Physical Review E **68** (2003) 056209 (*7 pages*)

Lyapunov exponent of ion motion in microplasmas

Pierre Gaspard
Center for Nonlinear Phenomena and Complex Systems
Université Libre de Bruxelles, Code Postal 231, Campus Plaine, B-1050 Brussels, Belgium

Dynamical chaos is studied in the Hamiltonian motion of ions confined in a Penning trap and forming so-called microplasmas. The dynamical chaos of the ion motion is characterized by the maximum Lyapunov exponent. Results are reported on the dependence of this exponent on the energy of the system, on the number of ions, as well as on the geometry of the trap. Different dynamical regimes are characterized from the crystaline state, to a strongly chaotic regime, and to quasiharmonic motion in the external potential of the trap. Across these regimes, the Lyapunov exponent increases, reaches a maximum value, and decreases as a function of energy. Besides, the maximum value of the Lyapunov exponent increases as a function of the number of ions.

PACS numbers: 05.20.-y; 05.45.-a; 05.45.Jn; 45.50.Jf

I. INTRODUCTION

Dynamical chaos provides a strong mechanism for the mixing of phase-space volumes and the loss of statistical correlations during time evolution, these latter being central properties in nonequilibrium statistical mechanism. Chaos is characterized by positive Lyapunov exponents which measure the rate of exponential separation between nearby trajectories [1]. A positive Lyapunov exponent is thus the signature of an exponential type of sensitivity to initial conditions. Several systems of statistical mechanics have been shown to present a full spectrum of positive Lyapunov exponents [2]. In particular, the characteristic quantities of chaos have been obtained for the hard-ball fluids thanks to the methods of statistical mechanics [3–5]. In those fluids, explicit values have been computed for the maximum Lyapunov exponent as well as for the Kolmogorov-Sinai entropy per unit time which is equal to the sum of positive Lyapunov exponents λ_j according to Pesin's theorem [1]

$$h_{\rm KS} = \sum_{\lambda_j > 0} \lambda_j \; , \tag{1}$$

which holds for bounded hyperbolic systems and is expected to apply also to more general bounded dynamical systems. As a consequence, the space-time entropy of the hard-ball fluids, i.e., the Kolmogorov-Sinai entropy per unit time and unit volume, turns out to be positive [3]. Moreover, numerical simulations show that the dynamics of typical systems of particles interacting by smooth potentials is also exponentially sensitive to initial conditions with positive Lyapunov exponents and a positive space-time entropy [6–11].

A remarkable fact is that a positive space-time entropy is the feature of typical stochastic processes of statistical mechanics. This is the case for the probabilistic cellular automata [12], as well as for the fluctuating Boltzmann equation [13] which can be simulated by Bird's direct simulation Monte Carlo method [14]. Such stochastic processes require the call of a pseudorandom generator at each time step in each space cell. This local stochasticity is characterized by a positive space-time entropy [15]. This dynamical entropy measures the amount of randomness which is generated by the process in each unit volume and unit time. Dynamical chaos is thus a mechanism which is compatible with both the positive space-time entropy of the typical stochastic processes of statistical mechanics and the determinism of the underlying microscopic dynamics.

Recent work has also drawn the attention to systems with zero Lyapunov exponents, presenting weaker mechanisms of mixing and sensitivity to initial conditions than dynamical chaos [16–19]. Such systems also sustain transport processes such as diffusion as well as some stochasticity.

A fundamental question is to know whether natural many-particle systems present strong or weak kinds of sensitivity to initial conditions. This question can be addressed not only by theoretical or numerical methods but also experimentally. The experimental determination of the type of sensitivity to initial conditions would require the direct observation of the trajectories of the particles and the measurement of Lyapunov exponents thanks to the Eckmann-Ruelle method [1]. During the last decade, the tracking of microscopic particles has become an experimental possibility in the so-called microplasmas, which are systems of atomic ions or charged microparticles confined in Paul or Penning traps [20–25]. The ions repel each other by Coulomb interaction and they are separated by distances of the order of micrometers or more, which allows the observation and tracking of their trajectory.

The purpose of the present paper is to anticipate possible future experimental measure of Lyapunov exponents in microplasmas with a theoretical and numerical study of these systems. Lyapunov exponents have already been investigated in spatially-extended plasmas [26–28] as well as in a one-dimensional wave-particle model of plasma [29, 30]. However, no work seems to have been devoted to the Lyapunov exponents of microplasmas confined in an electromagnetic trap. We here focus on microplasmas in Penning traps and investigate the behavior of the maximum Lyapunov exponent of the Hamiltonian motion of the ions. We consider a realistic Hamiltonian model which is three dimensional and contains a relatively small number of ions for which the tracking of trajectories of individual ions should remain feasible. Because of the long range of the Coulomb interaction between the ions, the Lyapunov exponents are expected to behave differently in microplasmas than in many-particle systems with short-ranged interaction such as the hard-ball fluids. Moreover, microplasmas are non-extensive dynamical systems with a finite number of degrees of freedom. In microplasmas composed of more than a few dozen ions, the behavior of the maximum Lyapunov exponent can nevertheless be understood thanks to statistical mechanics, as shown here below.

The plan of the paper is the following. The system and its Hamiltonian are defined in Sec. II. Its dynamical and statistical properties are described in Sec. III. Its sensitivity to initial conditions is characterized by the maximum Lyapunov exponent in Sec. IV. Conclusions are drawn in Sec. V.

II. THE SYSTEM

We consider a microplasma composed of N ions of mass m and electric charge q in a Penning trap with the electrostatic potential

$$\Phi(x,y,z) = V_0\,\frac{2z^2-x^2-y^2}{r_0^2+2z_0^2}\ , \tag{2}$$

and the magnetic field along the z-direction of vector potential

$$\mathbf{A}(x,y,z) = \frac{1}{2}(-By, Bx, 0)\ . \tag{3}$$

The Hamiltonian of the full system is given by

$$\mathcal{H} = \sum_{i=1}^{N}\left\{\frac{1}{2m}\left[\mathbf{p}_i - q\mathbf{A}(\mathbf{r}_i)\right]^2 + q\Phi(\mathbf{r}_i)\right\} + \sum_{1\leq i<j\leq N}\frac{q^2}{4\pi\varepsilon_0 r_{ij}}\ , \tag{4}$$

with $\mathbf{r}_i = (x_i, y_i, z_i)$ the position of the ion i, r_{ij} the distance between the ions i and j, and ε_0 the vacuum permitivity. In the Penning trap, the ions are submitted to a harmonic confinement in the z-direction of frequency

$$\omega_z = \sqrt{\frac{4qV_0}{m(r_0^2+2z_0^2)}}\ , \tag{5}$$

and in the perpendicular direction due to the cyclotron motion of frequency $\omega_{\rm c} = qB/m$. In a frame rotating around the z-axis at the Larmor frequency $\omega_{\rm L} = \omega_{\rm c}/2$, the ions feel a harmonic confinement of frequency

$$\omega_x = \omega_y = \sqrt{\frac{\omega_{\rm c}^2}{4} - \frac{\omega_z^2}{2}}\ , \tag{6}$$

in the direction perpendicular to the magnetic field. In the rescaled time $\tau = \omega_{\rm c} t$, position $\mathbf{R} = \mathbf{r}/a$, and energy $H = \mathcal{H}/(m\omega_{\rm c}^2 a^2)$ with $a = \left[q^2/(4\pi\varepsilon_0 m\omega_{\rm c}^2)\right]^{1/3}$, the Hamiltonian describing the motion in the Larmor rotating frame becomes

$$H = \sum_i\left[\frac{1}{2}\,\mathbf{P}_i^2 + \left(\frac{1}{8}-\frac{\gamma^2}{4}\right)\left(X_i^2+Y_i^2\right) + \frac{\gamma^2}{2}\,Z_i^2\right] + \sum_{i<j}\frac{1}{R_{ij}}\ , \tag{7}$$

with the positions $\mathbf{R}_i = (X_i, Y_i, Z_i)$, the canonically conjugated momenta $\mathbf{P}_i = (P_{Xi}, P_{Yi}, P_{Zi})$, and

$$\gamma = \frac{\omega_z}{\omega_{\rm c}}\ . \tag{8}$$

The ions are trapped in a bounded motion under the condition that

$$0 < |\gamma| < \frac{1}{\sqrt{2}} \, . \tag{9}$$

The trap is prolate if $0 < |\gamma| < 1/\sqrt{6}$, isotropic if $|\gamma| = 1/\sqrt{6}$, and oblate if $1/\sqrt{6} < |\gamma| < 1/\sqrt{2}$. The motion is quasi one-dimensional in the limit $\gamma \simeq 0$ and quasi two-dimensional in the limit $|\gamma| \simeq 1/\sqrt{2}$. The Z-direction is a symmetry axis so that the Z-component of angular momentum $L_Z = \sum_i (X_i P_{Yi} - Y_i P_{Xi})$ is conserved. We suppose from now on that this angular momentum vanishes, $L_Z = 0$.

III. DYNAMICAL AND STATISTICAL PROPERTIES

In the following, the motion is studied in the Larmor rotating frame. Figure 1 depicts typical trajectories of a system of $N = 20$ ions in an oblate trap in the XY-plane perpendicular to the Z-direction of the magnetic field. The ions have a periodic micromotion at frequency γ in the Z-direction which is not apparent in the XY-plane where the non-trivial motion is observed.

At zero temperature or kinetic energy, the system freezes in a crystaline state [20–25]. In the present case, the ion crystal is seen in Fig. 1a. It is composed of a central ion surrounded by an inner ring of 7 ions, and an outer ring of 12 ions. At nearly zero temperature, the ions have a quasiperiodic, quasiharmonic motion around their equilibrium position. It is the regime of normal modes of vibration.

At slightly positive temperatures, the system may already have bifurcated from the regime of quasiharmonic normal modes to another regime where local modes exist. This is already the case in Fig. 1a where we observe that the inner ring is animated by a slow collective motion, or soft mode, while the outer ring has a configuration which is essentially fixed. For this to occur the energy should be higher than the energy barriers for locking the rotation of the inner ring with respect to the outer ring. Such energy barriers are still very small with respect to barriers for exchange of ions within a ring or between rings as observed in Fig. 1b.

At temperatures high enough for exchanges of ions, their motion becomes erratic and the ion crystal melts, as seen in Figs. 1b and 1c.

At still higher temperature, the ions form a thermal cloud in which the mean Coulomb potential energy $\langle V_{\rm C} \rangle$ starts to become negligible with respect to the mean kinetic energy $\langle K \rangle$ and mean harmonic potential energy $\langle V_{\rm h} \rangle$. Indeed, the total energy $E = \langle K \rangle + \langle V_{\rm h} \rangle + \langle V_{\rm C} \rangle$ is then essentially shared between the kinetic energy and the harmonic potential energy:

$$\langle K \rangle \simeq \langle V_{\rm h} \rangle \simeq \frac{E}{2} \simeq \frac{3}{2} N T \, , \tag{10}$$

with a dimensionless temperature defined by $T \equiv k_{\rm B} \mathcal{T} / (m \omega_{\rm c}^2 a^2)$. This is the thermal regime which can be described by statistical mechanics as explained in Appendix A. The mean square position of the ions is then given by

$$\langle X_i^2 \rangle = \langle Y_i^2 \rangle \simeq \frac{4T}{1 - 2\gamma^2} \, , \quad \text{and} \quad \langle Z_i^2 \rangle \simeq \frac{T}{\gamma^2} \, . \tag{11}$$

The mean Coulomb energy thus decreases at high temperature as

$$\langle V_{\rm C} \rangle \sim \left\langle \frac{N^2}{R_{ij}} \right\rangle \sim \frac{N^2}{T^{1/2}} \, . \tag{12}$$

The thermodynamic entropy of the ion thermal cloud can also be estimated by neglecting the Coulomb interaction to get

$$\mathcal{S} \simeq k_{\rm B} N \, \ln \frac{{\rm e}^4 (k_{\rm B} \mathcal{T})^3}{N \hbar\omega_x \hbar\omega_y \hbar\omega_z} \, , \tag{13}$$

with ${\rm e} = \exp(1)$. This entropy is an increasing function of the temperature, indicating that the spatial distribution of the ions is more and more disordered as the temperature increases.

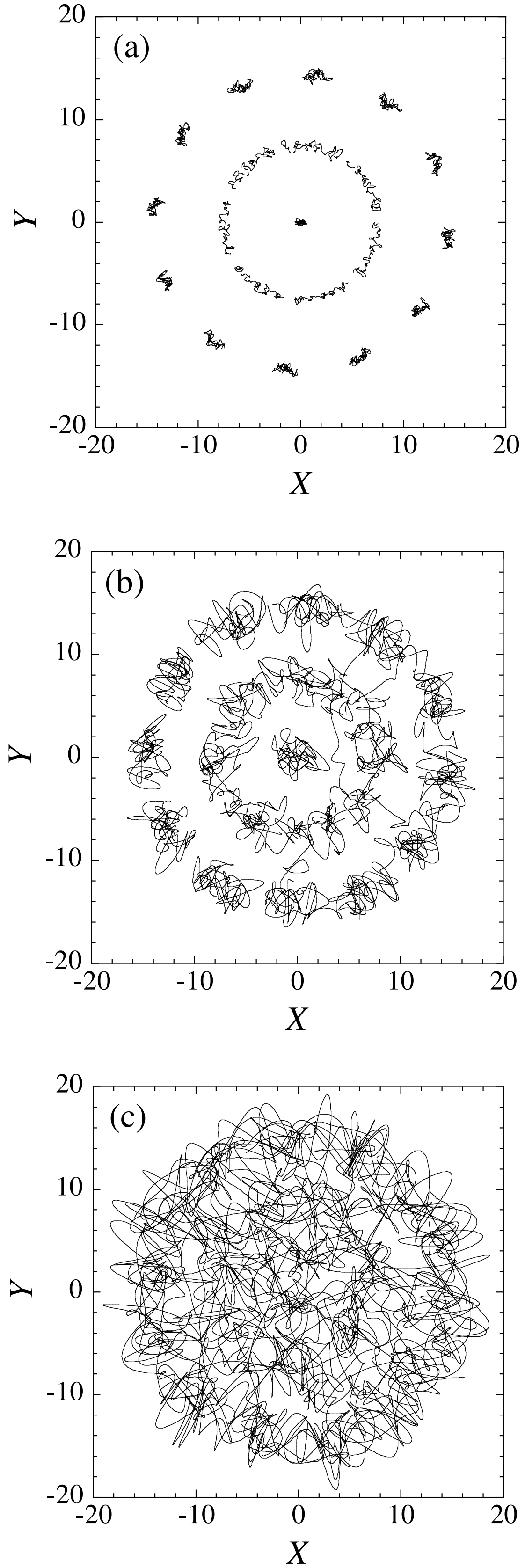

FIG. 1: Trajectories of a system of 20 ions with $L_Z = 0$ in an oblate Penning trap with $\gamma = 0.7$ at total energies: (a) $E = 21.6$ where $\lambda_1 = 0.001$; (b) $E = 22$ where $\lambda_1 = 0.021$; (c) $E = 23$ where $\lambda_1 = 0.047$. X and Y are dimensionless coordinates of position.

IV. SENSITIVITY TO INITIAL CONDITIONS

A. Theory

Sensitivity to initial conditions is characterized by the growth rate of an infinitesimal perturbation $\delta\Gamma$ on a trajectory of the system in the phase space of positions and momenta of all the particles: $\Gamma = \{\mathbf{R}_i, \mathbf{P}_i\}_{i=1}^N$. This growth rate is the so-called Lyapunov exponent [1]

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\delta\Gamma_t\|}{\|\delta\Gamma_0\|} \; , \tag{14}$$

where $\|\delta\Gamma\|$ is the magnitude of the perturbation $\delta\Gamma = \{\delta\mathbf{R}_i, \delta\mathbf{P}_i\}_{i=1}^N$. In many-particle systems, there exist as many Lyapunov exponents as phase-space dimensions. We shall here be concerned by the maximum Lyapunov exponent λ_1 computed with Eq. (14) starting from a typical initial perturbation $\delta\Gamma_0$. The time evolution of the perturbation $\delta\Gamma = \{\delta\mathbf{R}_i, \delta\mathbf{P}_i\}_{i=1}^N$ is ruled by the second variation of the Hamiltonian (7) of the system:

$$\begin{aligned} \delta^2 H \;&=\; \sum_i \left[\frac{1}{2}\,\delta\mathbf{P}_i^2 + \left(\frac{1}{8} - \frac{\gamma^2}{4}\right)\left(\delta X_i^2 + \delta Y_i^2\right) + \frac{\gamma^2}{2}\,\delta Z_i^2 \right] \\ &-\; \frac{1}{2}\sum_{i<j} \left[\frac{\delta\mathbf{R}_{ij}^2}{R_{ij}^3} - 3\frac{(\mathbf{R}_{ij}\cdot\delta\mathbf{R}_{ij})^2}{R_{ij}^5} \right] \; . \end{aligned} \tag{15}$$

The first terms describe the time evolution of the perturbation under the harmonic potential of the Penning trap. This motion is regular and would give vanishing Lyapunov exponents if the ions were not interacting via the Coulomb potential. Therefore, positive Lyapunov exponents come from the last Coulombic terms in R_{ij}^{-3}. As shown in Appendix A, the third inverse moment of the inter-particle distance decreases at high temperature as

$$\left\langle \frac{1}{R_{ij}^3} \right\rangle \sim \frac{\ln T}{T^{3/2}} \; . \tag{16}$$

If all the N^2 terms as R_{ij}^{-3} in Eq. (15) are supposed to contribute to the maximum Lyapunov exponent, we should have

$$\lambda_1 \sim \left\langle \frac{N^2}{R_{ij}^3} \right\rangle^{1/2} \sim N\, \frac{(\ln T)^{1/2}}{T^{3/4}} \; , \quad \text{for } T\to\infty \; . \tag{17}$$

The expectation is thus that the maximum Lyapunov exponent decreases for increasing temperature and increases with the number of ions in the high-temperature regime.

B. Dependence on energy and ion number

The maximum Lyapunov exponent has been computed numerically for microplasmas containing more and more ions in the oblate Penning trap of Fig. 1. The results are depicted in Fig. 2 where we observe the sharp increases of the Lyapunov exponent just above the minimum energy E_0 of the static ion crystal. The Lyapunov exponent increases up to a maximum value and then decreases. We observe that the maximum value shifts toward higher energies and higher values as the number of ions increases.

Let us describe in detail what happens near the minimum energy E_0. This energy is the total potential energy of the system at zero temperature when the kinetic energy vanishes. It is shown in Appendix A that this energy should increase as $E_0 \sim N^{\frac{5}{3}}$ with the number of particles, which is in agreement with the numerical results. For energies just above E_0, the ions are in quasiharmonic motion following the normal modes of vibration around the equilibrium position of the crystal, as discussed in the previous section. This explains that the Lyapunov exponent vanishes with the kinetic energy. As seen in Fig. 1 for the system with $N = 20$ ions, the ions soon have enough energy for their motion to be erratic. For instance, Fig. 1c depicts the motion of the $N = 20$ ions at the energy $E = 23$ where the Lyapunov exponent takes the value $\lambda_1 = 0.047$ significantly lower than its value $\lambda_1 = 0.114$ at the maximum occurring around the energy $E = 40$ in Fig. 2 for $N = 20$. This shows that the Lyapunov exponent reaches its maximum value for energies well above the melting seen in Fig. 1. Actually, the melting of the crystal does not leave a signature in the behavior of the Lyapunov exponent, except possibly in quasi one-dimensional systems (see below).

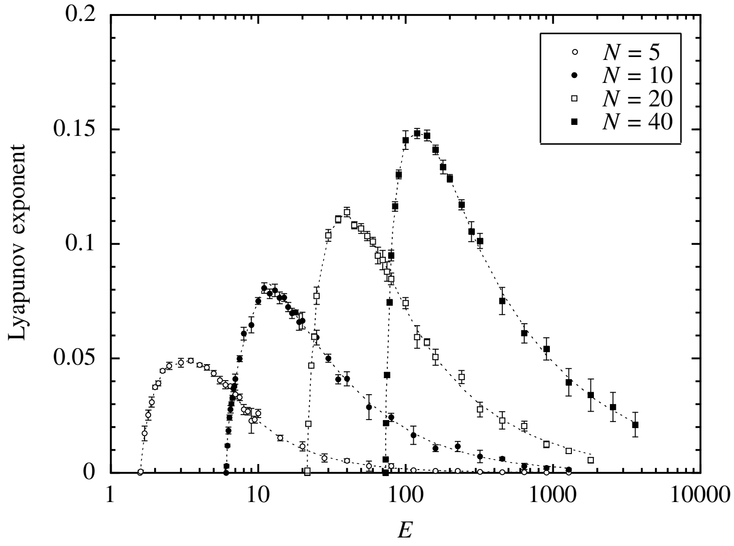

FIG. 2: Maximum Lyapunov exponent λ_1 versus total energy E for systems of 5, 10, 20, and 40 ions with $L_Z = 0$ in an oblate Penning trap with $\gamma = 0.7$. The dashed lines are fits to the data points. The plotted quantities are dimensionless (see text).

The maximum value of the Lyapunov exponent is numerically observed to happen at an energy scaling with N as $E_{\max} \sim N^{1.6}$, which is a power-law similar to the one of E_0. The maximum value of λ_1 is also increasing with N, as seen in Fig. 2. This maximum arises at an energy where the harmonic potential energy is of the same order of magnitude as the Coulomb potential energy. The confinement of the ions by the external potential of the trap precludes a possible thermodynamic limit as in translationally invariant systems. The external potential and the long range of the Coulomb interaction have for consequence the observed increase of the maximum value of λ_1 with N.

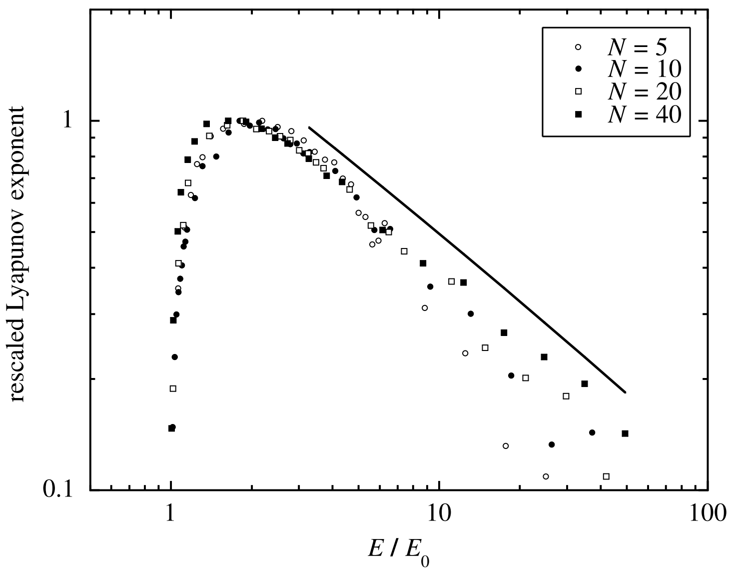

FIG. 3: Log-log plot of the rescaled Lyapunov exponent $\lambda_1/\max_E\{\lambda_1\}$ versus the rescaled total energy E/E_0 for the systems of 5, 10, 20, and 40 ions of Fig. 2. The continuous line is the theoretical prediction (17), shifted upward to avoid superposition with the numerical data. The plotted quantities are dimensionless.

In order to investigate the high-temperature behavior, the Lyapunov exponent is rescaled by its maximum value $\max_E\{\lambda_1\}$, where the maximum is taken over the dependence of λ_1 on the energy E. The rescaled Lyapunov exponent $\lambda_1/\max_E\{\lambda_1\}$ is depicted as a function of the rescaled energy E/E_0 in Fig. 3. At high temperature, this rescaled energy is proportional to the temperature T according to Eq. (10) so that Fig. 3 essentially depicts the rescaled

Lyapunov exponent versus temperature. It allows us to test the dependence of the Lyapunov exponent on the temperature. The theoretical expectation of Eq. (17) is shown as the solid line in Fig. 3, which agrees with the decrease of the Lyapunov exponent for our largest values of N. Deviations occur for smaller values of N, which may be due to the fact that the large-system limit required for the applicability of statistical mechanics is not yet reached at such small values of $N = 5$ or 10. Nevertheless, the numerical data are in agreement with the dependence on the temperature predicted by Eq. (17) for large values of N, as expected.

C. Dependence on trap geometry

Figure 4 depicts the Lyapunov exponent λ_1 versus energy for a microplasma of 10 ions in traps of different shapes. We observe that the motion in the isotropic trap is significantly more chaotic than in the extreme oblate and prolate traps. In a quasi one-dimensional (prolate) trap, the Lyapunov exponent is observed to slowly increases during the melting of the ion crystal in contrast to the multidimensional cases. The reason is that the ions keep their one-dimensional order up to a critical energy where ions can jump over each other, leading to chaotic motion. In all cases, the Lyapunov exponent decreases at high energy.

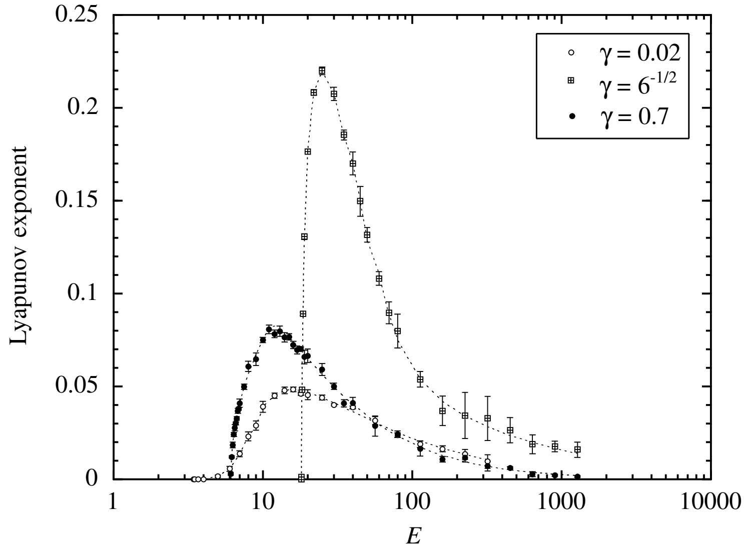

FIG. 4: Maximum Lyapunov exponent λ_1 versus total energy E for a system of 10 ions with $L_Z = 0$ in Penning traps with $\gamma = 0.02$ (prolate), $\gamma = 6^{-1/2} \simeq 0.4082$ (isotropic), and $\gamma = 0.7$ (oblate). The dashed lines are fits to the data points. The plotted quantities are dimensionless (see text).

V. CONCLUSIONS

In the present paper, we have shown that the motion of ions in microplasmas presents an exponential type of sensitivity to initial conditions characterized by a positive maximum Lyapunov exponent and we have studied the dependence of this exponent on the energy of the system, on the number of ions, as well as on the geometry of the trap.

At low kinetic energy where the microplasma forms an ion crystal, the dynamical chaos is much reduced because the motion is quasiharmonic around the crystal equilibrium configuration. In order to minimize the dynamical chaos in systems of ions for instance to build well-controlled quantum devices, our results show that high-dimensional behavior as well as the exchanges of ions should be avoided. Our study characterizes irregularity in the motion of ions in terms of the maximum Lyapunov exponent and shows under which conditions the Lyapunov exponent may remain small enough for the controlled manipulations of the ions. In this sense, the inverse of the Lyapunov exponent is an indicator of the controllability of the motion of ions. The larger the inverse Lyapunov exponent is, the longer is the interval of time when the motion is under control. Similarly, the control of the spatial distribution of the ions goes with the minimization of the standard thermodynamic entropy at low temperature.

The Lyapunov exponent turns out to decrease at high temperature because the Coulomb interaction becomes negligible and the microplasma forms a thermal cloud of nearly independent ions moving in the harmonic potential of the trap. At high temperature, the spatial disorder of these microplasmas characterized by the standard thermodynamic entropy (13) always increases although dynamical chaos decreases as (17). This decrease of the maximum Lyapunov exponent that we here observe in trapped microplasmas in reminiscent of a result obtained for a one-dimensional wave-particle model of plasma [29, 30], in which the maximum Lyapunov exponent even vanishes in the thermodynamic limit above a critical energy. In the three-dimensional trapped microplasmas we study here, the maximum Lyapunov exponent does not vanish as in the model of Refs. [29, 30] but decreases at high temperature according to Eq. (17) because of the quasiharmonic motion in the trap potential.

At intermediate values of energy, our results show that there is a regime of significant dynamical chaos which becomes broader and broader as the number of ions increases. In this intermediate regime, the Lyapunov exponent reaches a maximum value which turns out to increase as a function of the number of ions. This dynamical chaos could possibly be measured in experiments tracking the trajectories of the ions of the microplasma using the methods of Refs. [20–25]. Such an experimental measurement of a Lyapunov exponent would be a great achievement which could solve the fundamental question to know whether the sensitivity to initial conditions is strong or weak in the many-particle systems of statistical mechanics.

Acknowledgments. The author thanks Professors J. R. Dorfman and G. Nicolis for support and encouragement in this research. He is grateful to the FNRS Belgium for financial support.

APPENDIX A: STATISTICAL MECHANICS OF THE ION SYSTEM

In this appendix, we show how to calculate with statistical mechanics the properties of interest for our purposes.

The large-scale statistical properties can be described by the following free-energy functional of the density $n(\mathbf{R})$ of ions:

$$\begin{aligned} F &= E - TS \\ &= \int d\mathbf{R} \left[\frac{3}{2}T + \frac{\gamma_\perp^2}{2}(X^2+Y^2) + \frac{\gamma_\parallel^2}{2}Z^2\right] n(\mathbf{R}) \\ &\quad + \frac{1}{2}\int d\mathbf{R}\, d\mathbf{R}'\, \frac{n(\mathbf{R})\, n(\mathbf{R}')}{\|\mathbf{R}-\mathbf{R}'\|} \\ &\quad - T\int d\mathbf{R}\, n(\mathbf{R})\, \ln\frac{C}{n(\mathbf{R})} \, , \end{aligned} \tag{A1}$$

with $\gamma_\perp^2 = \frac{1}{4} - \frac{\gamma^2}{2}$, $\gamma_\parallel = \gamma$, and $C = \mathrm{e}^{\frac{5}{2}}(2\pi T)^{\frac{3}{2}}(a^2 m\omega_\mathrm{c} h^{-1})^3$, h being the Planck constant. The chemical potential is obtained from the first variation of the free energy with respect to the density as $\delta F = \int \mu\, \delta n\, d\mathbf{R}$, and should be constant at equilibrium so that we get the equilibrium density

$$n(\mathbf{R}) \sim \exp\left[-\frac{\gamma_\perp^2}{2T}(X^2+Y^2) - \frac{\gamma_\parallel^2}{2T}Z^2 - \frac{\Phi(\mathbf{R})}{T}\right] \, , \tag{A2}$$

with the mean-field potential

$$\Phi(\mathbf{R}) = \int d\mathbf{R}\, \frac{n(\mathbf{R}')}{\|\mathbf{R}-\mathbf{R}'\|} \tag{A3}$$

The mean-field potential typically decreases as R^{-1} at large distances and becomes negligible with respect to the trap harmonic potential at high temperature. Therefore, the density becomes Gaussian at high temperature.

At zero temperature $T=0$, the kinetic energy as well as the last term vanish in the functional (A1). Taking the Laplacian of the chemical potential and using the formula $\nabla^2 \frac{1}{\|\mathbf{R}\|} = -4\pi\delta(\mathbf{R})$ thus shows that the density is equal to a constant:

$$n(\mathbf{R}) = \frac{2\gamma_\perp^2 + \gamma_\parallel^2}{4\pi} = \frac{1}{8\pi} \, , \qquad \text{for} \quad T = 0 \, . \tag{A4}$$

This result implies that the radius of a zero-temperature spherical microplasma scales as $A \sim N^{\frac{1}{3}}$ as a function of the number N of ions and that its energy as

$$E = E_0 \sim N^{\frac{5}{3}} \, , \qquad \text{for} \quad T = 0 \, . \tag{A5}$$

In order to estimate the inverse moments of the interparticle distances $R_{ij} = \|\mathbf{R}_i - \mathbf{R}_j\|$, we need to take into account the Coulomb repulsion between the ions. For a dilute system, the two-particle density is given by [31]:

$$n_2(\mathbf{R},\mathbf{R}') \simeq n(\mathbf{R})\, n(\mathbf{R}')\, \exp\left(-\frac{1}{T\|\mathbf{R}-\mathbf{R}'\|}\right) , \tag{A6}$$

with

$$n(\mathbf{R}) \simeq \frac{N\,\gamma_\perp^2\gamma_\parallel}{(2\pi T)^{\frac{3}{2}}}\, \exp\left[-\frac{\gamma_\perp^2}{2T}(X^2+Y^2) - \frac{\gamma_\parallel^2}{2T}Z^2\right] . \tag{A7}$$

The moments are thus given by

$$\left\langle \sum_{i<j} \frac{1}{R_{ij}^\alpha} \right\rangle = \frac{1}{2}\int d\mathbf{R}\, d\mathbf{R}'\, \frac{n_2(\mathbf{R},\mathbf{R}')}{\|\mathbf{R}-\mathbf{R}'\|^\alpha} . \tag{A8}$$

Using the permutation symmetry and a change of variables from $\mathbf{R}$ and $\mathbf{R}'$ to their sum and difference, we obtain for an isotropic system with $\gamma_\perp = \gamma_\parallel = \gamma$ that

$$\left\langle \frac{1}{R_{ij}^\alpha} \right\rangle \simeq \frac{\gamma^3}{2\sqrt{\pi}T^{\frac{3}{2}}} \int_0^\infty \frac{dr}{r^{\alpha-2}}\, \exp\left(-\frac{\gamma^2 r^2}{4T} - \frac{1}{Tr}\right) , \tag{A9}$$

with $r = \|\mathbf{R}-\mathbf{R}'\|$. The first term in the exponential comes from the trap harmonic potential and the second term from the Coulomb repulsion. In the limit $T \to \infty$, the integral is dominated by the trap harmonic potential for the moments with $\alpha < 3$ and by the Coulomb repulsion for the moments with $\alpha > 3$. For $\alpha = 3$, the integral can be evaluated by splitting it in two incomplete Gamma functions at some value r_0, leading to an extra $\ln T$ factor in the limit $T \to \infty$. Whereupon, we obtain the following asymptotic behavior

$$\left\langle \frac{1}{R_{ij}^\alpha} \right\rangle \sim \begin{cases} \frac{1}{T^{\frac{\alpha}{2}}} , & \text{for} \quad \alpha < 3 , \\ \frac{\ln T}{T^{\frac{3}{2}}} , & \text{for} \quad \alpha = 3 , \\ T^{\alpha - \frac{9}{2}} , & \text{for} \quad \alpha > 3 , \end{cases} \tag{A10}$$

as $T \to \infty$. These results give, in particular, the mean Coulomb energy of Eq. (12) and the third inverse moment of Eq. (16) at high temperature.

---

[1] J.-P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[2] Ya. G. Sinai, Int. J. Bifurc. Chaos **6**, 1137 (1996).
[3] H. van Beijeren, J. R. Dorfman, H. A. Posch, and Ch. Dellago, Phys. Rev. E **56**, 5272 (1997).
[4] R. van Zon, H. van Beijeren, and Ch. Dellago, Phys. Rev. Lett. **80**, 2035 (1998).
[5] J. R. Dorfman, A. Latz, and H. van Beijeren, Chaos **8**, 444 (1998).
[6] R. Livi, A. Politi, and S. Ruffo, J. Phys. A: Math. Gen. **19**, 2033 (1986).
[7] H. A. Posch and W. G. Hoover, *Phys. Rev. A* **38**, 473 (1988).
[8] H. A. Posch and W. G. Hoover, *Phys. Rev. A* **39**, 2175 (1989).
[9] D. J. Evans and G. P. Morriss, *Statistical Mechanics of Nonequilibrium Liquids* (Academic Press, London, 1990).
[10] D. J. Searles, D. J. Evans, and D. J. Isbister, Physica A **240**, 96 (1997).
[11] C. Forster and H. A. Posch, preprint (2003).
[12] J. L. Lebowitz, C. Maes, and E. R. Speer, J. Stat. Phys. **59**, 117 (1990).
[13] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981) pp. 358-364.
[14] G. A. Bird, Phys. Fluids **6**, 1518 (1963).
[15] P. Gaspard, *Chaos, Scattering, and Statistical Mechanics* (Cambridge University Press, Cambridge UK, 1998).
[16] C. P. Dettmann, E. G. D. Cohen, and H. van Beijeren, *Nature* **401**, 875 (1999).
[17] C. P. Dettmann and E. G. D. Cohen, *J. Stat. Phys.* **101**, 775 (2000).
[18] B. Li, L. Wang, and B. Hu, Phys. Rev. Lett. **88**, 223901 (2002).
[19] P. Grassberger, W. Nadler, and L. Yang, Phys. Rev. Lett. **89**, 180601 (2002).

[20] I. Siemers, R. Blatt, Th. Sauter, and W. Neuhauser, Phys. Rev. A **38**, 5121 (1988).
[21] J. Hoffnagle, R. G. DeVoe, L. Reyna, and R. G. Brewer, Phys. Rev. Lett. **61**, 255 (1988).
[22] R. Blümel, J. M. Chen, E. Peik, W. Quint, W. Schleich, Y. R. Shen, and H. Walther, Nature **334**, 309 (1988).
[23] L. R. Brewer, J. D. Prestage, J. J. Bollinger, W. M. Itano, D. J. Larson, and D. J. Wineland, Phys. Rev. A **38**, 859 (1988).
[24] M. G. Raizen, J. M. Gilligan, J. C. Bergquist, W. M. Itano, and D. J. Wineland, Phys. Rev. A **45**, 6493 (1992).
[25] J. Hoffnagle and R. G. Brewer, Phys. Rev. Lett. **71**, 1828 (1993).
[26] D. M. Barnett, T. Tajima, K. Nishihara, Y. Ueshima, and H. Furukawa, Phys. Rev. Lett. **76**, 1812 (1996).
[27] D. M. Barnett and T. Tajima, Phys. Rev. E **54**, 6084 (1996).
[28] Y. Ueshima, K. Nishihara, D. M. Barnett, T. Tajima, and H. Furukawa, Phys. Rev. E **55**, 3439 (1997).
[29] M.-C. Firpo, Phys. Rev. E **57**, 6599 (1998).
[30] V. Latora, A. Rapisarda, and S. Ruffo, Phys. Rev. Lett. **80**, 692 (1998).
[31] D. A. McQuarrie, *Statistical Mechanics* (Harper & Row, New York, 1976).