Physics Letters A **168** (1992) 13-17

Diffusion in uniformly hyperbolic one-dimensional maps and Appell polynomials

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Abstract

We construct the eigenpolynomials and the eigendistributions associated with Ruelle's resonances in a piecewise-linear one-dimensional map model of deterministic diffusion which is uniformly hyperbolic. We show that the eigenpolynomials belong to the class of Appell polynomials and that the eigendistributions are given by series of derivatives of the Dirac distribution. The expansion on the eigenpolynomials is shown to converge for initial density which are entire functions of exponential type.

PACS numbers: 02.30; 02.50; 05.45.

1. Introduction. We have shown in Ref. 1 that the spectrum of Ruelle's resonances [2] in simple dynamical models of deterministic diffusion approaches the spectrum of the phenomenological diffusion equation. This fundamental result shows that a mechanical evolution law and transport properties may be more directly related in chaotic dynamical systems than previously thought. Several recent works have extended our results to the study of the eigenstates associated with Ruelle's resonances in uniformly hyperbolic systems in order to obtain an explicit decomposition of the time evolution in terms of the decaying eigenmodes [3-7].

In this Letter, we would like to show that this problem can be rigourously solved in the case of diffusion thanks to several results on the polynomial expansion of analytic functions [8]. Accordingly, we obtain the left eigendistributions which are associated with Ruelle's resonances and their right eigenpolynomials. Furthermore, the functional space of test functions in which the distributions are defined is completely characterized. In the following, we shall follow the approach of Ref. 8 which differs from Ref. 7 in its departure from the use of orthonormal basis.

2. The 1D map and the evolution operators. We are concerned with the uniformly hyperbolic piecewise linear map of the real line

$$\phi(x) \;=\; \begin{cases} 4x \;-\; 3k \;-1 \;, & k \;\le\; x \;<\; k+1/2 \;, \\ 4x \;-\; 3k \;-2 \;, & k+1/2 \;\le\; x \;<\; k+1 \;, \end{cases} \tag{1}$$

for all integer k. Similar maps have also been considered in the context of deterministic diffusion [9]. The particular map (1) arises in the study of the area-preserving multibaker map introduced in Ref. 1. It is known that this map generates a diffusive motion with a diffusion coefficient $\mathcal{D} = 1/4$. If we used the phenomenological diffusion equation, we would obtain the dispersion relation $\tilde{\gamma} = \mathcal{D}q^2$ where $\tilde{\gamma}$ is the decay rate of the diffusion eigenmode of wavenumber q. We may expect that such a relation is valid in the limit of small wavenumbers q but corrections of higher degrees in q are ignored in this limit. In the present Letter, we shall show that it is possible to derive rigourously the exact dispersion relation and, moreover, to carry out the decomposition into the diffusion eigenmodes of Eq. (1).

We introduce the Frobenius-Perron operator associated with the map (1)

$$(\hat{P}f)(x) \;=\; \sum_{x=\phi(y)} \frac{f(y)}{|\phi'(y)|}$$

$$= \frac{1}{4}\left[f\left(\frac{x-1}{4}\right) + f\left(\frac{x+1}{4}\right) + f\left(\frac{x+2}{4}\right) + f\left(\frac{x+4}{4}\right)\right] \qquad \text{for} \quad 0 < x < 1 \; , \tag{2}$$

acting on functions $f(x)$ on the defined on the real line. Because of the translational invariance $\hat{T}\hat{P} = \hat{P}\hat{T}$ with $\hat{T}f(x) = f(x+1)$, the motion along the real line can be decomposed into a reduced map $\phi_r(x)$ describing the motion in a single lattice cell accompanying the particle combined with discrete jumps from one lattice cell to the next. For the map (1), the reduced map $\phi_r(x)$ is the 4-adic map of the unit interval.

On the other hand, translational invariance also implies a decomposition of the functional space $\{f(x)\}$ into the eigenspaces $\mathcal{E}_q$ of the translation operator: $\hat{T}f_q = \exp(iq)f_q$ for wavenumber q. The projectors on the eigenspaces $\mathcal{E}_q$ are

$$\hat{E}_q \;=\; \sum_{n=-\infty}^{+\infty} e^{-iqn}\; \hat{T}^n \; , \tag{3}$$

and they satisfy the relations

$$\hat{E}_q\; \hat{E}_{q'} \;=\; 2\pi\; \delta(q-q')\; \hat{E}_q \qquad \text{and} \qquad \int_{\pi}^{+\pi} dq\; \hat{E}_q \;=\; \hat{I} \; . \tag{4}$$

In the eigenspace $\mathcal{E}_q$, the Frobenius-Perron operator $\hat{P}$ reads [7]

$$(\hat{P}f)(x) \;=\; \frac{1}{4}\left[e^{+iq} f\left(\frac{x}{4}\right) + f\left(\frac{x+1}{4}\right) + f\left(\frac{x+2}{4}\right) + e^{-iq}\, f\left(\frac{x+3}{4}\right)\right] . \tag{5}$$

In Ref. 7, it is observed that $\hat{P}$ is the square of the operator

$$(\hat{Q}f)(x) \;=\; \frac{1}{2}\left[e^{+iq/2} f\left(\frac{x}{2}\right) + e^{-iq/2}\, f\left(\frac{x+1}{2}\right)\right] . \tag{6}$$

When $q = 0$, Eqs. (5) and (6) are respectively the Frobenius-Perron operators of the 4-adic and dyadic maps, the eigenvalue problem of which was solved in Refs. 3 and 6.

3. The eigenvalues. We are looking after eigenpolynomials for $\hat{Q}$ such that

$$\hat{Q}\ \varphi_n \ = \ \chi_n\ \varphi_n\ . \tag{7}$$

Then $\hat{P}$ will have the same eigenpolynomials but with the eigenvalues

$$\hat{P}\ \varphi_n \ = \ \lambda_n\ \varphi_n\ , \qquad \text{with} \qquad \lambda_n \ = \ \chi_n^2\ , \tag{8}$$

since $\hat{P} = \hat{Q}^2$. A basis set for the eigenpolynomials is formed by the monomials $\{x^k\}$. In this basis, the Frobenius-Perron operator is a triangular matrix where the eigenvalues

$$\chi_n \ = \ \frac{1}{2^n}\ \cos(q/2) \qquad n \ = \ 0, 1, 2, \ldots \tag{9}$$

are found on the diagonal.

4. The eigenpolynomials and their generating function. The first few eigenpolynomials are given in Ref. 7. We shall here calculate the generating function of all the eigenpolynomials defined like

$$G(x,t) \ = \ \sum_{n=0}^{\infty}\ \varphi_n(x)\ t^n\ . \tag{10}$$

Because of the eigenvalue equation (7), G must satisfy

$$(\hat{Q}G)(x,t) \ = \ \cos(q/2)\ G(x,t/2)\ . \tag{11}$$

Using the Frobenius-Perron operator (6), we observe that the generating function has an exponential dependence on its first argument x and, moreover, that

$$G(x,t) \ = \frac{\exp(xt)}{C(t)}\ . \tag{12}$$

The function $C(t)$ must satisfy

$$C(2t) \ = \ \frac{\exp(t) + \exp(iq)}{1 + \exp(iq)}\ C(t)\ . \tag{13}$$

When $q = 0$, we recover the case of the r-adic map where $C(t) = [\exp(t) - 1]/t$ and where the eigenpolynomials are the Bernoulli polynomials [6, 8, 10]. The function $C(t)$ can be resolved into a Taylor series

$$C(t) \;=\sum_{n=0}^{\infty} c_n\; t^n \qquad c_0 \;=\; 1\;. \tag{14}$$

Inserting in Eq. (13), we obtain the recurrence formula

$$c_n \;=\; \frac{\mu}{2^n \;-\; 1}\;\sum_{k=1}^{n}\frac{c_{n-k}}{k!} \qquad c_0 \;=\; 1 \qquad \text{where} \qquad \mu \;\equiv\; (1+e^{iq})^{-1}\;. \tag{15}$$

We can then show by recurrence that the following inequality holds

$$|c_n| \;\le\; \frac{(|\mu|+1)^n}{n!} \qquad \text{for} \qquad n \ge 0\;, \tag{16}$$

where $|\mu| \;=\; [2|\cos(q/2)|]^{-1}$. Therefore, the function $C(t)$ is of exponential type itself without pole at finite distance if $q \neq (2\ell\pi+\pi)$

$$|C(t)| \;\le\; \exp\Bigg[(|\mu|+1)|t|\Bigg]\;. \tag{17}$$

Moreover, the nonvanishing of c_0 in the Taylor series (14)-(15) shows that $C(t)$ is not vanishing in the vicinity of $t=0$. Using (13), the function $C(t)$ can then be written as an infinite product

$$C(t) \;=\; \prod_{n=1}^{\infty}\frac{\exp(iq)+\exp\left(\frac{t}{2^n}\right)}{\exp(iq)+1}\;, \tag{18}$$

which shows that the zeros of $C(t)$ are

$$t_{n,\ell} \;=\; i\;2^n\;[q+\pi(2\ell+1)] \qquad n \;\in\; \mathbf{N}_0\;, \quad \ell \;\in\; \mathbf{Z}\;. \tag{19}$$

The first zero appears at a radius

$$\rho_q \;=\; 2\left[(\pi-|\mathrm{Re}\;q|)^2 \;+\; |\mathrm{Im}\;q|^2\right]^{1/2}\;. \tag{20}$$

Let us remark that this radius decreases to zero when $q \to \pm\pi$, which is related to the fact that the eigenfunctions of $\hat{Q}$ corresponding to the eigenvalue $\chi=0$ are constant in this limit.

5. The null spaces. For an arbitrary wavenumber q, the Frobenius-Perron operator (6) admits zero as eigenvalue. The corresponding eigenfunctions are $\exp(xt_{1,\ell})$ $(\ell \in \mathbf{Z})$. We can then define the null spaces $\mathcal{N}_q^{(n)}$ generated by linear combinaison of the functions $\exp(xt_{n,\ell})$ with $\ell \in \mathbf{Z}$. They are mapped successively onto each others under the Frobenius-Perron operator (6) according to [11]

$$0 \overset{\hat{Q}}{\longleftarrow} \mathcal{N}_q^{(1)} \overset{\hat{Q}}{\longleftarrow} \mathcal{N}_q^{(2)} \overset{\hat{Q}}{\longleftarrow} \mathcal{N}_q^{(3)} \overset{\hat{Q}}{\longleftarrow} \mathcal{N}_q^{(4)} \cdots \tag{21}$$

We shall see hereafter that the null spaces form the border of the functional space on which the expansion into the eigenpolynomials is convergent.

6. The eigendistributions. The preceding results imply that the eigenpolynomials fall into the class of Appell polynomials which are defined in the context of polynomial expansions of analytic functions [8]. We have the

Theorem[12]: *If $\varphi_n(x)$ are the Appell polynomials defined by (10) and (12), if $C(t)$ has no zero in the region Ω of the complex plane t, and if Δ is the largest circular disk, with center at 0, in Ω, and Δ has radius ρ_q, then every entire function of exponential type $\tau < \rho_q$ has the representation*

$$f(x) = \sum_{n=0}^{\infty} \mathcal{L}_n(f)\ \varphi_n(x)\ , \tag{22}$$

where

$$\mathcal{L}_n(f) = \frac{1}{2\pi i} \int_\Gamma t^n\ C(t)\ F(t)\ dt\ , \tag{23}$$

$$F(t) \equiv \int_0^\infty e^{-tx}\ f(x)\ dx\ , \qquad f(x) = \frac{1}{2\pi i} \int_\Gamma e^{xt}\ F(t)\ dt\ , \tag{24}$$

and Γ is a circumference $|t| = \sigma$ with $\tau < \sigma < \rho_q$ on which $C(t)$ is regular.

This theorem provides us with the decomposition of the initial density $f(x)$ into the decaying eigenmodes of the dynamical system. Since $f(x)$ and $F(t)$ are related by a Laplace transform (24) we have

$$f^{(n)}(0) = \frac{1}{2\pi i} \int_\Gamma t^n\ F(t)\ dt\ , \tag{25}$$

so that the coefficients $\mathcal{L}_n(f)$ of the expansion are given by

$$\mathcal{L}_n(f) \;=\; \mathcal{L}_0(f^{(n)}) \;=\; \sum_{k=n}^{\infty} c_{k-n}\; f^{(k)}(0) \;=\; \int_{-\infty}^{+\infty} L_n(x)\; f(x)\; dx \;, \tag{26}$$

with the distributions

$$L_n(x) \;\equiv\; \sum_{k=n}^{\infty} c_{k-n}\; (-)^k \delta^{(k)}(x) \;, \tag{27}$$

where $f^{(n)}(x)$ and $\delta^{(n)}(x)$ denote respectively the n^{th} derivatives of the function $f(x)$ and of the Dirac distribution $\delta(x)$. We infer that, for every entire function $f(x)$ of exponential type τ such that

$$|f^{(n)}(0)| \;\leq\; \kappa_\epsilon\; (\tau \;+\; \epsilon)^n \qquad \forall\, \epsilon > 0 \;, \tag{28}$$

the value of the distribution $\mathcal{L}_n$ is finite and bounded like

$$|\mathcal{L}_n(f)| \;\leq\; \kappa_\epsilon\; (\tau \;+\; \epsilon)^n\; \exp\Big[(|\mu| + 1)(\tau + \epsilon)\Big] \;. \tag{29}$$

Eq. (20) gives the radius ρ_q of the disk Δ in which the integration contour Γ must be considered. We conclude that the functions $f(x)$ for which the expansion (22) converges are entire functions of exponential type $\tau < \rho_q$ forming a Fréchet functional space $\mathcal{E}_q(\tau)$ [6]. For $q = 0$ and $\mu = 1/2$, Eq. (20) gives the critical radius $\rho_q = 2\pi$ and we recover the results known about the Bernoulli polynomials and the Euler summation formula for which $\tau < 2\pi$ [6, 8, 10]. For $q = \pm\pi$, zero is the only eigenvalue and an expansion is possible using the null spaces [8].

7. Discussion and conclusions. In this Letter, we have solved the eigenvalue problem for the Frobenius-Perron operator of the 1D map model of deterministic diffusion. According to Eqs. (4), (8), and (22), the density at time $t(\in \mathbf{Z})$ can be written like

$$(\hat{P}^t f)(x) \;= \int_{-\pi}^{+\pi} dq\; e^{-iq[x]} \sum_{n=0}^{\infty} \lambda_{nq}^t\; \mathcal{L}_{nq}(\hat{E}_q f)\; \varphi_{nq}(x - [x]) \;. \tag{30}$$

For a given q, the series converge under weaker conditions for large t since the eigenvalues λ_{nq} decreases exponentially with n. It is then enough to require that $\hat{E}_q f$ is of exponential type $\tau < 4^t \rho_q$. However, the integral over q is convergent at the end points $\pm\pi$ only for t large enough and for a restricted class of the class of functions f(x) which are polynomials on intervals $[m, n]$ with integers m and n but which are zero outside this interval.

The long time dynamics is dominated by the slowest among the eigenvalues (8) with (9) i. e.

$$\gamma \;=\; - \;\ln\;\chi_0 \;=\; -2\;\ln\;\cos\frac{q}{2} \;=\; \frac{q^2}{4} \;+\; \frac{q^4}{96} \;+\;\cdots \tag{31}$$

which should be compared with the approximate diffusion dispersion relation mentioned in the introduction. In (31), we find not only the diffusion coefficient $\mathcal{D} = 1/4$ but also the Burnett coefficient $\mathcal{B} = -1/96$, as well as higher order diffusion coefficients.

The decaying eigenmodes that govern the diffusion process on long times and large spatial scales are the modes associated with this eigenvalue χ_0. One such mode exists in each eigenspace of given wavenumber q. The corresponding eigenpolynomial is always the constant function and the expected wavenumber dependence is removed in the corresponding eigendistribution $\mathcal{L}_0$ defined by (26)-(27) with $n = 0$. This eigendistribution is expressed in Ref. 7 as $\lim_{n\to\infty} \hat{U}^n g_q$ for the Koopman operator corresponding to the map (1). This limit does not exist as a function as soon as $q \neq 0$ since the image of any smooth function g_q under the Koopman operator presents several discontinuities. This problem is reminiscent of the famous problem of statistical mechanics where each nonequilibrium density cannot converge to the equilibrium density in the sense of pointlike convergence in phase space. We have here shown that it is possible to define this limiting eigendistribution as the series (26) on the appropriate space of test functions which is $\mathcal{E}_q(\tau)$ $(\tau < \rho_q)$.

Acknowledgements. The authors thanks Prof. G. Nicolis for support and encouragements. P. G. is “Chercheur Qualifié” at the National Fund for Scientific Research (Belgium).

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