_{\rm p} = r\mathcal{T}_{\rm p}(E = 0)$ of the macroscopic deterministic system. The contributions to the trace (83) of the propagator at time t are given by the intersections of the dashed horizontal line t with the periods $r\mathcal{T}_{\rm p}(E)$, according to Eq. (84). As time increases, the dashed horizontal line moves upward and the intersections $E_{r{\rm p}}(t)$ successively cross the vertical line $E = 0$ corresponding to the macroscopic deterministic system without noise.

This condition selects different pseudo-energies $E_{r{\rm p}}(t)$ such that $\mathcal{T}_{\rm p}\left[E_{r{\rm p}}(t)\right] = t/r$ (see Fig. 2).

If we define the reduced action of the periodic orbit as

$$V_{\rm p}(E) = \oint_{\rm p} \mathbf{p}\cdot d\mathbf{x} = \int_0^{\mathcal{T}_{\rm p}(E)} \mathbf{p}\cdot d\mathbf{x} \tag{85}$$

the prime period is given by [24, 39]

$$\mathcal{T}_{\rm p}(E) = \partial_E V_{\rm p}(E) \tag{86}$$

and we have the relationships

$$\partial_t^2 W\left[\mathcal{T}_{\rm p}(E)\right] = -\frac{1}{\partial_E^2 V_{\rm p}(E)} = -\frac{1}{\partial_E \mathcal{T}_{\rm p}(E)} \tag{87}$$

which are consequences of the Legendre transform given by Eqs. (63), (64), and (65).

The symbol $\mathsf{J}_{\rm p}$ denotes the matrix of a linearized Poincaré map of the full Hamiltonian flow in the pseudo-energy shell $H = E$ of the doubled phase space $(\mathbf{x}, \mathbf{p})$. This square matrix is of dimension $(2d-2)\times(2d-2)$ because it acts in the $(2d-2)$-dimensional linear subspace which is tangent to the doubled phase space of dimension $2d$, to the energy shell of codimension one, and to a Poincaré surface of section also of codimension one and transverse to the periodic orbit. The matrix $\mathsf{J}_{\rm p}$ can alternatively be obtained by linearizing Hamilton's equations (38). The fundamental matrix of this linear system of $2d$ differential equations when integrated over the prime period $\mathcal{T}_{\rm p}(E)$ has the $2d$ eigenvalues $\{\Lambda_1, \Lambda_2, ..., \Lambda_{2d-2}, 1, 1\}$. Because of the symplectic character of the Hamiltonian flow, these eigenvalues obey the pairing rule that Λ_i^{-1} is an eigenvalue if Λ_i is. Both unit eigenvalues correspond to the direction of the flow and to the direction transverse to the energy shell, $H = E$. The $2d-2$ other eigenvalues are the same as the $2d-2$ eigenvalues of the matrix $\mathsf{J}_{\rm p}$.

As aforementioned, the periodic orbits which produce the dominant contribution are the ones in the invariant subspace $\mathbf{p} = 0$ because the action vanishes for these periodic orbits: $W = 0$. Fig. 2 shows that the successive repetitions of the prime period intersect the invariant subspace at $E = 0$ as the time t increases. These intersections occurs when $E_{rp}(t) = 0$, i.e., at each repetition $t = rT_{\rm p}$ of the prime period of the periodic orbit of the invariant subspace $\mathbf{p} = 0$: $T_{\rm p} = \mathcal{T}_{\rm p}(E = 0)$. At these times $t = rT_{\rm p}$, the corresponding term of the sum (83) has its action which vanishes, $W(t = rT_{\rm p}) = 0$, so that this term is thus expected to have a large contribution to the sum (83). In order to determine this contribution, we can expand the action in a power series of the difference $t - rT_{\rm p}$ between the current time t and the repetition $rT_{\rm p}$ of the prime period as

$$W(t) = \underbrace{W(rT_{\rm p})}_{=0} + \underbrace{\frac{\partial W}{\partial t}(rT_{\rm p})}_{=0} \, (t - rT_{\rm p}) + \frac{1}{2}\, \frac{\partial^2 W}{\partial t^2}(rT_{\rm p}) \, (t - rT_{\rm p})^2 + \cdots \tag{88}$$

The zeroth-order term of this expansion vanishes, $W(rT_{\rm p}) = 0$, because the action vanishes for all the orbits of the invariant subspace $\mathbf{p} = 0$. The first-order term also vanishes because

$$\frac{\partial W}{\partial t} = -E \tag{89}$$

by the condition (64) for the Legendre transform (63) and because $E = 0$. Finally, we remain with the quadratic term of this expression which has the coefficient

$$\frac{\partial^2 W}{\partial t^2}(rT_{\rm p}) = -\frac{1}{r\, \partial_E T_{\rm p}} \tag{90}$$

where $\partial_E T_{\rm p} = \partial_E \mathcal{T}_{\rm p}(E = 0)$. Here, we have the property:

For a typical periodic orbit of the invariant subspace $\mathbf{p} = 0$*:*

$$\partial_E T_{\rm p} < 0 \tag{91}$$

Indeed, if this derivative was positive, $\partial_t^2 W$ would be negative by Eq. (90) and a contradiction would appear because $\exp(-W/\varepsilon) \simeq \exp\left[-\partial_t^2 W (t - rT_{\rm p})^2/(2\varepsilon)\right]$ would become arbitrarily large for $\varepsilon \to 0$. Moreover, for a typical periodic orbit, we do not expect the equality $\partial_E T_{\rm p} = 0$, hence the property. Q. E. D.

Furthermore, for the periodic orbits of the invariant subspace $\mathbf{p} = 0$, we have the remarkable property that

$$\det(\mathsf{J} - \mathsf{I}) = \det(\mathsf{m} - \mathsf{I}) \; \det(\mathsf{m}^{-1} - \mathsf{I}) \tag{92}$$

where $\mathsf{J} = \mathsf{J}_{\rm p}^r$ is the $(2d-2) \times (2d-2)$ matrix of the $r^{\rm th}$ iterate of the linearized Poincaré map in the *doubled* phase space of dimension $2d$, while $\mathsf{m} = \mathsf{m}_{\rm p}^r$ is the $(d-1) \times (d-1)$ matrix of the $r^{\rm th}$ iterate of the linearized Poincaré map in the invariant subspace $\mathbf{p} = 0$, i.e., in the *original* phase space of dimension d where the macroscopic deterministic flow (39) is defined.

The property (92) is a crucial observation for the following results and it is proved as follows. In the doubled space phase, the linearized stability of a periodic orbit of the invariant subspace $\mathbf{p} = 0$ is analyzed by the linearized system of equations:

$$\begin{cases} \delta\dot{\mathbf{x}} = \dfrac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}) \cdot \delta\mathbf{x} + 2\; \mathsf{Q}(\mathbf{x}) \cdot \delta\mathbf{p} \\ \delta\dot{\mathbf{p}} = -\dfrac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x})^{\rm T} \cdot \delta\mathbf{p} \end{cases} \tag{93}$$

The solutions of these equations are

$$\begin{cases} \delta\mathbf{x}(t) = \mathsf{M}(t) \cdot \left[\delta\mathbf{x}(0) + 2 \displaystyle\int_0^t d\tau\; \mathsf{M}(\tau)^{-1} \cdot \mathsf{Q}\left[\mathbf{x}(\tau)\right] \cdot \mathsf{M}(\tau)^{-1\,{\rm T}} \cdot \delta\mathbf{p}(0)\right] \\ \delta\mathbf{p}(t) = \mathsf{M}(t)^{-1\,{\rm T}} \cdot \delta\mathbf{p}(0) \end{cases} \tag{94}$$

where $\mathsf{M}(t)$ is the fundamental $d \times d$ matrix which is solution of

$$\dot{\mathsf{M}}(t) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\left[\mathbf{x}(t)\right] \cdot \mathsf{M}(t) \qquad \text{from} \qquad \mathsf{M}(0) = \mathsf{I} \tag{95}$$

Therefore, the solution over r prime periods takes the form of the following $2d \times 2d$ matrix

$$\begin{pmatrix} \delta\mathbf{x}(t) \\ \delta\mathbf{p}(t) \end{pmatrix} = \begin{pmatrix} \mathsf{M}(t) & \mathsf{N}(t) \\ 0 & \mathsf{M}(t)^{-1\,{\rm T}} \end{pmatrix} \cdot \begin{pmatrix} \delta\mathbf{x}(0) \\ \delta\mathbf{p}(0) \end{pmatrix} \tag{96}$$

with the $d \times d$ matrix $\mathsf{N}(t)$ given by the second term of the first line of Eq. (94). The eigenvalues of the $d \times d$ matrix $\mathsf{M}(t = rT_{\rm p})$ are expected to be given by $\{\Lambda_1^r, \Lambda_2^r, ..., \Lambda_{d-1}^r, 1\}$ where $\{\Lambda_i\}_{i=1}^{d-1}$ are the eigenvalues of the $(d-1) \times (d-1)$ matrix $\mathsf{m}_{\rm p}$ of the linearized Poincaré map of the macroscopic system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ in the original phase space of dimension d. Accordingly, the full $2d \times 2d$ matrix in Eq. (96) have the $2d$ eigenvalues $\{\Lambda_1^r, \Lambda_2^r, ..., \Lambda_{d-1}^r, 1, \Lambda_1^{-r}, \Lambda_2^{-r}, ..., \Lambda_{d-1}^{-r}, 1\}$. If we eliminate both unit eigenvalues which correspond to the direction of the flow and to the conjugated direction transverse to the energy shell $H = E$ we remain with the $2d-2$ eigenvalues of the matrix $\mathsf{J} = \mathsf{J}_{\rm p}^r$. This line of reasoning proves that the $2d-2$ eigenvalues of J are the $d-1$ eigenvalues of m together with the $d-1$ eigenvalues of m^{-1}. The equality (92) between the determinants follows since the determinant of a matrix is given in terms of the eigenvalues of the matrix. Q. E. D.

Besides, for periodic orbits in the invariant subspace $\mathbf{p} = 0$, we have the following property which is a consequence of Eq. (59)

$$\exp\left[-\frac{1}{2}\int_0^t {\rm tr}\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{p}}\, d\tau\right] = \exp\left[-\frac{1}{2}\int_0^t {\rm div}\mathbf{F}\, d\tau\right] = |\det \mathsf{m}|^{-\frac{1}{2}} \tag{97}$$

for $t = rT_{\rm p}$ because the time average of the divergence of the vector field $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is known to be the sum of all the Lyapunov exponents. Moreover, the Lyapunov exponents λ_i of a periodic orbit are related to its stability eigenvalues Λ_i according to $\lambda_i = \ln|\Lambda_i|/T_{\rm p}$ and the Lyapunov exponents corresponding to the direction of the flow or to conserved quantities vanish, hence Eq. (97). Q. E. D.

Combining the properties (92) and (97), we obtain the remarkable identity

$$\frac{{\rm e}^{-\frac{1}{2}\int_0^t {\rm tr}\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{p}}\, d\tau}}{|\det(\mathsf{J}-\mathsf{I})|^{\frac{1}{2}}} = \frac{1}{|\det(\mathsf{m}-\mathsf{I})|} \tag{98}$$

for $t = rT_{\rm p}$.

For periodic orbits in the invariant subspace $\mathbf{p} = 0$, we also have the property that

$${\rm e}^{\int_0^t \mathbf{G}\cdot\mathbf{p}\, d\tau} = 1 \tag{99}$$

Combining the properties (88), (90), (98), and (99) into Eq. (83), we finally find that the leading contribution of a periodic orbit of the invariant subspace $\mathbf{p} = 0$ to the trace of the Fokker-Planck evolution operator takes the form

$${\rm tr}\, {\rm e}^{\hat{L}t}\Big|_{\rm p} = \int d\mathbf{x}\; K(\mathbf{x},\mathbf{x},t)\Big|_{\rm p} \simeq \sum_{r=1}^{\infty} \frac{T_{\rm p}}{|\det(\mathsf{m}_{\rm p}^r - \mathsf{I})|}\, \frac{{\rm e}^{-\frac{(t-rT_{\rm p})^2}{2\varepsilon r|\partial_E T_{\rm p}|}}}{\sqrt{2\,\pi\,\varepsilon\, r\, |\partial_E T_{\rm p}|}} \tag{100}$$

for $\varepsilon \to 0$. We observe the remarkable feature that the doubling of the phase-space dimension in the Onsager-Machlup-Freidlin-Wentzell scheme has for consequence that $|\det(\mathsf{m}_{\rm p}^r - \mathsf{I})|$ appears with the classical exponent 1 although the full factor $|\det(\mathsf{J}_{\rm p}^r - \mathsf{I})|$ appears with the exponent $\frac{1}{2}$ in Eq. (83) which is closer in this respect to the quantum mechanical expression. In the present context of stochastic systems, we expect to recover instead the exponent 1 of the trace formula of the Frobenius-Perron evolution operator of classical mechanics. Actually, the formula (100) has been anticipated by Nicolis and Gaspard [25] in analogy with the classical trace formula obtained by Cvitanović and Eckhardt [26] and assuming an extra phase diffusion due to the presence of the noise. Because of the phase diffusion, we expect that the Gaussian function in the trace of the evolution operator should become broader and broader as the repetition number r increases (see Fig. 3). By phase diffusion, the width of the Gaussian should increase as $\sqrt{r}$. We have here given a complete derivation of the stochastic trace formula, obtaining the precise expression for the coefficient $\alpha_{\rm p} = 1/|\partial_E T_{\rm p}|$ which gives the effect of phase diffusion along the periodic orbit. This effect is thus determined by the derivative of the period with respect to the pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme.

In the noiseless limit $\varepsilon = 0$, we recover the Cvitanović-Eckhardt classical trace formula [26]

$$\lim_{\varepsilon\to 0}\; {\rm tr}\, {\rm e}^{\hat{L}t}\Big|_{\rm p} = \sum_{r=1}^{\infty} \frac{T_{\rm p}\, \delta(t - rT_{\rm p})}{|\det(\mathsf{m}_{\rm p}^r - \mathsf{I})|} \tag{101}$$

as expected. However, the quality factor of the periodic orbits in the presence of noise is determined in an essential way by the expression (100) before taking the noiseless limit $\varepsilon = 0$, as we shall explain in the next section.

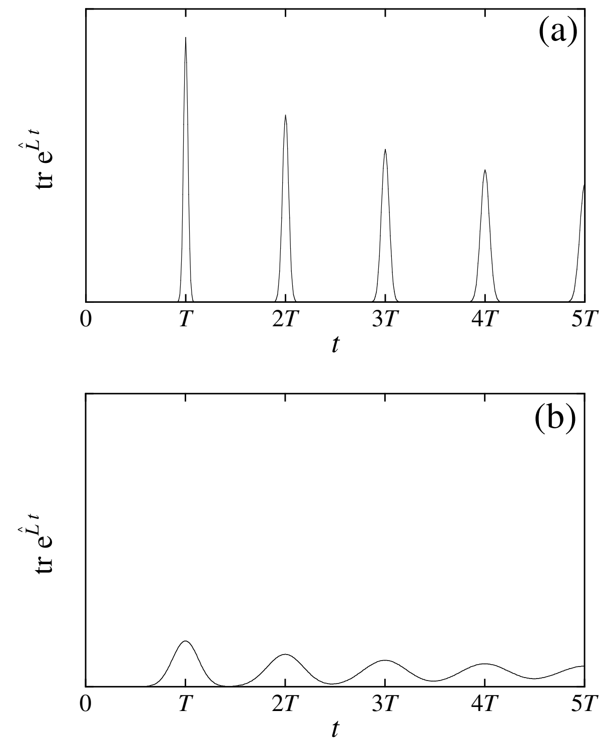

FIG. 3: Schematic behavior of the contribution (100) of a periodic orbit of prime period $T = T_{\rm p}$ to the trace of the evolution operator of the Fokker-Planck operator as a function of time for the case of a weak noise (a) and a stronger noise (b).

V. EIGENRATES

If the system is time-independent, the relaxation rates of the time-correlation functions are given by the eigenvalues of the Fokker-Planck operator. These eigenvalues are in turn given by the poles of the trace of the resolvent of the Fokker-Planck operator. The weak-noise approximation of the trace of the resolvent can be obtained from the results of the previous section. We notice that both stationary and periodic orbits may contribute to this trace

$$\mathrm{tr}\, \frac{1}{s-\hat{L}} \simeq \sum_{\rm s} \mathrm{tr}\, \frac{1}{s-\hat{L}}\Big\vert_{\rm s} + \sum_{\rm p} \mathrm{tr}\, \frac{1}{s-\hat{L}}\Big\vert_{\rm p} \tag{102}$$

in the weak-noise limit $\varepsilon \to 0$.

A. Stationary attractor

We focus on the contribution of a stationary state in the invariant subspace $\mathbf{p} = 0$ because they are expected to dominate the behavior of systems away from bifurcations.

In the weak-noise limit $\varepsilon \to 0$, Eq. (82) leads to the result

$$\mathrm{tr}\, \frac{1}{s-\hat{L}}\Big\vert_{\rm s} \simeq \sum_{\mathbf{l},\mathbf{m}=0}^{\infty} \frac{1}{s-s_{\mathbf{lm}}} \tag{103}$$

with the eigenvalues

$$\text{stationary state :} \qquad s_{\mathbf{lm}} = \sum_{{\rm Re}\,\xi_i<0} l_i\, \xi_i \; - \sum_{{\rm Re}\,\xi_j>0} (m_j+1)\, \xi_j + \mathcal{O}(\varepsilon) \tag{104}$$

for $l_i, m_j = 0, 1, 2, 3, ...$ We have here assumed that the stationary state does not undergo any bifurcation so that ${\rm Re}\,\xi_k \neq 0$ for all $k = 1, 2, ..., d$.

If the stationary state is unstable, its leading eigenvalue is given by $\mathbf{l} = \mathbf{m} = 0$:

$$s_{00\cdots 0} = -\sum_{\mathrm{Re}\,\xi_j>0} \xi_j + \mathcal{O}(\varepsilon) \tag{105}$$

which has a strictly negative real part. Accordingly, the corresponding eigenmode located near an unstable stationary state is always decaying.

If the stationary state is stable (i.e., $\mathrm{Re}\,\xi_i < 0$ for all $i = 1, 2, ..., d$), it constitutes a stationary attractor. The eigenvalues of the Fokker-Planck operator are now given by

$$\text{stationary attractor :} \qquad s_{\mathbf{l}} = \sum_{\mathrm{Re}\,\xi_i<0} l_i\, \xi_i + \mathcal{O}(\varepsilon) \qquad \text{with} \quad l_i = 0, 1, 2, 3, ... \tag{106}$$

The leading eigenvalue given by $\mathbf{l} = 0$ vanishes, $s_{00\cdots 0} = 0$, as expected for a stable stationary state which attracts the invariant probability density in its neighborhood. This invariant probability density is the eigenfunction of the Fokker-Planck operator corresponding to the eigenvalue $s_{00\cdots 0} = 0$. If the stability eigenvalues are ordered as:

$$0 > \lambda_1 = \mathrm{Re}\,\xi_1 \geq \lambda_2 = \mathrm{Re}\,\xi_2 \geq \cdots \geq \lambda_d = \mathrm{Re}\,\xi_d \tag{107}$$

the slowest relaxation mode corresponds to the Fokker-Planck eigenvalue

$$s_{10\cdots 0} = \xi_1 + \mathcal{O}(\varepsilon) = \lambda_1 + \mathrm{i}\,\omega_1 + \mathcal{O}(\varepsilon) = -\frac{1}{\tau} + \mathrm{i}\,\omega \tag{108}$$

which shows that the relaxation time of this mode is inversely proportional to the maximum Lyapunov exponent λ_1 of the stationary state. This relaxation time also gives the leading decay rate of the time-correlation functions in a system with such a stationary attractor. We have therefore obtained the following:

Theorem. The correlation time of the fluctuations around a non-bifurcating stationary attractor is given in the weak-noise limit $\varepsilon \to 0$ by

$$\tau = \frac{1}{|\lambda_1|} + \mathcal{O}(\varepsilon) \tag{109}$$

where $\lambda_1 < 0$ is the maximum Lyapunov exponent of the stationary attractor.

If the stationary attractor bifurcates, the maximum Lyapunov exponents λ_1 vanishes and the formula (109) predicts a growing lifetime of the correlations as the bifurcation is approached. This growth is however limited by the presence of the noise according to Eq. (108) and further methods not developed in the present paper are required near bifurcations [41–44].

B. Periodic attractor

Here again, we focus on the contribution from a periodic orbit of the invariant subspace $\mathbf{p} = 0$, as explained above.

The Laplace transform of the contribution (100) of such a periodic orbit to the trace of the Fokker-Planck evolution operator is given in the weak-noise limit $\varepsilon \to 0$ by

$$\mathrm{tr}\, \frac{1}{s - \hat{L}}\Big|_{\mathrm{p}} \simeq \sum_{r=1}^{\infty} \frac{T_{\mathrm{p}}}{|\det(\mathsf{m}_{\mathrm{p}}^r - \mathsf{I})|} \left(\mathrm{e}^{-T_{\mathrm{p}} s + \frac{\varepsilon}{2}|\partial_E T_{\mathrm{p}}| s^2}\right)^r \tag{110}$$

We suppose that the periodic orbit does not bifurcate so that the stability eigenvalues of the matrix m_{p} are $\{\Lambda^{(s)}_{\mathrm{p},1}, ..., \Lambda^{(s)}_{\mathrm{p},d_s}, \Lambda^{(u)}_{\mathrm{p},1}, ..., \Lambda^{(u)}_{\mathrm{p},d_u}\}$ where

$$|\Lambda^{(s)}_{\mathrm{p},i}| < 1 < |\Lambda^{(u)}_{\mathrm{p},j}| \tag{111}$$

with $i = 1, ..., d_s$, $j = 1, ..., d_u$, and $d_s + d_u = d - 1$. By standard methods [26, 29, 45], the expression (110) can be rewritten as the logarithmic derivative

$$\mathrm{tr}\, \frac{1}{s - \hat{L}}\Big|_{\mathrm{p}} \simeq \frac{\partial}{\partial s}\, \ln\, Z_{\mathrm{p}}(s) \tag{112}$$

of the zeta function

$$\text{periodic orbit :} \qquad Z_{\rm p}(s) = \prod_{\mathbf{l},\mathbf{m}=0}^{\infty} \left[1 - \frac{\exp\left(-T_{\rm p}s + \frac{\varepsilon}{2}|\partial_E T_{\rm p}|s^2\right)\prod_{i=1}^{d_s}\Lambda_{{\rm p},i}^{(s)\;l_i}}{\prod_{j=1}^{d_u}|\Lambda_{{\rm p},j}^{(u)}|\,\Lambda_{{\rm p},j}^{(u)\;m_j}} \right] \tag{113}$$

with $l_i, m_j = 0, 1, 2, 3, ...$, always in the weak-noise limit $\varepsilon \to 0$.

We notice that, in the noiseless limit $\varepsilon = 0$, we recover the classical zeta function for the Pollicott-Ruelle resonances of the deterministic dynamics [26–29]. In the presence of a weak noise and assuming that the periodic orbit does not bifurcate, the eigenvalues of the Fokker-Planck operator are shifted with respect to the Pollicott-Ruelle resonances of the deterministic dynamics by a quantity of order ε. If a Pollicott-Ruelle resonance has a non-vanishing real part, the shift due to the weak noise appears as a small perturbation to the decay rate. However, if the real part of the Pollicott-Ruelle resonance vanishes, this shift may have a considerable effect.

This is the case for a periodic attractor or stable limit cycle. Such periodic attractors are known to constitute models of nonequilibrium oscillators [1]. In the noiseless limit $\varepsilon = 0$, the oscillations of the periodic attractor are perfectly correlated in time which is expressed by an infinite family of Pollicott-Ruelle resonances with a vanishing real part: $s_n = {\rm i}\omega n$ with $n \in \mathbb{Z}$ and $\omega = 2\pi/T_{\rm p}$. These Pollicott-Ruelle resonances correspond to all the harmonics of the fundamental pulsation $s_1 = {\rm i}\omega$.

In the presence of noise, the oscillations are no longer perfectly correlated in time because of the phase diffusion induced by the noise. As a consequence, the aforementioned family of resonances acquire a real part which is proportional to the noise amplitude ε. Accordingly, the correlation time of the oscillations turns out to be finite since it is given by the inverse of the leading nonvanishing real part of the Fokker-Planck eigenvalues.

In order to calculate this correlation time, we have thus to consider the zeta function for a periodic attractor. Such an attractor contains a single periodic orbit which is stable so that there is no unstable eigenvalue: $d_s = d - 1$ and $d_u = 0$. Whereupon, the zeta function (113) takes the form

$$\text{periodic attractor :} \qquad Z_{\rm p}(s) = \prod_{\mathbf{l}=0}^{\infty} \left[1 - \exp\left(-T_{\rm p}s + \frac{\varepsilon}{2}|\partial_E T_{\rm p}|s^2\right)\prod_{i=1}^{d-1}\Lambda_{{\rm p},i}^{(s)\;l_i} \right] = 0 \tag{114}$$

with $l_i = 0, 1, 2, 3, ...$

Eq. (112) shows that the poles of the resolvent of the Fokker-Planck operator and, thus, the eigenvalues themselves are given by the zeroes of the zeta function. Using Eq. (114) and the identity $1 = \exp(2\pi{\rm i}n)$ for $n \in \mathbb{Z}$, we obtain the eigenvalues of the Fokker-Planck operator in the weak-noise limit $\varepsilon \to 0$ as

$$\mathbf{l} = 0: \qquad s = {\rm i}\,\frac{2\pi}{T_{\rm p}}\,n - \frac{\varepsilon}{2}\,\frac{|\partial_E T_{\rm p}|}{T_{\rm p}}\,\left(\frac{2\pi n}{T_{\rm p}}\right)^2 + \mathcal{O}(\varepsilon^2) \tag{115}$$

$$\mathbf{l} \neq 0: \qquad s = {\rm i}\,\frac{2\pi}{T_{\rm p}}\,n - \sum_{i=1}^{d-1}\frac{l_i}{T_{\rm p}}\,\ln\frac{1}{\Lambda_{{\rm p},i}^{(s)}} + \mathcal{O}(\varepsilon) \tag{116}$$

with $n \in \mathbb{Z}$ and $l_i = 1, 2, 3, ...$ for all $i = 1, 2, ..., d-1$.

We observe that the leading eigenvalues with $\mathbf{l} = 0$ and $n \in \mathbb{Z}$ have a real part only due to the presence of the noise. Their real part vanishes in the noiseless limit $\varepsilon = 0$, as explained above. In contrast, the non-leading eigenvalues with $\mathbf{l} \neq 0$ (i.e., with at least one integer $l_i \neq 0$) already have a real part due to the deterministic dynamics and the noise only shifts their real part by a correction of order ε. These eigenvalues are deeper in the complex plane of the Laplace variable s and they rule the fast transients of the relaxation of the probability density attracted by the limit cycle. Fig. 4 depicts a typical spectrum of Fokker-Planck eigenvalues for a noisy periodic attractor, as compared with the spectrum of Pollicott-Ruelle resonances for the noiseless deterministic system.

Going back to the leading eigenvalues with $\mathbf{l} = 0$ given by Eq. (115), we observe that the eigenvalue with $n = 0$ vanishes, $s = 0$, as expected. Indeed, this eigenvalue is associated with the invariant probability density of the periodic attractor for which there is no decay and no oscillation. Several works have shown that this invariant probability density has the well-known form of a crater when depicted as a 3D plot over a two-dimensional phase space [11, 46]. The next eigenvalues with $n = \pm 1$ in Eq. (115) control the long-time relaxation toward the invariant probability density:

$$\mathbf{l} = 0, n = \pm 1: \qquad s = \pm\,{\rm i}\,\frac{2\pi}{T_{\rm p}} - \frac{\varepsilon}{2}\,\frac{|\partial_E T_{\rm p}|}{T_{\rm p}}\,\left(\frac{2\pi}{T_{\rm p}}\right)^2 + \mathcal{O}(\varepsilon^2) = \pm\,{\rm i}\,\omega - \frac{1}{\tau} \tag{117}$$

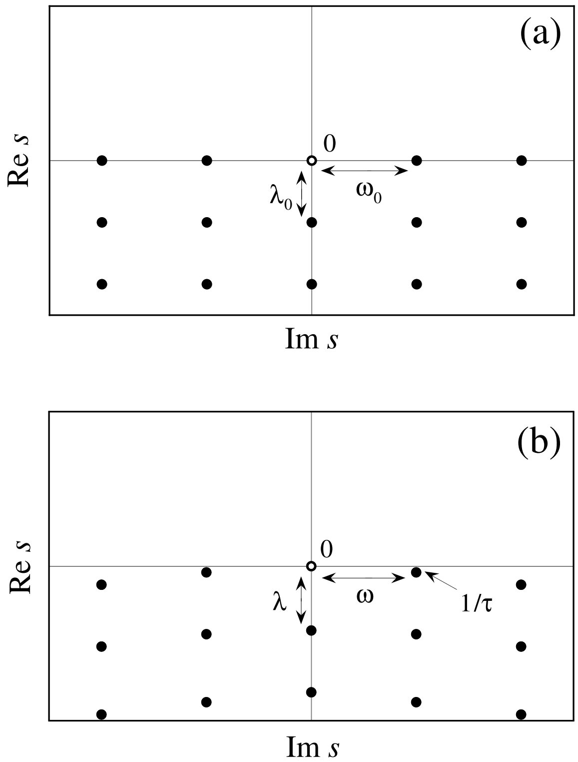

FIG. 4: (a) Spectrum of Pollicott-Ruelle resonances for a noiseless periodic attractor in a two-variable system ($d = 2$). $\lambda_0 = \ln|\Lambda^{(s)}_{\mathrm{p},1}|/T_{\mathrm{p}}$ denotes the non-vanishing Lyapunov exponent and $\omega_0 = 2\pi/T_{\mathrm{p}}$ the pulsation of the limit cycle. (b) Spectrum of the eigenvalues of the Fokker-Planck operator of the periodic attractor in the presence of noise. $s = \lambda = \lambda_0 + \mathcal{O}(\varepsilon)$ is the next-to-leading eigenvalue with $n = 0$ and $l = 1$ in Eq. (116). ω and τ are respectively the pulsation and the lifetime of the leading eigenvalue given by Eq. (117).

The further eigenvalues with $\mathbf{l} = 0$ and $n = \pm 2, \pm 3, ...$ have larger real parts in absolute value and, thus, correspond to faster relaxations. We notice that, for increasing values of $|n|$, the real part decreases quadratically with respect to the imaginary part, which is characteristic of phase diffusion on the limit cycle (see Fig. 4). This phase diffusion causes a damping of the oscillations in the time-correlation functions.

The long-time features of such a time-correlation function can therefore be characterized in terms of the leading eigenvalues (117). This correlation function presents oscillations with the pulsation $\omega = 2\pi/T_{\mathrm{p}}$ and these oscillations are damped at a rate given by the real part of (117). We can conclude that:

Theorem. The correlation time of a noisy oscillator of period T is given in the weak-noise limit $\varepsilon \to 0$ by

$$\tau = \frac{T^3}{2\,\pi^2\,|\partial_E T|\,\varepsilon} + \mathcal{O}(\varepsilon^0) \tag{118}$$

In Eq. (118), $\partial_E T$ denotes the derivative of the period T with respect to the pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme at $E = 0$. Since the pseudo-energy is the product of the noise amplitude by the fixed Laplace variable s according to Eq. (66) we verify that the expression (118) has indeed the physical unit of a time.

In the weak-noise limit $\varepsilon \to 0$, we observe that the correlation time (118) increases without bound as the inverse of the noise amplitude ε. This effect is very important and shows that the oscillations of the noisy oscillator will be more and more correlated if the oscillator is more and more macroscopic with respect to the thermal fluctuations. The quality factor of the noisy oscillator is thus given by

$$\mathcal{Q} \equiv 2\,\pi\,\frac{\tau}{T} \simeq \frac{T^2}{\pi\,|\partial_E T|\,\varepsilon} \tag{119}$$

The oscillator can act as a clock as long as the quality factor is large enough, $\mathcal{Q} > 1$ essentially. For smaller values of the quality factor, the time-correlation function of the oscillator drops after a single period to a value less than 0.2%.

Such a noisy oscillator may still present pulsating oscillations but at so irregular time intervals that the oscillations are no longer correlated. By this reasoning, we obtain the limit

$$\varepsilon < \varepsilon_c = \frac{T^2}{\pi\,|\partial_E T|} \tag{120}$$

on the noise amplitude below which the oscillator remains time correlated.

C. Chaotic attractor

Let us consider a chaotic attractor of axiom-A type for the time-independent macroscopic system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, in which all the periodic orbits are unstable and isolated, and which contains no stationary state. In the presence of noise, the process is supposed to be described by the Itô stochastic differential equation (2). The resolvent of the Fokker-Planck operator (5) can be expressed as the logarithmic derivative

$$\mathrm{tr}\,\frac{1}{s-\hat{L}} \simeq \sum_{\mathrm{p}} \mathrm{tr}\,\frac{1}{s-\hat{L}}\Big|_{\mathrm{p}} = \frac{\partial}{\partial s}\,\ln\, Z(s) \tag{121}$$

of the overall zeta function

$$Z(s) = \prod_{\mathrm{p}} Z_{\mathrm{p}}(s) \tag{122}$$

given in terms of the zeta functions (113) associated with the individual periodic orbits. In the weak-noise limit, the eigenvalues of the Fokker-Planck operator can thus be approximated as the zeroes of the zeta function (122). The zeta function (122) has the form of a product over all the periodic orbits

$$Z(s) = \prod_{\mathrm{p}} \prod_{\mathbf{l},\mathbf{m}=0}^{\infty} (1 - t_{\mathrm{p}\mathbf{lm}}) \tag{123}$$

with

$$t_{\mathrm{p}\mathbf{lm}} = \frac{\exp\left(-T_{\mathrm{p}}s + \frac{\varepsilon}{2}|\partial_E T_{\mathrm{p}}|s^2\right) \prod_{i=1}^{d_s} \Lambda_{\mathrm{p},i}^{(s)\;l_i}}{\prod_{j=1}^{d_u} |\Lambda_{\mathrm{p},j}^{(u)}|\; \Lambda_{\mathrm{p},j}^{(u)\;m_j}} \tag{124}$$

The approximate values of the zeroes of the zeta function can be obtained by first expanding the zeta function into a sum over pseudo-cycles which are distinct non-repeating combinations of prime cycles [45]:

$$Z(s) = \sum_{k=0}^{\infty} (-1)^k \sum_{\mathrm{p}_1\mathbf{l}_1\mathbf{m}_1} \cdots \sum_{\mathrm{p}_k\mathbf{l}_k\mathbf{m}_k} t_{\mathrm{p}_1\mathbf{l}_1\mathbf{m}_1} \cdots t_{\mathrm{p}_k\mathbf{l}_k\mathbf{m}_k} \tag{125}$$

and then by truncating.

In the noiseless limit $\varepsilon = 0$, this zeta function reduces to the known zeta function for classical systems [26] and its zeros are the Pollicott-Ruelle resonances which describe the relaxation of the time-correlation functions under the deterministic dynamics. Assuming that the deterministic system is mixing implies that no Pollicott-Ruelle resonance has a vanishing real part except the one corresponding to the unique ergodic invariant measure. In the presence of noise, the nontrivial Fokker-Planck eigenvalues with $s \neq 0$ are shifted with respect to the Pollicott-Ruelle resonances by a correction of order ε. This correction due to noise can be estimated by substituting into the zeta function (122) an expansion of the eigenvalues in powers of the noise amplitude ε:

$$s(\varepsilon) = s^{(0)} + s^{(1)}\,\varepsilon + \cdots \tag{126}$$

starting from the Pollicott-Ruelle resonance $s^{(0)}$ which satisfies

$$Z(s^{(0)})\Big|_{\varepsilon=0} = 0 \tag{127}$$

Substituting the expansion (126) into the condition $Z(s) = 0$, we get

$$s^{(1)} = -\frac{\partial_\varepsilon Z}{\partial_s Z}\Big|_{s=s^{(0)},\varepsilon=0} \tag{128}$$

which can be expressed in terms of the pseudo-cycle expansion (125) as

$$s^{(1)} = \frac{s^{(0)\,2}}{2} \frac{\sum_{k=0}^{\infty}(-1)^k \sum_{\mathrm{p}_1\mathbf{l}_1\mathbf{m}_1\dots\mathrm{p}_k\mathbf{l}_k\mathbf{m}_k} \left(|\partial_E T_{\mathrm{p}_1}| + \cdots + |\partial_E T_{\mathrm{p}_k}|\right) t_{\mathrm{p}_1\mathbf{l}_1\mathbf{m}_1}\dots t_{\mathrm{p}_k\mathbf{l}_k\mathbf{m}_k}}{\sum_{k=0}^{\infty}(-1)^k \sum_{\mathrm{p}_1\mathbf{l}_1\mathbf{m}_1\dots\mathrm{p}_k\mathbf{l}_k\mathbf{m}_k} \left(T_{\mathrm{p}_1} + \cdots + T_{\mathrm{p}_k}\right) t_{\mathrm{p}_1\mathbf{l}_1\mathbf{m}_1}\dots t_{\mathrm{p}_k\mathbf{l}_k\mathbf{m}_k}} \tag{129}$$

This expression can be rewritten as the ratio of two averages over the conditionally invariant measure associated with the Pollicott-Ruelle resonance $s^{(0)}$ as

$$s^{(1)} = \frac{s^{(0)\,2}}{2} \frac{\langle |\partial_E T| \rangle_{s^{(0)}}}{\langle T \rangle_{s^{(0)}}} \tag{130}$$

Such averages can be evaluated by considering the flow $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ as a suspended flow based on a Poincaré surface of section [29]. $\langle T \rangle_{s^{(0)}}$ is the average of the first return time in the Poincaré surface of section, while $\langle |\partial_E T| \rangle_{s^{(0)}}$ is the average of the derivative of the first return time T with respect to the pseudo-energy at $E = 0$.

The formula (130) would also apply to the escape rate of a noisy fractal repeller. In this case, the noiseless escape rate $\gamma = -s^{(0)}$ is known to be the leading Pollicott-Ruelle resonance and the average $\langle \cdot \rangle_{s^{(0)}}$ is carried out over a statistical ensemble of trajectories which still remain near the repeller after a long time t and taking the limit $t \to \infty$ after averaging [29]. We observe in Eq. (130) that the first-order correction $s^{(1)}$ to the Pollicott-Ruelle resonance $s^{(0)} = -\gamma$ is positive (the noise amplitude ε being positive by definition). Accordingly, the escape rate of a chaotic repeller should be reduced by the presence of noise and the lifetime be lengthened. The positivity of the correction (130) has its origin in the inequality (91), which itself arises from the positivity of the Onsager-Machlup Lagrangian (34). We notice that noise corrections to the escape rate of one-dimensional maps have been studied in Refs. [30–35] in which case the noise can enhance or inhibit the escape [31].

VI. CONCLUSIONS AND PERSPECTIVES

In the present paper, we have studied the effect of a weak noise on time-continuous nonlinear dynamical systems. We have assumed that these noisy systems are described by a Itô stochastic differential equation (2) with a Gaussian white noise or, equivalently, by a Fokker-Planck equation (5). We have moreover supposed that the system is time-independent which has allowed us to study the decay of the time-correlation functions in terms of the eigenvalues of the Fokker-Planck operator. We have here also assumed that the deterministic dynamics does not undergo bifurcations and that the noise is weak. Under these assumptions and using the methods of Onsager-Machlup and Freidlin-Wentzell as well as Gutzwiller's theory, we have been able to derive approximate values for the Fokker-Planck eigenvalues based on the Pollicott-Ruelle resonances of the noiseless macroscopic system. In this way, we have also shown explicitly that the Fokker-Planck eigenvalues converge to the Pollicott-Ruelle resonances in the noiseless limit under the aforementioned assumptions.

In the weak-noise limit, we have thus obtained a trace formula for the evolution operator of the Fokker-Planck operator, as well as a zeta function, which both takes into account of the effect of a weak noise. We consider the cases of stationary, periodic, and chaotic attractors. For a stationary attractor, we have shown that the time-correlation functions decay according to the properties of linear stability of the macroscopic system around the stationary state. For periodic and chaotic attractors, the noise induces a phase diffusion along each periodic orbit, which results into a Gaussian spreading of the recurrences at the repetitions $t = rT_{\mathrm{p}}$ of the period. Our central result is the precise calculation of the effect of the phase diffusion on each periodic orbit. This effect is given in terms of the derivative $\partial_E T_{\mathrm{p}}$ of the period with respect to the pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme. This central result has allowed us to obtain the formula

$$\tau \simeq \frac{T^3}{2\,\pi^2\,|\partial_E T|\,\varepsilon} \tag{131}$$

for the correlation time of a noisy oscillator of period T submitted to a noise amplitude ε. This formula gives the coefficient of proportionality between the correlation time τ and the inverse ε^{-1} of the noise amplitude.

This result has many consequences. In particular, we have obtained in this way an efficient formula (119) for the quality factor of a noisy oscillator. This result applies to many nonequilibrium oscillators (or rotators) in hydrodynamics, nonlinear optics, electronics, or nanomechanics, as well as in biophysics to molecular motors or in nonlinear chemistry to chemical and biochemical clocks, in particular, to the case of circadian rhythms in single cells [47]. The methods developed here also allow us to obtain the effect of noise on the correlation times in chaotic systems and, in particular, on the escape rate of chaotic scattering systems.

Higher-order corrections can be obtained by expanding systematically in powers of the noise amplitude ε and by using semiclassical methods, as already developed for noisy mappings in Refs. [30–35].

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APPENDIX A: METHOD OF CALCULATION OF $\partial T/\partial E$

The correlation time of a noisy oscillator involves the derivative $\partial T/\partial E$ of the period of the periodic orbit with respect to the pseudo-energy at $E = 0$. Therefore, we need to determine the behavior of the periodic orbit in the vicinity of the invariant subspace of zero momenta $\mathbf{p} = 0$. Since we only need the first derivative of the period with respect to the pseudo-energy we can use the linear stability properties determined by integrating the macroscopic nonlinear system (39) together with the linearized equations (93). The solution of these linearized equations is given by Eq. (96).

The perturbed periodic orbit at pseudo-energy δE has the period $T + \delta T$. The conditions of periodicity are thus $\mathbf{x}(T + \delta T) = \mathbf{x}(0)$ and $\mathbf{p}(T + \delta T) = \mathbf{p}(0)$. Expanding in terms of the perturbations with respect to the unperturbed periodic orbit, we get

$$\begin{cases} \dot{\mathbf{x}}(T)\ \delta T + \mathsf{M}(T) \cdot \delta\mathbf{x}(0) + \mathsf{N}(T) \cdot \delta\mathbf{p}(0) = \delta\mathbf{x}(0) \\ \dot{\mathbf{p}}(T)\ \delta T + \mathsf{M}(T)^{-1\,\mathrm{T}} \cdot \delta\mathbf{p}(0) = \delta\mathbf{p}(0) \end{cases} \tag{A1}$$

For a periodic orbit of the invariant subspace, we also have that $\dot{\mathbf{x}}(T) = \dot{\mathbf{x}}(0) = \mathbf{F}\left[\mathbf{x}(0)\right]$ and $\dot{\mathbf{p}}(T) = \dot{\mathbf{p}}(0) = 0$. Moreover, the perturbed periodic orbit has the pseudo-energy

$$H \simeq \mathbf{F}\left[\mathbf{x}(0)\right]^{\mathrm{T}} \cdot \delta\mathbf{p}(0) = \delta E \tag{A2}$$

The conditions that the perturbations of the initial conditions have to satisfy are therefore

$$\mathbf{F}\,\delta T + \mathsf{M} \cdot \delta\mathbf{x}(0) + \mathsf{N} \cdot \delta\mathbf{p}(0) = \delta\mathbf{x}(0) \tag{A3}$$

$$\mathsf{M}^{-1\,\mathrm{T}} \cdot \delta\mathbf{p}(0) = \delta\mathbf{p}(0) \tag{A4}$$

$$\mathbf{F}^{\mathrm{T}} \cdot \delta\mathbf{p}(0) = \delta E \tag{A5}$$

where we have dropped the dependence on the period T to simplify the notations.

However, the matrix M has the properties that

$$\mathbf{F} = \mathsf{M} \cdot \mathbf{F} \qquad \text{and} \qquad \mathbf{F}^{\mathrm{T}} \cdot \mathsf{M}^{-1\,\mathrm{T}} = \mathbf{F}^{\mathrm{T}} \tag{A6}$$

so that the system of Eqs. (A3)-(A5) is degenerate. To cure this problem, we can delete a line in Eq. (A4) and set $\delta x^i(0) = 0$ in Eq. (A3) for a specific component (e.g. taking the plane $x^i(0) = a$ with a constant a as a Poincaré surface of section).

The fundamental matrix M of the unperturbed periodic orbit is supposed to have the following spectral decomposition, which is the generic case:

$$\mathsf{M} = \sum_{k=1}^{d} \mathbf{e}_k \Lambda_k \mathbf{f}_k^{\mathrm{T}} \qquad \text{with} \qquad \mathbf{f}_k^{\mathrm{T}} \cdot \mathbf{e}_l = \delta_{kl} \tag{A7}$$

where Λ_k are the stability eigenvalues of the periodic orbit and $\lambda_k = \ln|\Lambda_k|/T$ its Lyapunov exponents. The Lyapunov exponent corresponding to the direction of the flow always vanishes so that $\lambda_1 = 0$, $\Lambda_1 = 1$, and we can take $\mathbf{e}_1 = \mathbf{F} = \dot{\mathbf{x}}$. We have the following expansions for the initial perturbations:

$$\delta\mathbf{x}(0) = \sum_k \delta\xi_k\ \mathbf{e}_k \qquad \text{and} \qquad \delta\mathbf{p}(0) = \sum_k \delta\pi_k\ \mathbf{f}_k \tag{A8}$$

The equations for $\delta\mathbf{p}(0)$ are (A4) and (A5) from which we get that $\delta\pi_1 = \delta E$ and $\delta\pi_k = 0$ if $\Lambda_k \neq 1$ so that the perturbation on the momenta is in the direction adjoint to the direction of the flow:

$$\delta\mathbf{p}(0) = \delta E\ \mathbf{f}_1 \tag{A9}$$

Substituting the expansions (A8) in Eq. (A3), we obtain

$$\delta T \; \mathbf{e}_1 + \sum_k \Lambda_k \; \delta\xi_k \; \mathbf{e}_k + \delta E \; \mathsf{N} \cdot \mathbf{f}_1 = \sum_k \delta\xi_k \; \mathbf{e}_k \tag{A10}$$

Taking the scalar product with $\mathbf{f}_1$, we get

$$\frac{\partial T}{\partial E} = -\frac{\mathbf{f}_1^{\rm T} \cdot \mathsf{N}(T) \cdot \mathbf{f}_1}{\mathbf{f}_1^{\rm T} \cdot \mathbf{e}_1} \tag{A11}$$

where

$$\mathsf{N}(T) = 2 \; \mathsf{M}(T) \cdot \int_0^T \mathsf{M}(\tau)^{-1} \cdot \mathsf{Q}\left[\mathbf{x}(\tau)\right] \cdot \mathsf{M}(\tau)^{-1 \; \rm T} \; d\tau \tag{A12}$$

according to Eq. (94). On the other hand, Eq. (96) shows that $\delta\tilde{\mathbf{x}}(T) = \mathsf{N}(T) \cdot \mathbf{f}_1$ is solution of the linearized system (93) from the special initial conditions $\delta\tilde{\mathbf{x}}(0) = 0$ and $\delta\tilde{\mathbf{p}}(0) = \mathbf{f}_1$. Therefore, we can obtain the derivative by knowing this special solution $\delta\tilde{\mathbf{x}}(T)$ of Eq. (93), together with the right- and left-eigenvectors of the fundamental matrix $\mathsf{M}(T)$ corresponding to the unit eigenvalue:

$$\frac{\partial T}{\partial E} = -\frac{\mathbf{f}_1^{\rm T} \cdot \delta\tilde{\mathbf{x}}(T)}{\mathbf{f}_1^{\rm T} \cdot \mathbf{e}_1} \tag{A13}$$

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