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Maps

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A *map* is a dynamical system with discrete time. Such dynamical systems are defined by iterating a transformation $\boldsymbol{\phi}$ of points $\mathbf{x}$

$$\mathbf{x}_{n+1} = \boldsymbol{\phi}(\mathbf{x}_n) \; , \tag{1}$$

from a space $\mathcal{M}$ of dimension d (or a domain of this space) onto itself. $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ are thus points belonging to this so-called *phase space* $\mathcal{M}$, which can be a Euclidean space such as the space $\mathbb{R}^d$ of d-uples of real numbers or a manifold such as a circle, a sphere, or a torus $\mathbb{T}^d$ (or a domain such as an interval or a square).

An *endomorphism* is a surjective (i.e., many-to-one) transformation $\boldsymbol{\phi}$ of the space $\mathcal{M}$ onto itself. Thus, the transformation $\boldsymbol{\phi}$ is not invertible. Examples of endormorphisms are one-dimensional maps of the interval such as the logistic map $\phi(x) = 1 - ax^2$ on $-1 \leq x \leq 1$, the Bernoulli map $\phi(x) = rx$ (modulo 1) with integer r (also called r-adic map), or the Gauss map $\phi(x) = 1/x$ (modulo 1) which generates continuous fractions. Both latter maps are defined onto the unit interval $0 \leq x \leq 1$. There also exist multidimensional examples such as the exact map $\boldsymbol{\phi}(x, y) = (3x + y, x + 3y)$ (modulo 1) on the torus $\mathbb{T}^2$ (Lasota & Mackey 1985).

An *automorphism* is a one-to-one (i.e., invertible) transformation $\boldsymbol{\phi}$ of the space $\mathcal{M}$ onto itself. Automorphisms for which the one-to-one transformation $\boldsymbol{\phi}$ is continuous on $\mathcal{M}$ are called *homeomorphisms*. We speak about $\mathcal{C}^r$-*diffeomorphisms* if $\boldsymbol{\phi}$ is r-times differentiable and $\frac{\partial^r \boldsymbol{\phi}}{\partial \mathbf{x}^r}$ is continuous on $\mathcal{M}$. Examples of automorphisms are the circle maps defined with a monotonously increasing function $\phi(x) = \phi(x + 1) - 1$ onto the circle,

$$\text{the baker map:} \qquad \boldsymbol{\phi}(x, y) = \begin{cases} (2x, \frac{y}{2}) & \text{if} \quad 0 \leq x < 1/2 \; , \\ (2x - 1, \frac{y+1}{2}) & \text{if} \quad 1/2 \leq x < 1 \; , \end{cases} \tag{2}$$

onto the unit square (Hopf 1937);

$$\text{the cat map:} \qquad \boldsymbol{\phi}(x, y) = (x + y, x + 2y) \quad (\text{modulo } 1) \; , \tag{3}$$

onto the torus $\mathbb{T}^2$ (Arnold & Avez 1968);

$$\text{the quadratic map:} \qquad \boldsymbol{\phi}(x, y) = (y + 1 - ax^2, bx) \; , \tag{4}$$

onto $\mathbb{R}^2$ also called the *Hénon map* (Hénon 1976, Gumowski & Mira 1980); among many others.

Iterating a noninvertible transformation $\boldsymbol{\phi}$ generates a semigroup of endomorphisms $\{\boldsymbol{\phi}^n(\mathbf{x})\}_{n \in \mathbb{N}}$, where $\mathbb{N}$ is the set of nonnegative integers.

Iterating an invertible transformation $\boldsymbol{\phi}$ generates a group of automorphisms $\{\boldsymbol{\phi}^n(\mathbf{x})\}_{n \in \mathbb{Z}}$, where $\mathbb{Z}$ is the set of all the integers.

Such groups or semigroups are deterministic dynamical systems with discrete time, called *maps*.

Link between maps and flows. Maps naturally arise in continuous-time dynamical systems (i.e., flows) defined with $d + 1$ ordinary differential equations

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) \; , \tag{5}$$

by considering the successive intersections of the trajectories $\mathbf{X}(t)$ with a codimension-one Poincaré section $\sigma(\mathbf{X}) = 0$. If $\mathbf{x}$ denotes d coordinates which are intrinsic to the Poincaré section, the successive intersections $\{\mathbf{X}_n = \mathbf{X}(t_n)\}_{n \in \mathbb{Z}}$ of the trajectory correspond to a sequence of points $\{\mathbf{x}_n\}_{n \in \mathbb{Z}}$ and return times $\{t_n\}_{n \in \mathbb{Z}}$ in the Poincaré section. According to Cauchy's theorem which guarantees the unicity of the trajectory $\mathbf{X}(t)$ issued from a given initial condition $\mathbf{X}(0)$ (i.e., by the determinism of the flow), the successive points and return times are related by

$$\begin{cases} \mathbf{x}_{n+1} = \boldsymbol{\phi}(\mathbf{x}_n) \; , \\ t_{n+1} = t_n + T(\mathbf{x}_n) \; , \end{cases} \tag{6}$$

where $\boldsymbol{\phi}(\mathbf{x})$ is the so-called *Poincaré map* and $T(\mathbf{x})$ the return-time (or ceiling) function. The knowledge of the Poincaré map and its associated return-time function allows us to recover the flow and its properties.

We consider a similar construction for ordinary differential equations which are periodic in time

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t + T) \ , \tag{7}$$

in which case the return-time function reduces to the period T in Eq. (6) and the Poincaré map becomes a *stroboscopic map.*

Examples of Poincaré maps are the Birkhoff maps in the case of billiards. Billiards are systems of particles in free flights (or more generally following Hamiltonian trajectories) interrupted by elastic collisions. The knowledge of the collisions suffices to reconstruct the full trajectories. The Birkhoff map is thus the transformation ruling the dynamics of billiards from collision to collision.

PROPERTIES. Maps can be classified according to different properties. An important question is to know if a map is locally volume-preserving or not. If the map is differentiable, the volume preservation holds if the absolute value of its Jacobian determinant is equal to unity everywhere in $\mathcal{M}$:

$$\left| \det \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}} \right| = 1 \ . \tag{8}$$

This is the case for the baker map (2), the cat map (3), and the quadratic map (4) if $b = \pm 1$.

Maps that contract phase-space volumes on average are said to be *dissipative.* In the limit $b \to 0$, the two-dimensional automorphism (4) contracts the phase-space areas so much that it becomes an endomorphism given by the one-dimensional logistic map. This explains why highly dissipative dynamical systems are often very well described in terms of endomorphisms such as the logistic map.

A map is *symplectic* if its Jacobian matrix satisfies

$$\left(\frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}} \right)^{\mathrm{T}} \cdot \boldsymbol{\Sigma} \cdot \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}} = \boldsymbol{\Sigma} \ , \tag{9}$$

where $^{\mathrm{T}}$ denotes the transpose and $\boldsymbol{\Sigma}$ is an antisymmetric constant matrix: $\boldsymbol{\Sigma}^{\mathrm{T}} = -\boldsymbol{\Sigma}$. Symplectic maps act onto phase spaces of even dimension. Poincaré maps of Hamiltonian systems as well as Birkhoff maps are symplectic in appropriate coordinates. Symplectic maps are volume-preserving. Area-preserving maps are symplectic but there exist volume-preserving maps which are not symplectic in dimension higher than two.

A map is symmetric under a group $\mathcal{G}$ of transformations $\mathbf{g} \in \mathcal{G}$ if

$$\mathbf{g} \circ \boldsymbol{\phi} = \boldsymbol{\phi} \circ \mathbf{g} \ . \tag{10}$$

A map is said to be *reversible* if there exists an involution, that is, a transformation $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}^2 = 1$, which transforms the map into its inverse:

$$\boldsymbol{\theta} \circ \boldsymbol{\phi} \circ \boldsymbol{\theta} = \boldsymbol{\phi}^{-1} \ . \tag{11}$$

There exist reversible maps which are not volume-preserving (Roberts & Quispel 1992).

INVARIANT SUBSETS. These are subsets of the phase space which are invariant under the action of the map. They include the fixed points $\boldsymbol{\phi}(\mathbf{x}_*) = \mathbf{x}_*$ (which correspond to periodic orbits of a corresponding flow) and the periodic orbits of prime period n defined as trajectories from initial condition $\mathbf{x}_p$ such that $\boldsymbol{\phi}^n(\mathbf{x}_p) = \mathbf{x}_p$ but $\boldsymbol{\phi}^j(\mathbf{x}_p) \neq \mathbf{x}_p$ for $0 < j < n$. Tori may also be invariant as in the case of KAM quasiperiodic motion.

An invariant subset $\mathcal{I}$ of the map is *attracting* if there exists an open neighborhood $\mathcal{U}$ such that $\boldsymbol{\phi}\mathcal{U} \subset \mathcal{U}$ and $\mathcal{I} = \cap_{n\in\mathbb{N}} \boldsymbol{\phi}^n \mathcal{U}$. The open set $\mathcal{B} = \cup_{n\in\mathbb{N}} \boldsymbol{\phi}^{-n} \mathcal{U}$ is called the *basin of attraction* of $\mathcal{I}$. An *attractor* is an attracting set which cannot be decomposed into smaller ones. A set which is not attracting is said to be *repelling.*

A closed invariant subset $\mathcal{I}$ is *hyperbolic* if: (i) the tangent space $\mathcal{T}_{\mathbf{x}}\mathcal{M}$ of the phase space $\mathcal{M}$ splits into stable and unstable linear subspaces $\mathcal{E}_{\mathbf{x}}^{(s)}$ and $\mathcal{E}_{\mathbf{x}}^{(u)}$ depending continuously on $\mathbf{x} \in \mathcal{I}$,

$$\mathcal{T}_{\mathbf{x}}\mathcal{M} = \mathcal{E}_{\mathbf{x}}^{(s)} \oplus \mathcal{E}_{\mathbf{x}}^{(u)} \ ; \tag{12}$$

(ii) the linearized dynamics preserves these subspaces; and (iii) the vectors of the stable (resp. unstable) subspace are contracted (resp. expanded) by the linearized dynamics (Ott 1993). Hyperbolicity implies sensitivity to initial conditions of exponential type, characterized by positive Lyapunov exponents. By extension, a map is said to be *hyperbolic* if its invariant subsets are hyperbolic.

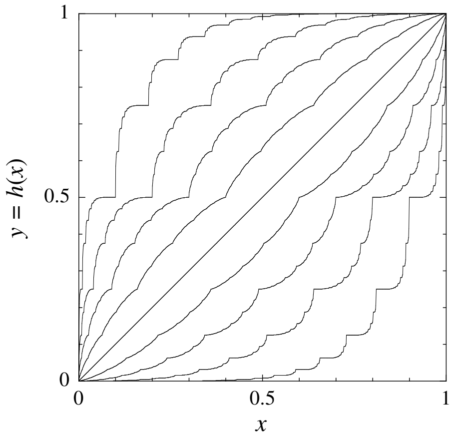

FIG. 1: Homeomorphisms $y = h(x)$ transforming the dyadic map into the maps (15) with $p = 0.1, 0.2, ..., 0.9$.

For the baker map (2), the unstable linear subspace is the x-direction while the stable one is the y-direction and the unit square is hyperbolic with a positive Lyapunov exponent. Moreover, the dynamics of the baker map can be shown to be equivalent to a so-called Bernoulli shift, that is, a symbolic dynamics acting as a simple shift on all the possible infinite sequences of symbols 0 and 1, so that most of its trajectories are random. The baker map is thus an example of hyperbolic fully chaotic map.

A diffeomorphism is said to have the *Anosov* property if its whole compact phase space $\mathcal{M}$ is hyperbolic. Examples of Anosov diffeomorphisms are the cat map (3) and its nonlinear perturbations:

$$\begin{cases} x_{n+1} = x_n + y_n + f(x_n, y_n) \ , & \text{(modulo 1)} \ , \\ y_{n+1} = x_n + 2y_n + g(x_n, y_n) \ , & \text{(modulo 1)} \ , \end{cases} \tag{13}$$

with small enough periodic functions $f(x, y)$ and $g(x, y)$ defined on the torus. We notice that these nonlinear perturbations of the cat map are generally not area-preserving.

Dissipative maps may have chaotic attractors (i.e., attractor with positive Lyapunov exponents) which are not necessarily hyperbolic. This is the case for a set of $a > 0$ values of positive Lebesgue measure in the logistic map (Jakobson 1981), as well as in the quadratic map (4) if $b > 0$ is sufficiently small (Benedicks & Carleson 1991).

Maps may also have sensitivity to initial conditions of stretched-exponential type (with vanishing Lyapunov exponent) as it is the case for the intermittent maps $\phi(x) = x + ax^{\zeta}$ (modulo 1) with $\zeta > 2$.

Conjugacy between maps. In the study of maps, it is often important to modify the analytic form of the map by a change of variables $\mathbf{y} = \mathbf{h}(\mathbf{x})$. Such a conjugacy would transform the map (1) into

$$\mathbf{y}_{n+1} = \boldsymbol{\psi}(\mathbf{y}_n) \ , \qquad \text{with} \quad \boldsymbol{\psi} = \mathbf{h} \circ \boldsymbol{\phi} \circ \mathbf{h}^{-1} \ . \tag{14}$$

The Kolmogorov-Sinai entropy per iteration is known to remain invariant if the conjugacy $\mathbf{h}$ is a diffeomorphism, but it is only the topological entropy per iteration which is invariant if the conjugacy is a homeomorphism. For instance, the logistic map $\phi(x) = 1 - 2x^2$ is conjugated to the tent map $\psi(y) = 1 - 2|y|$ by the conjugacy $y = -1 + \frac{4}{\pi} \arcsin \sqrt{\frac{x+1}{2}}$, both maps having their Kolmogorov-Sinai entropy equal to $\ln 2$. On the other hand, the dyadic map $\phi(x) = 2x$ (modulo 1) is conjugated to the map

$$\psi(y) = \begin{cases} \frac{y}{p} & \text{if} \quad 0 \leq y \leq p \ , \\ \frac{y-p}{1-p} & \text{if} \quad p < y \leq 1 \ , \end{cases} \tag{15}$$

with $p \neq \frac{1}{2}$ by a homeomorphism $h(x)$ which is not differentiable (see Fig. 1). These two maps have their topological entropy equal to $\ln 2$ but different Kolmogorov-Sinai entropies.

Conjugacies are also important to transform a circle map such as $\phi(x) = x + \alpha + \varepsilon \sin(2\pi x)$ with $|\varepsilon| < \frac{1}{2\pi}$ into a pure rotation $\psi(y) = y + \omega$ of rotation number

$$\omega = \lim_{n\to\infty} \frac{1}{n}(x_n - x_0) \ . \tag{16}$$

According to Denjoy theory, such a conjugacy is possible if the rotation number is irrational, in which case the motion is quasiperiodic and nonchaotic. The circle map also illustrates the phenomenon of synchronization to the external frequency α, which occurs when the rotation number ω takes rational values corresponding to periodic motions. The motion may become chaotic if $|\varepsilon| > \frac{1}{2\pi}$.

AREA-PRESERVING MAPS. Periodic, quasiperiodic, and chaotic motions are also the feature of area-preserving maps which can be considered as Poincaré maps of Hamiltonian systems with two degrees of freedom. Area-preserving maps can be derived from a variational principle based on some Lagrangian generating function $\ell(x_{n+1}, x_n)$. The variational principle requires that the trajectories are extremals of the action

$$W = \sum_{n\in\mathbb{Z}} \ell(x_{n+1}, x_n) \ . \tag{17}$$

The vanishing of the first variation, $\delta W = 0$, leads to the second-order recurrence equation

$$\frac{\partial \ell(x_{n+1}, x_n)}{\partial x_n} + \frac{\partial \ell(x_n, x_{n-1})}{\partial x_n} = 0 \ . \tag{18}$$

This recurrence can be rewritten in the form of a two-dimensional map by expliciting the equations for the momenta

$$\begin{cases} p_{n+1} = \frac{\partial \ell(x_{n+1},x_n)}{\partial x_{n+1}} \ , \\ p_n = -\frac{\partial \ell(x_{n+1},x_n)}{\partial x_n} \ . \end{cases} \tag{19}$$

The Birkhoff map of a billiard is recovered if ℓ is the distance traveled by the particle in free flight between collisions and x_n is the arc of perimeter at which the collision occurs. If a free particle or rotor is periodically kicked by an external driving, the Lagrangian function takes the form

$$\ell = \frac{1}{2}(x_{n+1} - x_n)^2 - V(x_n) \ . \tag{20}$$

A famous map is the so-called *standard map*

$$\begin{cases} p_{n+1} = p_n + K \sin x_n \ , \\ x_{n+1} = x_n + p_{n+1} \quad (\text{modulo } 2\pi) \ , \end{cases} \tag{21}$$

obtained for the kicked rotor with the potential $V(x) = K \cos x$ in Eq. (20). The motivation for studying the standard map goes back to works on the origin of stochasticity in Hamiltonian systems (Chirikov 1979, Lichtenberg & Lieberman 1983, MacKay & Meiss 1987).

Phase portraits of an area-preserving typically present closed curves of KAM quasiperiodic motion, which form elliptic islands. Hierarchical structures of elliptic islands develops on smaller and smaller scales. The elliptic islands are surrounded by chaotic zones extending over finite area (see Fig. 2).

Typical area-preserving maps such as the standard map (21) or the one of Fig. 2 display a variety of motions which interpolate between two extremes, namely, the fully chaotic behavior of hyperbolic area-preserving maps such as the baker and cat maps and the fully regular motion of integrable maps such as the one given by the second-order recurrence

$$x_{n+1} - 2x_n + x_{n-1} = -2i \ln \frac{\kappa^2 + \mathrm{e}^{-ix_n}}{\kappa^2 + \mathrm{e}^{+ix_n}} \ , \tag{22}$$

(Faddeev & Volkov 1994).

Some area-preserving maps may have a repelling Smale horseshoe as the only invariant subset at finite distance. This is the case in the quadratic map (4) for $b = -1$ and large enough values of $a > 0$. Such horseshoes often arise in open two-degree-of-freedom Hamiltonian systems describing the chaotic scattering of a particle in some time-periodic potential.

COMPLEX MAPS. Such maps are defined with some analytic function $\phi(z)$ of $z = x+iy \in \mathbb{C}$ or some multidimensional generalizations of it. Complex maps are generally endomorphisms. An example is the complex logistic map $\phi(z) = z^2 + c$. Other examples are given by the Newton-Raphson method of finding the roots of a function $f(z) = 0$:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \ . \tag{23}$$

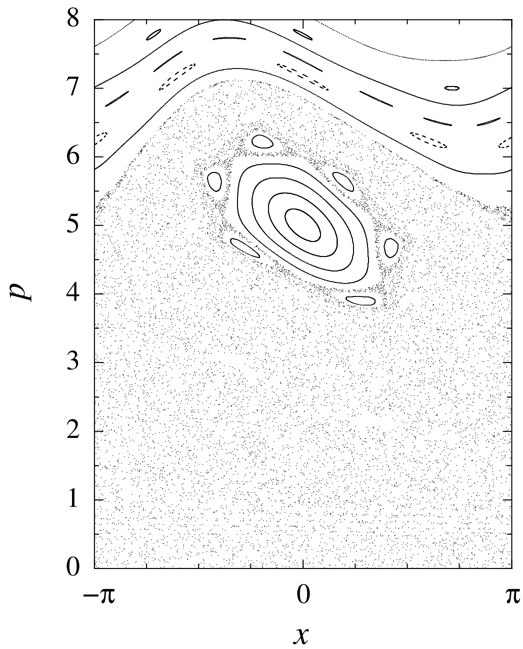

FIG. 2: Phase portrait of the Fermi-Ulam area-preserving map, $p_{n+1} = |p_n + \sin x_n|$, $x_{n+1} = x_n + 2\pi M/p_{n+1}$ (modulo 2π), ruling the motion of a ball bouncing between a fixed wall and a moving wall oscillating sinusoidally in time, in the limit where the amplitude of the oscillations is much smaller than the distance between the walls. p is the velocity of the ball in units proportional to the maximum velocity of the moving wall. x is the phase of the moving wall at the time of collision. The parameter M is proportional to the ratio of the distance between the walls to the amplitude of the oscillations of the moving wall. (See Lichtenberg & Lieberman 1983 for more detail.)

Complex maps have invariant subsets called Julia sets which are defined as the closure of the set of repelling periodic orbits (Devaney 1986). The motion is typically chaotic on the Julia set which is repelling and often separates the basins of attraction of the attractors. For instance, the map

$$z_{n+1} = \frac{z_n}{2} + \frac{1}{2z_n} \ , \tag{24}$$

derived from Eq. (23) with $f(z) = z^2 - 1$ has the attractors $z = \pm 1$. Their respective basins of attraction $x > 0$ and $x < 0$ are separated by the line $x = 0$ where the dynamics is ruled by Eq. (24) with $z = iy$. This one-dimensional map is conjugated to the dyadic map

$$\chi_{n+1} = \begin{cases} 2\chi_n - \frac{\pi}{2} & \text{if} \quad 0 \leq \chi_n \leq \frac{\pi}{2} \ , \\ 2\chi_n + \frac{\pi}{2} & \text{if} \quad -\frac{\pi}{2} < \chi_n \leq 0 \ , \end{cases} \tag{25}$$

by the transformation $y = \tan \chi$, which shows that the dynamics is chaotic on this Julia set.

However, the boundaries between the basins of attraction are typically fractal (Ott 1993) as it is the case for the Newton-Raphson map (23) with $f(z) = \mathrm{e}^z - 1$ (see Fig. 3).

Maps and probability. An important issue is to understand how maps evolve probability in their phase space. The time evolution of probability densities is ruled by the so-called Frobenius-Perron equation (Lasota & Mackey 1985). The probability density at the current point $\mathbf{x}$ comes from all the points $\mathbf{y}$ which are mapped onto $\mathbf{x}$. Since the inverse of an endomorphism is not unique, the Frobenius-Perron equation is composed of the sum

$$\rho_{n+1}(\mathbf{x}) = \sum_{\mathbf{y}:\, \boldsymbol{\phi}(\mathbf{y})=\mathbf{x}} \frac{\rho_n(\mathbf{y})}{\left| \det \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}}(\mathbf{y}) \right|} \ . \tag{26}$$

For an automorphism, the sum reduces to the single term corresponding to the unique inverse. An invariant probability measure is obtained as a solution of the Frobenius-Perron equation such that $\rho_{n+1}(\mathbf{x}) = \rho_n(\mathbf{x})$. The study of invariant measures is the subject of ergodic theory (Hopf 1937, Arnold & Avez 1968, Cornfeld et al. 1982). The knowledge

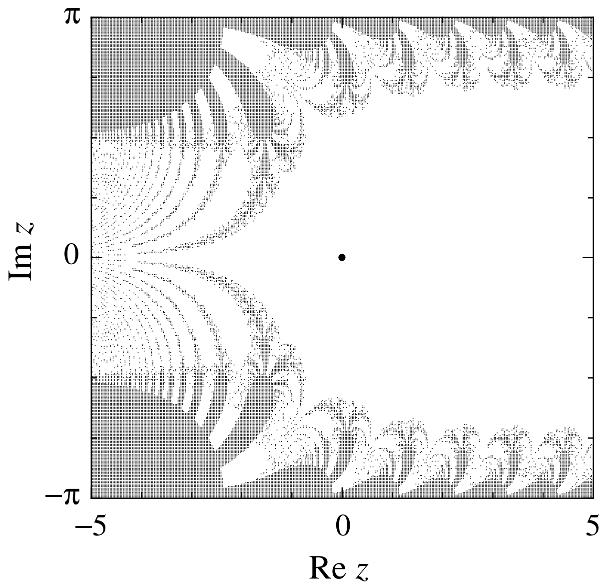

FIG. 3: Complex map $z_{n+1} = z_n - 1 + \exp(-z_n)$: basin of attraction of the point at infinity in grey and of the attractors $z = 0, \pm 2\pi i, \pm 4\pi i, \ldots$ in white. The dot is the attractor $z = 0$.

of the ergodic invariant measure provides us with the statistics of the quantities of interest: observables, correlation functions, Lyapunov exponents, the Kolmogorov-Sinai entropy, etc. With these tools, transport properties such as normal and anomalous diffusion can also be studied in maps (Lichtenberg & Lieberman 1983).

Some applications. Dissipative maps are used to study chaos in hydrodynamics (Lorenz 1963), chemical kinetics (Scott 1991), biology (Olsen & Degn 1985, Murray 1993), nonlinear optics (Ikeda et al. 1980), and beyond. In particular, systems with time delay in some feedback can be approximated by maps, as in the nonlinear optics of a ring cavity (see Fig. 4).

Dissipative maps are also used to study complex systems composed of many interacting units. The units may form a lattice or a graph and interact with each other by diffusive or global couplings. These high-dimensional maps are often called *coupled map lattices* in reference to their spatial extension.

Besides, area-preserving and symplectic maps have become a fundamental tool to study the long-term evolution of the Solar system (Murray & Dermott 1999).

See also Anosov and axiom A systems; Attractors; Aubry-Mather theory; Billiards; Cat map; Chaotic dynamics; Coupled map lattice; Denjoy theory; Entropy; Ergodic theory; Hamiltonian systems; Horseshoe and hyperbolicity in dynamical systems; Kolmogorov-Arnold-Moser theorem; Lyapunov exponents; Maps in the complex plane; One-dimensional maps; Phase space; Symbolic dynamics.

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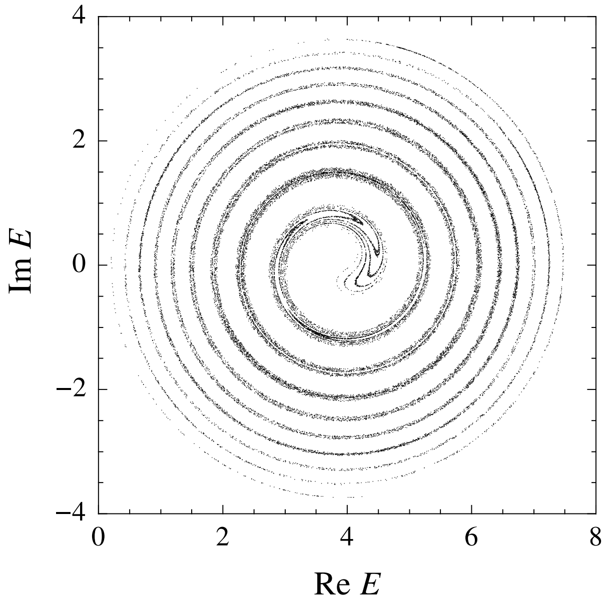

FIG. 4: Chaotic attractor of the dissipative Ikeda map $E_{n+1} = a + bE_n \exp(i|E_n|^2 - ic)$ ruling the complex amplitude $E_n \in \mathbb{C}$ of the electric field of light transmitted in a ring cavity containing a nonlinear dielectric medium, at each passage along the ring (Ikeda et al. 1980). The parameters take the values $a = 3.9$, $b = 0.5$, and $c = 1$.

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